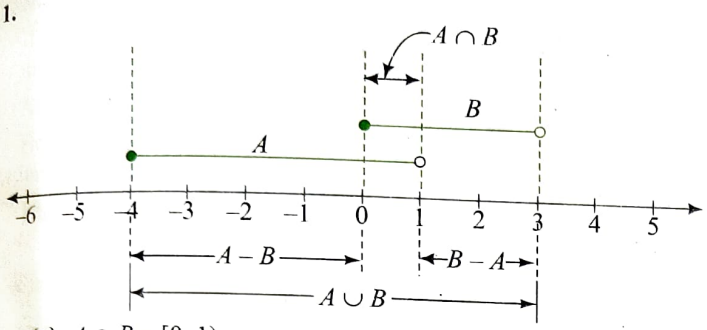
Solutions

Chapter 1

Concept Application Exercises

Exercise 1.1

1.



(a) $A \cap B = [0, 1)$

(b) $A \cup B = [-4, 3)$

(c) $A - B = [-4, 0)$

(d) $B - A = [1, 3)$

(e) $(A \cup B)' = (-\infty, -4) \cup [3, \infty)$

(f) $(A \cap B)' = (-\infty, 0) \cup [1, \infty)$

2. (a) $-5 \le x \le -1$

$\Rightarrow \quad x^2 \in [1, 5]$

(b) $3 < x < 6$

$\Rightarrow \quad x^2 \in (9, 36)$

(c) $-2 < x \le 3$

$\Rightarrow \quad -2 < x < 0$ or $0 \le x \le 3$

For $-2 < x < 0$, $x^2 \in (0, 4)$ (1)

and for $0 \le x \le 3$, $x^2 \in [0, 9]$ (2)

From (1) and (2), $x^2 \in [0, 9]$

Alternatively, $x \in (-2, 3]$, now least value of x^2 is 0 which occurs when $x = 0$

Greatest value of x^2 is 9 for $x = 3$

$\Rightarrow \quad x^2 \in [0, 9]$

(d) $(-3, \infty)$

Here least value of x^2 is 0 for $x = 0$, and when x goes up to infinity, x^2 also goes up to infinity

$\Rightarrow \quad x^2 \in [0, \infty)$

(e) $(-\infty, 4)$

Here least value of x^2 is 0 for $x = 0$ and $x^2 \to \infty$, when $x \to -\infty$

Hence, $x^2 \in [0, \infty)$

3. (a) $2 < x < 5$

$\Rightarrow \quad \frac{1}{2} > \frac{1}{x} > \frac{1}{5}$

$\Rightarrow \quad \frac{1}{x} \in \left(\frac{1}{5}, \frac{1}{2}\right)$

(b) $-5 \le x < -1$

$\Rightarrow \quad -\frac{1}{5} \ge \frac{1}{x} > -1$

$\Rightarrow \quad \frac{1}{x} \in \left(-1, -\frac{1}{5}\right]$

(c) $x > 3$ or $3 < x < \infty$

$\Rightarrow \quad \frac{1}{3} > \frac{1}{x} > 0$

$\Rightarrow \quad \frac{1}{x} \in \left(0, \frac{1}{3}\right)$

(d) $x \le -2$ or $-\infty < x \le -2$

$\Rightarrow \quad 0 > \frac{1}{x} \ge -\frac{1}{2}$

$\Rightarrow \quad \frac{1}{x} \in \left[-\frac{1}{2}, 0\right)$

(e) $x \in [-3\ 4]$ or $-3 \le x \le 4$

Now for $\frac{1}{x}$ to get defined, we must have $-3 \le x < 0$ or $0 < x \le 4$

$\Rightarrow \quad -\frac{1}{3} \ge \frac{1}{x} > -\infty$ or $\frac{1}{4} \le \frac{1}{x} < \infty$

$\Rightarrow \quad \frac{1}{x} \in \left(-\infty, -\frac{1}{3}\right] \cup \left[\frac{1}{4}, \infty\right)$

4. (a) $4x^2 + 28x + 41 = (2x + 7)^2 - 8$

Now, $(2x + 7)^2 \ge 0 \ \forall \ x \in R$.

$\Rightarrow \quad (2x + 7)^2 - 8 \ge -8 \ \ \forall \ x \in R$

So, the values of the expression are $[-8, \infty)$.

For $x \in [-3, 2]$

$-3 \le x \le 2$

$\Rightarrow \quad -6 \le 2x \le 4$

$\Rightarrow \quad 1 \le 2x + 7 \le 11$

$\Rightarrow \quad 1 \le (2x + 7)^2 \le 121$

$\Rightarrow \quad -7 \le (2x + 7)^2 - 8 \le 113$

Thus, for $x \in [-3, 2]$, the values of the expression are $[-7, 113]$.

(b) $1 + 6x - x^2 = 10 - (x - 3)^2$

Now, $(x - 3)^2 \ge 0 \ \ \forall \ x \in R$.

$\Rightarrow \quad -(x - 3)^2 \le 0 \ \forall \ x \in R$

$\Rightarrow \quad 10 - (x - 3)^2 \le 10 \ \forall \ x \in R$

So, the values of the expression are $(-\infty, 10]$.

For $x \in [-3, 2]$

$-3 \le x \le 2$

$\Rightarrow \quad -6 \le x - 3 \le -1$

$\Rightarrow \quad 1 \le (x - 3)^2 \le 36$

$\Rightarrow \quad -36 \le -(x - 3)^2 \le -1$

$\Rightarrow \quad -26 \le 10 - (x - 3)^2 \le 9$

Thus, for $x \in [-3, 2]$ values of the expression are $[-26, 9]$.

5. $\frac{2 + x^2}{4 - x^2} = \frac{6 - (4 - x^2)}{4 - x^2} = \frac{6}{4 - x^2} - 1$

Now, $x^2 \ge 0$

$\therefore \quad -x^2 \le 0$

$\Rightarrow \quad 4 - x^2 \le 4$

$$\Rightarrow \quad \frac{4-x^2}{6} \le \frac{2}{3}$$

$$\Rightarrow \quad -\infty < \frac{4-x^2}{6} < 0 \text{ or } 0 < \frac{4-x^2}{6} \le \frac{2}{3}$$

$$\Rightarrow \quad 0 > \frac{6}{4-x^2} > -\infty \text{ or } \infty > \frac{6}{4-x^2} \ge \frac{3}{2}$$

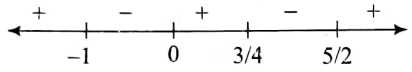
$$\Rightarrow \quad \frac{6}{4-x^2} \in (-\infty, 0) \cup \left[\frac{3}{2}, \infty\right)$$

$$\Rightarrow \quad \left(\frac{6}{4-x^2} - 1\right) \in (-\infty, -1) \cup \left[\frac{1}{2}, \infty\right)$$

6. $\dfrac{x(3-4x)(x+1)}{(2x-5)} < 0$

or $\quad -\dfrac{x(4x-3)(x+1)}{(2x-5)} < 0$

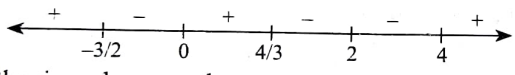
or $\quad \dfrac{x(4x-3)(x+1)}{(2x-5)} > 0$



$\Rightarrow \quad x \in (-\infty, -1) \cup (0, 3/4) \cup (5/2, \infty)$

7. $\dfrac{(2x+3)(4-3x)^3(x-4)}{(x-2)^2x^5} \le 0$

or $\quad \dfrac{(2x+3)(3x-4)^3(x-4)}{(x-2)^2x^5} \ge 0$



From the sign scheme, we have

$x \in (-\infty, -3/2) \cup (0, 4/3] \cup [4, \infty)$

8. $\dfrac{5x+1}{(x+1)^2} - 1 < 0$

or $\quad \dfrac{5x+1-(x+1)^2}{(x+1)^2} < 0$

or $\quad \dfrac{-x^2+3x}{(x+1)^2} < 0$

or $\quad \dfrac{x(x-3)}{(x+1)^2} > 0$

$\Rightarrow \quad x < 0$ or $x > 3, x \ne -1$

9. $\sqrt{-x^2+10x-16} < x-2$

We must have $-x^2 + 10x - 16 \ge 0$

or $\quad x^2 - 10x + 16 \le 0$

$\Rightarrow \quad 2 \le x \le 8 \qquad (1)$

Also $-x^2 + 10x - 16 < x^2 - 4x + 4$

or $\quad 2x^2 - 14x + 20 > 0$

or $\quad x^2 - 7x + 10 > 0$

$\Rightarrow \quad x > 5$ or $x < 2 \qquad (2)$

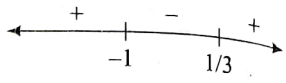
From (1) and (2), $5 < x \le 8 \Rightarrow x = 6, 7, 8$

10. $f(x) = y = \dfrac{1-x^2}{x^2+3}$

or $\quad yx^2 + 3y = 1 - x^2$

or $\quad x^2 = \dfrac{1-3y}{y+1}$

Now $x^2 \ge 0$



$\Rightarrow \quad \dfrac{1-3y}{y+1} \ge 0$

$\Rightarrow \quad \dfrac{3y-1}{y+1} \le 0$

Hence, $y \in (-1, 1/3]$

11. $|x-2| = \begin{cases} x-2, & x \ge 2 \\ 2-x, & x < 2 \end{cases}$

$\Rightarrow \quad |x-2| - (x-2) = \begin{cases} 0, & x \ge 2 \\ 4-2x, & x < 2 \end{cases}$

Therefore, given expression is defined for $(-\infty, 2)$

12. $|4 - |x-1|| = 3$

$\Rightarrow \quad |x-1| - 4 = \pm 3$

$\Rightarrow \quad |x-1| = 7, |x-1| = 1$

$\Rightarrow \quad x = -6, 0, 2, 8$

13. $f(x) = x + \sqrt{x^2} = x + |x|$

$$= \begin{cases} x+x, & x \ge 0 \\ x-x, & x < 0 \end{cases}$$

$$= \begin{cases} 2x, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

Hence, $f(x)$ takes values $[0, \infty)$

14. **(a)** $1 \le |x-2| \le 3$

$\Rightarrow -3 \le x-2 \le -1$ or $1 \le x-2 \le 3$

$\Rightarrow -1 \le x \le 1$ or $3 \le x \le 5$

(b) $0 < |x-3| \le 5$

$\Rightarrow -5 \le x-3 \le 5, x-3 \ne 0$

$\Rightarrow -2 \le x \le 8, x \ne 3$

(c) $|x-2| + |2x-3| = |x-1|$

$\Rightarrow |x-2| + |2x-3| = |(2x-3) - (x-2)|$

$\Rightarrow (x-2)(2x-3) \le 0$

$\Rightarrow 3/2 \le x \le 2$

(d) $\left|\dfrac{x-3}{x+1}\right| \le 1$

$\Rightarrow \quad -1 \le \dfrac{x-3}{x+1} \le 1$

$\Rightarrow \quad 0 \le \dfrac{x-3}{x+1} + 1$ and $\dfrac{x-3}{x+1} - 1 \le 0$

$\Rightarrow \quad 0 \le \dfrac{2x-2}{x+1}$ and $\dfrac{-4}{x+1} \le 0$

$\Rightarrow \quad x \le -1$ or $x \ge 1$ and $x > -1$

$\Rightarrow \quad x \ge 1$

15. $\sqrt{1-\sqrt{x^2-6x+9}}$

$$= \sqrt{1-\sqrt{(x-3)^2}}$$

$$= \sqrt{1-|x-3|} \in [0, 1]$$

Exercise 1.2

1. **(a)** $n(A \times B) = n(A) \times n(B)$

$\therefore \quad 42 = 6 \times n(B) \Rightarrow n(B) = 7.$

(b) For minimum value of $n(A \times B)$, $A = \{x, p, r\}$ and $B = \{p, q, s\}$. So, minimum value of $n(A \times B) = 9$.

2. Given relation R from set A to itself is $R = \{(x, y) : x = 3y$, where $x, y \in A\}$

$\therefore$ $R = \{(3, 1), (6, 2), (9, 3), (12, 4)\}$

Domain of R is $\{3, 6, 9, 12\}$ and range of R is $\{1, 2, 3, 4\}$

Codomain of R is set A.

3. It is given that $A = \{1, 2, 3\}$.

An equivalence relation is reflexive, symmetric and transitive.

The smallest equivalence relation containing $(1, 2)$ is given by,

$R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2,1)\}$

Now, we are left with only four pairs i.e., $(2, 3)$, $(3, 2)$, $(1, 3)$, and $(3, 1)$.

If we add any one pair [say $(2, 3)$] to R_1, then for symmetry we must add $(3, 2)$.

Also, for transitivity we are required to add $(1, 3)$ and $(3, 1)$.

Hence, the only equivalence relation (bigger than R_1) is the universal relation.

This shows that the total number of equivalence relations containing $(1, 2)$ is two.

4. Since $1 + a \cdot a = 1 + a^2 > 0, \forall\, a \in S, \therefore (a, a) \in R$

$\therefore$ R is reflexive

Also $(a, b) \in R \Rightarrow 1 + ab > 0 \Rightarrow 1 + ba > 0 \Rightarrow (b, a) \in R$,

$\therefore$ R is symmetric

$\because$ $(a, b) \in R$ and $(b, c) \in R$ need not imply $(a, c) \in R$.

As $(0.25, -3) \in R$ and $(-3, -4) \in R$, but $(0.25, -4) \notin R$.

So, R is not transitive

5. The given relation is $R = \{(a, b) : a, b \in Z$ and $a - b$ is divisible by $5\}$.

We shall prove that R is reflexive, symmetric and transitive.

(i) R is reflexive as for any $x \in Z$, we have $x - x = 0$ and 0 is divisible by 5

$\Rightarrow$ $x - x$ is divisible by 5

$\Rightarrow$ $(x, x) \in R, \forall\, x \in Z$

$\therefore$ R is reflexive.

(ii) R is symmetric As $(x, y) \in R$, where $x, y \in Z$

$\Rightarrow$ $x - y$ is divisible by 5 [By definition of R]

$\Rightarrow$ $x - y = 5\lambda$ for some $\lambda \in Z$

$\Rightarrow$ $y - x = 5(-\lambda)$

$\Rightarrow$ $y - x$ is also divisible by 5

$\therefore$ R is symmetric.

(iii) R is transitive As $(x, y) \in R$, where $x, y \in Z$

$\Rightarrow$ $x - y$ is divisible by 5

$\Rightarrow$ $x - y = 5\lambda_1$ for some $\lambda_1 \in Z$

Again, for $(y, z) \in R$ where, $y, z \in Z$

$\Rightarrow$ $y - z$ is divisible by 5

$\Rightarrow$ $y - z = 5\lambda_2$ for some $\lambda_2 \in Z$

Now, $(x - y) + (y - z) = 5\lambda_1 + 5\lambda_2$

$\Rightarrow$ $x - z = 5(\lambda_1 + \lambda_2)$

$\Rightarrow$ $x - z$ is divisible by 5 for some $(\lambda_1 + \lambda_2) \in Z$

$\therefore$ R is transitive.

Since, R is reflexive, symmetric and transitive.

Therefore, it is an equivalence relation.

Exercise 1.3

1. (a) $f(x) = \dfrac{1}{\sqrt{x-2}}$ is defined if $x - 2 > 0$ or $x > 2$.

Therefore, domain is $(2, \infty)$.

(b) $f(x) = \dfrac{1}{x^3 - x}$ is not defined if $x^3 - x = 0$ or $x(x - 1)(x + 1) = 0$ or $x = -1, 0, 1$.

Therefore, domain is $R - \{-1, 0, 1\}$.

(c) $f(x) = \sqrt[3]{x^2 - 2}$. We know that cube roots are defined for any real value. So, $x^2 - 2$ can take any real value.

So, x can take any real value. Therefore, domain is set R.

2. (a) $f(x) = 5 - 7x$ or $y = 5 - 7x$. So, $x = \dfrac{5 - y}{7}$.

Clearly, for any real value of y there exists a real number x.

So, range is set R.

(b) $f(x) = 5 - x^2$ or $y = 5 - x^2$. So $x^2 = 5 - y$. Since, $x^2 \geq 0, 5 - y \geq 0$.

So, $y \leq 5$. Therefore, range is $(-\infty, 5]$.

(c) $f(x) = \dfrac{x^2}{x^2 + 1} = y$.

$\therefore$ $x^2 = yx^2 + y \Rightarrow x^2 = \dfrac{y}{1 - y}$.

Since $x^2 \geq 0$ for all real x, $\dfrac{y}{1-y} \geq 0$ or $\dfrac{y}{y-1} \leq 0$.

So, $0 \leq y < 1$.

Therefore, range is $[0, 1)$

3. $f(x) = y = \dfrac{2 - 5x}{3x - 4}$

$\Rightarrow$ $3yx - 4y = 2 - 5x$

$\Rightarrow$ $x = \dfrac{4y + 2}{3y + 5}$

Hence $x \in R - \{4/3\}$ and $y \in R - \{-5/3\}$

4. $f(x) = y = \sqrt{4 - 16x^2}$

We must have $4 - 16x^2 \geq 0 \Rightarrow x^2 \leq 1/4$

$\Rightarrow$ $x \in [-1/2, 1/2]$

Hence domain is $[-1/2, 1/2]$

Also, minimum value of function is 0 and maximum value 2 when $16x^2 = 4$

Hence range is $[0, 2]$

5. $f(x) = \dfrac{x^2 - x - 6}{x - 3} = \dfrac{(x - 3)(x + 2)}{x - 3} = x + 2, x \neq 3$

Hence range is $R - \{5\}$

6. Given relation is function if $2(2) - 3 = 2^3 - a$ or $a = 7$

7. π is irrational number, so $f(\pi) = 2$.

$\dfrac{22}{7}$ is rational number. So, $f\left(\dfrac{22}{7}\right) = 1$.

$\therefore$ $f(\pi) + f\left(\dfrac{22}{7}\right) = 2 + 1 = 3$

8. $f(x) = \begin{cases} x^2 - 4x + 3, & x < 3 \\ x - 4, & x \geq 3 \end{cases}$

$= \begin{cases} x^2 - 4x + 3, & x < 3 \\ x - 4, & 3 \leq x < 4 \\ x - 4, & x \geq 4 \end{cases}$ (1)

$g(x) = \begin{cases} x - 3, & x < 4 \\ x^2 + 2x + 2, & x \geq 4 \end{cases}$

$$= \begin{cases} x-3, & x<3 \\ x-3, & 3 \le x<4 \\ x^2+2x+2, & x \ge 4 \end{cases} \quad (2)$$

From (1) and (2), we have

$$\frac{f(x)}{g(x)} = \begin{cases} \dfrac{x^2-4x+3}{x-3}, & x<3 \\ \dfrac{x-4}{x-3}, & 3<x<4 \\ \dfrac{x-4}{x^2+2x+2}, & x \ge 4 \end{cases}$$

Clearly, $f(x) / g(x)$ is not defined at $x = 3$. Hence, the domain is $R - \{3\}$.

9. $f(x) = \sqrt{x^2-4x+4} = \sqrt{(x-2)^2} = |x-2|$

$$g(x) = |x| - |2| = |x| - 2 = \begin{cases} -x-2, & x<2 \\ x-2, & x \ge 2 \end{cases}.$$

Thus g(x) is not same as $|x-2|$

$$h(x) = \frac{|x-2|^2}{|x-2|} = |x-2|, x \ne 2.$$

This is not same as $|x-2|$ as $h(x)$ has domain $R - \{2\}$

$$t(x) = \left|\frac{x^2-x-2}{x+1}\right| = \left|\frac{(x-2)(x+1)}{x+1}\right| = |x-2|, x \ne -1.$$

Thus $t(x)$ is not same as $|x-2|$ as $t(x)$ has domain $R - \{-1\}$.

Exercise 1.4

1. (b) Clearly, $f(x)$ must be $x + 2$ as for this function, each image has its preimage and each image has one and only one preimage.

2. When n is even, let

$$f(2m_1) = f(2m_2)$$

or $$-\frac{2m_1}{2} = -\frac{2m_2}{2}$$

or $$m_1 = m_2$$

When n is odd, let

$$f(2m_1+1) = f(2m_2+1)$$

or $$\frac{2m_1+1-1}{2} = \frac{2m_2+1-1}{2} \quad \text{or } m_1 = m_2$$

Therefore, $f(x)$ is one-one.

Also, when n is even, $-\frac{n}{2} = -\frac{2m}{2} = -m$.

When n is odd, $\frac{n-1}{2} = \frac{2m+1-1}{2} = m$.

Hence, the range of the function is Z.

Therefore, function is onto.

3. $f(x) = f(-x)$. So, f is many-one.

Also, $f(x) = 1 - \frac{5}{x^2+1} \in [-4, 1)$ So, f is into.

4. $f(x) = \sin x - \sqrt{3}\cos x + 1$

$$= 2\left(\sin x \frac{1}{2} - \cos x \frac{\sqrt{3}}{2}\right) + 1$$

$$= 2\left(\sin x \cos\frac{\pi}{3} - \cos x \sin\frac{\pi}{3}\right) + 1$$

$$= 2\sin\left(x - \frac{\pi}{3}\right) + 1$$

Clearly, f is onto, when the interval of S is $[-1, 3]$.

5. $g(x)$ is surjective if

$$\frac{1}{2} \le \frac{x^2-k}{1+x^2} < 1 \ \forall x \in R$$

or $$\frac{1}{2} \le 1 - \frac{(k+1)}{x^2+1} < 1 \ \forall x \in R$$

or $$-\frac{1}{2} \le -\frac{(k+1)}{x^2+1} < 0 \ \forall x \in R$$

or $$0 < \frac{(k+1)}{x^2+1} \le \frac{1}{2} \ \forall x \in R$$

or $k + 1 > 0$;

So, $k > -1$ (1)

and $$\frac{k+1}{x^2+1} \le \frac{1}{2} \ \forall x \in R$$

or $x^2 - (2k+1) \ge 0 \ \forall x \in R$

or $4(2k+1) \le 0$

$\therefore$ $k \le -\frac{1}{2}$ (2)

From (1) and (2), $k \in \left(-1, -\frac{1}{2}\right]$.

6. $f(x) = e^{x^2} + \cos x$

Clearly, $f(1) = f(-1), f(2) = f(-2)$ etc.

So, $f(x)$ is many-one.

Range of e^{x^2} is $[1, \infty)$ and range of $\cos x$ is $[-1, 1]$.

Thus $f(x)$ does not take negative values.

Hence $f(x)$ is into.

7. We have $f(x) = 2x + \cos x + \sin x$

$\therefore$ $f'(x) = 2 - \sin x + \cos x > 0 \ \forall x \in R$.

Also, $f(x)$ exists continuously.

Hence $f(x)$ is one-one.

Clearly range of $f(x)$ is R.

Therefore $f(x)$ is onto.

8. We have $f(x) = x^3 + px^2 + qx + r$

$\therefore$ $f'(x) = 3x^2 + 2px + q, \forall x \in R$

$\Rightarrow$ Discriminant, $D \le 0$ (as coefficient of x^2 is positive)

$\Rightarrow$ $4p^2 - 12q \le 0$

$\Rightarrow$ $p^2 \le 3q$

Exercise 1.5

1. $y = x^2 + (k-1)x + 9 = \left(x + \frac{k-1}{2}\right)^2 + 9 - \left(\frac{k-1}{2}\right)^2$

For entire graph to be above x-axis, we should have

$$9 - \left(\frac{k-1}{2}\right)^2 > 0$$

$\Rightarrow$ $k^2 - 2k - 35 < 0$

$\Rightarrow$ $(k-7)(k+5) < 0$

$\Rightarrow$ $-5 < k < 7$

2. Let $\frac{x^2+34x-71}{x^2+2x-7} = y$

or $x^2(1-y) + 2(17-y)x + (7y-71) = 0$

For the real value of x,

$b^2 - 4ac \geq 0$

or $4(17 - y)^2 - 4(1 - y)(7y - 71) \geq 0$

or $y^2 - 14y + 45 \geq 0$

i.e., $y \leq 5$ or $y \geq 9$

Hence, range is $(-\infty, 5] \cup [9, \infty)$.

3. Let $y = \sqrt{x-1} + \sqrt{5-x}$

or $y^2 = x - 1 + 5 - x + 2\sqrt{(x-1)(5-x)}$

or $y^2 = 4 + 2\sqrt{-x^2 - 5 + 6x}$

or $y^2 = 4 + 2\sqrt{4 - (x-3)^2}$

Then y^2 has minimum value 4 [when $4 - (x-3)^2 = 0$] and maximum value 8 when $x = 3$.

Therefore, $y \in [2, 2\sqrt{2}]$.

4. $f(x) = \sqrt{x^2 + ax + 4}$ is defined for all x. Therefore,

$x^2 + ax + 4 \geq 0$ for all x

or $D = a^2 - 16 \leq 0$

or $a \in [-4, 4]$

5. $f(x) = \sqrt{3 - 2x - x^2}$ is defined if

$3 - 2x - x^2 \geq 0$

or $x^2 + 2x - 3 \leq 0$

or $(x-1)(x+3) \leq 0$

or $x \in [-3, 1]$

Also, $f(x) = \sqrt{4 - (x+1)^2}$ which has maximum value when $x + 1 = 0$.

Hence, the range is $[0, 2]$.

Exercise 1.6

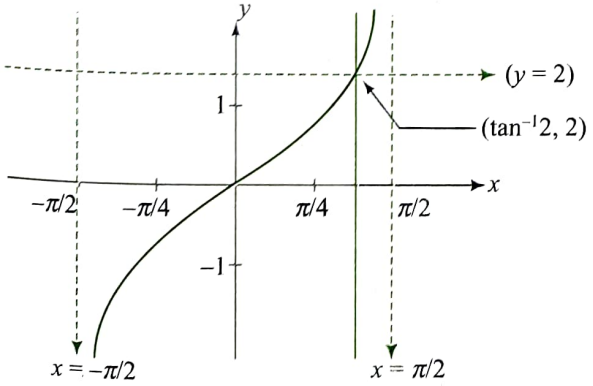
1. $f(x) = \sqrt{\sin x} + \sqrt{16 - x^2}$

or $\sin x \geq 0$ and $16 - x^2 \geq 0$

or $2n\pi \leq x \leq (2n+1)\pi$ and $-4 \leq x \leq 4$

Therefore, domain is $[-4, -\pi] \cup [0, \pi]$.

2. **(a)** We know that $\tan x$ is periodic with period π. So, check the solution in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.



It is clear from the figure that $\tan x < 2$ when

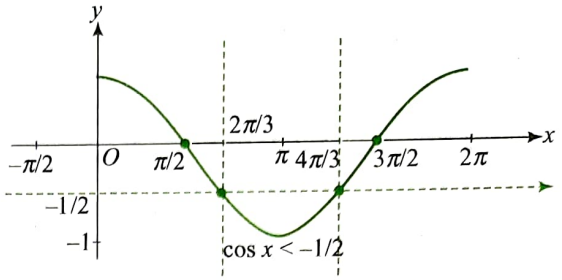
$$-\frac{\pi}{2} < x < \tan^{-1} 2.$$

Therefore, general solution is

$$\bigcup_{n \in Z}\left(n\pi - \frac{\pi}{2}, n\pi + \tan^{-1} 2\right)$$

(b) $\cos x$ is periodic with period 2π.

So, check the solution in $[0, 2\pi]$.



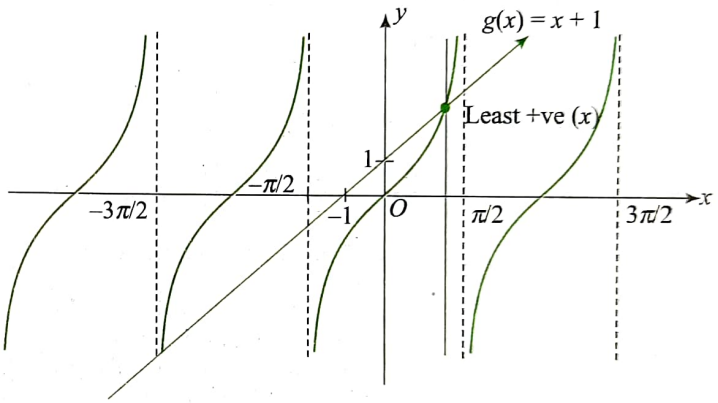
It is clear from the figure that $\cos x \leq -\frac{1}{2}$ when

$$\frac{2\pi}{3} \leq x \leq \frac{4\pi}{3}.$$

On generalizing the above solution, we get

$$x \in \bigcup_{n \in Z}\left[2n\pi + \frac{2\pi}{3}, 2n\pi + \frac{4\pi}{3}\right]$$

3. Let $f(x) = \tan x$ and $g(x) = x + 1$, which can be shown as follows.



From the figure, $\tan x = x + 1$ has infinitely many solutions but the least positive value of x lies in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

4. $f(x) = \sec\left(\frac{\pi}{4}\cos^2 x\right)$

We know that $0 \leq \cos^2 x \leq 1$, i.e.,

$$0 \leq \frac{\pi}{4}\cos^2 x \leq \frac{\pi}{4}$$

For the above value of $\theta = \frac{\pi}{4}\cos^2 x$, $\sec x$ is an increasing function.

$$\therefore \quad f(x) \in \left[f(0), f\left(\frac{\pi}{4}\right)\right] \text{ or } f(x) \in [1, \sqrt{2}]$$

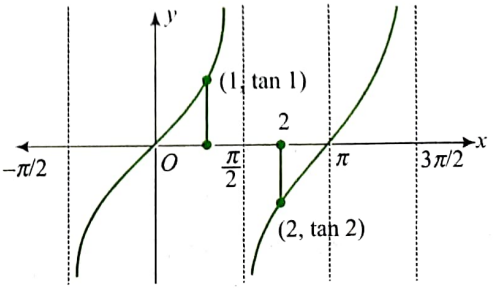
5. $f(x) = \tan x, x \in [1, 2]$

Here, the limited values of x are given.

The best way to get the range of $\tan x$ for such values of x is graphical one.

Consider the graph of $f(x) = \tan x$ for $x \in [1, 2]$.

Clearly, from the graph, $\tan x \in (-\infty, \tan 2] \cup [\tan 1, \infty)$.



6. $f(x) = \dfrac{1}{1-3\sqrt{1-\sin^2 x}} = \dfrac{1}{1-3\sqrt{\cos^2 x}}$

$= \dfrac{1}{1-3|\cos x|}$

Now, $-3|\cos x| \in [-3, 0]$

or $1 - 3|\cos x| \in [-2, 1]$

or $\dfrac{1}{1-3|\cos x|} \in (-\infty, -1/2] \cup [1, \infty)$

7. $f(x) = \dfrac{2\sin^2 x + 2\sin x + 3}{\sin^2 x + \sin x + 1}$

$= 2 + \dfrac{1}{\left(\sin x + \dfrac{1}{2}\right)^2 + \dfrac{3}{4}}$

$f(x)_{\text{min.}} = 2 + \dfrac{1}{\frac{9}{4} + \frac{3}{4}} = \dfrac{7}{3}$

$f(x)_{\text{max.}} = 2 + \left(\dfrac{1}{3/4}\right) = \dfrac{10}{3}$

Hence, range is $\left[\dfrac{7}{3}, \dfrac{10}{3}\right]$.

8. $y = (\sin 2x)\sqrt{1+\tan^2 x}$

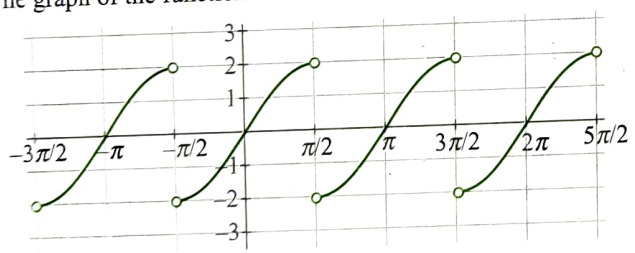
$= (2\sin x \cos x)|\sec x|$

$= \begin{cases} 2\sin x, & \cos x > 0 \\ -2\sin x, & \cos x < 0 \end{cases}$

$= \begin{cases} 2\sin x, & x \in \text{1st and 4th quadrant} \\ -2\sin x, & x \in \text{2nd and 3rd quadrant} \end{cases}$

Clearly, domain is $R - \left\{(2n+1)\dfrac{\pi}{2}, n \in Z\right\}$.

The graph of the function is shown in the following figure.



From the graph, range of the function is $(-2, 2)$.

Exercise 1.7

1. (a) $f(x)$ is defined if $x \in [-1, 1]$ and $x \neq 0$, i.e.,

$x \in [-1, 0) \cup (0, 1]$

(b) $f(x) = \sin^{-1}(|x-1| - 2)$

Since the domain of $\sin^{-1} x$ is $[-1, 1]$, $f(x)$ is defined if

$-1 \le |x-1| - 2 \le 1$

or $1 \le |x-1| \le 3$

i.e., $-3 \le x - 1 \le -1$ or $1 \le x - 1 \le 3$

i.e., $-2 \le x \le 0$ or $2 \le x \le 4$

or domain $= [-2, 0] \cup [2, 4]$

(c) $-1 \le 1 + 3x + 2x^2 \le 1$

or $2x^2 + 3x + 1 \ge -1$

or $2x^2 + 3x + 2 \ge 0$ (1)

and $2x^2 + 3x \le 0$ (2)

From equation (2), $2x^2 + 3x \le 0$ or $2x\left(x + \dfrac{3}{2}\right) \le 0$

or $\dfrac{-3}{2} \le x \le 0$ or $x \in \left[-\dfrac{3}{2}, 0\right]$

In equation (1), we get imaginary root for $2x^2 + 3x + 2 = 0$ and $2x^2 + 3x + 2 \ge 0$ for all x. Therefore,

domain of function $= \left[-\dfrac{3}{2}, 0\right]$

(d) To define $f(x)$, $9 - x^2 > 0$ or $-3 < x < 3$ (1)

$-1 \le (x-3) \le 1$ or $2 \le x \le 4$ (2)

From equations (1) and (2), $2 \le x < 3$, i.e., $x \in [2, 3)$.

(e) $f(x) = \cos^{-1}\left(\dfrac{6-3x}{4}\right) + \text{cosec}^{-1}\left(\dfrac{x-1}{2}\right)$

For $\cos^{-1}\left(\dfrac{6-3x}{4}\right)$, $-1 \le \dfrac{6-3x}{4} \le 1$ (1)

or $-4 \le 6 - 3x \le 4$

or $-10 \le -3x \le -2$

or $2/3 \le x \le 10/3$

For $\text{cosec}^{-1}\left(\dfrac{x-1}{2}\right)$, $\dfrac{x-1}{2} \le -1$ or $\dfrac{x-1}{2} \ge 1$

i.e., $x \le -1$ or $x \ge 3$ (2)

From equations (1) and (2), $x \in \left[3, \dfrac{10}{3}\right]$.

(f) $f(x) = \sqrt{\sec^{-1}\left(\dfrac{2-|x|}{4}\right)}$

$\sec^{-1}$ function always takes positive values which are $[0, \pi] - \{\pi/2\}$.

Hence, the given function is defined if

$\dfrac{2-|x|}{4} \le -1$ or $\dfrac{2-|x|}{4} \ge 1$

i.e., $|x| \ge 6$ or $|x| \le -2$, i.e., $x \in (-\infty, -6] \cup [6, \infty)$

2. $f(x) = \tan^{-1}\left(\sqrt{(x-1)^2 + 1}\right)$

Now, $(x-1)^2 + 1 \in [1, \infty)$

or $\tan^{-1}\left(\sqrt{(x-1)^2 + 1}\right) \in \left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

3. $y = (x^2 - 1)^2 + 2 \ge 2$

or $\log_{0.5}(x^4 - 2x^2 + 3) \le -1$

or $\cot^{-1}\log_{0.5}\left(x^4 - 2x^2 + 3\right) \in \left[\dfrac{3\pi}{4}, \pi\right)$

4. We have $f(x) = \sqrt{\sin^{-1}(2x) + \dfrac{\pi}{6}}$

We must have $\sin^{-1}(2x) + \dfrac{\pi}{6} \ge 0$

$\Rightarrow \quad \sin^{-1}(2x) \ge -\dfrac{\pi}{6}$ ...(1)

But $-\dfrac{\pi}{2} \le \sin^{-1}(2x) \le \dfrac{\pi}{2}$ ...(2)

From (1) and (2), we have

$-\dfrac{\pi}{6} \le \sin^{-1}(2x) \le \dfrac{\pi}{2}$

$\Rightarrow \quad \sin\left(-\frac{\pi}{6}\right) \le 2x \le \sin\frac{\pi}{2}$

$\Rightarrow \quad -\frac{1}{2} \le 2x \le 1$

$\Rightarrow \quad -\frac{1}{4} \le x \le \frac{1}{2}$

Hence, domain is $\left[-\frac{1}{4}, \frac{1}{2}\right]$

5. We have $f(x) = \sin^{-1}x + \tan^{-1}x + \sec^{-1}x$

Domain of $\sin^{-1}x$ is $[-1, 1]$.

Domain of $\tan^{-1}x$ is R.

Domain of $\sec^{-1}x$ is $R - (-1, 1)$

So, domain $f(x)$ is $\{-1, 1\}$

$f(-1) = \sin^{-1}(-1) + \tan^{-1}(-1) + \sec^{-1}(-1)$

$= -\pi/2 - \pi/4 + \pi = \pi/4$

$f(1) = \sin^{-1}(1) + \tan^{-1}(1) + \sec^{-1}(1)$

$= \pi/2 + \pi/4 + 0 = 3\pi/4$

Hence range is $\{\pi/4, 3\pi/4\}$

ercise 1.8

1. $4^x + 8^{\frac{2}{3}(x-2)} - 13 - 2^{2(x-1)} \ge 0$

or $\quad 4^x + \frac{4^x}{16} - \frac{4^x}{4} \ge 13$

or $\quad 4^x \ge 4^2 \quad$ or $\quad x \in [2, \infty)$

2. $f(x) = \sin^{-1}(\log_2 x)$

Since the domain of $\sin^{-1}x$ is $[-1, 1]$, $f(x) = \sin^{-1}(\log_2 x)$ is defined if

$-1 \le \log_2 x \le 1$

or $\quad 2^{-1} \le x \le 2^1$

or $\quad \frac{1}{2} \le x \le 2$

or $\quad$ domain $= \left[\frac{1}{2}, 2\right]$

3. $f(x) = \log_{(x-4)}(x^2 - 11x + 24)$.

$f(x)$ is defined if

$x - 4 > 0$ and $\ne 1$ and $x^2 - 11x + 24 > 0$

or $\quad x > 4$ and $\ne 5$ and $(x - 3)(x - 8) > 0$

i.e., $\quad x > 4$ and $\ne 5$ and $x < 3$ or $x > 8$

or $\quad x > 8$

or domain $(y) = (8, \infty)$

4. $f(x) = \frac{3}{4 - x^2} + \log_{10}(x^3 - x)$

f is defined when

$x \ne \pm 2$ and $x^3 - x > 0$

or $\quad x \ne \pm 2$ and $x(x^2 - 1) > 0$

or $\quad x \ne \pm 2, x \in (-1, 0) \cup (1, \infty)$

or $\quad x \in (-1, 0) \cup (1, 2) \cup (2, \infty)$

5. $f(x) = \sqrt{\frac{\log_{0.3}|x-2|}{|x|}}$. Here, $|x| > 0 \ \forall\ x \in R - \{0\}$ (1)

Therefore, for $f(x)$ to get defined,

$\log_{0.3}|x - 2| \ge 0$

or $\quad 0 < |x - 2| \le 1$

or $\quad |x - 2| \le 1$ and $x \ne 2$

or $\quad -1 \le x - 2 \le 1$ and $x \ne 2$

or $\quad 1 \le x \le 3$ and $x \ne 2$

or $\quad x \in [1, 2) \cup (2, 3]$

6. $f(x) = \sqrt{\log_{10}\left\{\frac{\log_{10}x}{2(3 - \log_{10}x)}\right\}}$.

Clearly, $f(x)$ is defined if

$\log_{10}\left\{\frac{\log_{10}x}{2(3 - \log_{10}x)}\right\} \ge 0, \ \frac{\log_{10}x}{2(3 - \log_{10}x)} > 0$, and $x > 0$

or $\quad \frac{\log_{10}x}{2(3 - \log_{10}x)} \ge 1, \ \frac{\log_{10}x}{\log_{10}x - 3} < 0$, and $x > 0$

or $\quad \frac{3(\log_{10}x - 2)}{2(\log_{10}x - 3)} \le 0, \ \frac{\log_{10}x}{\log_{10}x - 3} < 0$, and $x > 0$

or $\quad 2 \le \log_{10}x < 3, \ 0 < \log_{10}x < 3$, and $x > 0$

or $\quad 10^2 \le x < 10^3, \ 10^0 < x < 10^3$, and $x > 0$

or $\quad x \in [10^2, 10^3)$

7. $f(x) = \frac{1}{\sqrt{\log_{1/2}(x^2 - 7x + 13)}}$ exists if

$\log_{1/2}(x^2 - 7x + 13) > 0$

or $\quad x^2 - 7x + 13 < 1$ (1)

and $\quad x^2 - 7x + 13 > 0$ (2)

or $\quad x^2 - 7x + 12 < 0$ and $\left(x - \frac{7}{2}\right)^2 + \frac{3}{4} > 0$

or $\quad 3 < x < 4$ and $x \in R$

or $\quad 3 < x < 4$

8. $-\sqrt{2} \le \sin x - \cos x \le \sqrt{2}$

or $\quad 2\sqrt{2} \le \sin x - \cos x + 3\sqrt{2} \le 4\sqrt{2}$

or $\quad 2 \le \frac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}} \le 4$

or $\quad \log_2 2 \le \log_2\left(\frac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}}\right) \le \log_2 4$

or $\quad 1 \le \log_2\left(\frac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}}\right) \le 2$

9. $f(x) = \sqrt{\log_2(4\sin^2 x - 2\sqrt{3}\sin x - 2\sin x + \sqrt{3} + 1)}$ is defined if

$\log_2(4\sin^2 x - 2(\sqrt{3}\sin x + \sin x) + (\sqrt{3} + 1)) \ge 0$

or $\quad 4\sin^2 x - 2\sin x(\sqrt{3} + 1) + \sqrt{3} + 1 \ge 1$

or $\quad \sin^2 x - \sin x\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right) + \frac{\sqrt{3}}{4} \ge 0$

or $\quad \left(\sin x - \frac{\sqrt{3}}{2}\right)\left(\sin x - \frac{1}{2}\right) \ge 0$

i.e., $-1 \le \sin x \le \frac{1}{2}$ or $\frac{\sqrt{3}}{2} \le \sin x \le 1$

or $\quad x \in \left[-\pi, \frac{\pi}{6}\right] \cup \left[\frac{\pi}{3}, \frac{2\pi}{3}\right] \cup \left[\frac{5\pi}{6}, \pi\right]$

Exercise 1.9

1. $[x]^2 - 5[x] + 6 = 0$

or $\quad [x] = 2, 3$

or $\quad x \in [2, 4)$

2. $y = 3[x] + 1 = 4[x-1] - 10 = 4[x] - 14$

or $[x] = 15$ and $y = 3(15) + 1 = 46$

or $[x + 2y] = 2y + [x] = 2(46) + 15 = 107$

3. (a) We have $f(x) = \dfrac{1}{\sqrt{x-[x]}}$

We know that $0 \le x - [x] < 1$ for all $x \in R$. Also, $x - [x] = 0$ for $x \in Z$.

Now, $f(x) = \dfrac{1}{\sqrt{x-[x]}}$ is defined if

$x - [x] > 0$

or $x \in R - Z$ $\quad \left[\because x-[x]=0 \text{ for } x \in Z \text{ and } 0 < x-[x] < 1 \text{ for } x \in R-Z\right]$

Hence, domain $= R - Z$.

(b) $f(x) = \dfrac{1}{\log[x]}$

We must have $[x] > 0$ and $[x] \ne 1$ (as for $[x] = 1$, $\log [x] = 0$).

Therefore, $[x] \ge 2$ or $x \in [2, \infty)$.

(c) $f(x) = \log \{x\}$ is defined if $\{x\} > 0$ which is true for all real numbers except integers.

Hence, the domain is $R - Z$.

4. $f(x) = \dfrac{1}{\sqrt{|[|x|-1]|-5}}$ is defined. Therefore,

$|[|x| - 1]| - 5 > 0$

or $|[|x| - 1]| > 5$

i.e., $[|x| - 1] < -5$ or $[|x| - 1] > 5$

i.e., $|x| - 1 < -5$ or $|x| - 1 \ge 6$

or $|x| \ge 7$

or $x \in (-\infty, -7] \cup [7, \infty)$

5. (a) $1 - \sin x \ge 0$ or $\sin x \le 1$ or $x \in R$

(b) $1 - 4x^2 > 0$ or $x \in (-1/2, 1/2)$

(c) $\log_5(1 - 4x^2) \ne 0$ or $1 - 4x^2 \ne 1$ or $x \ne 0$

(d) $-1 \le 1 - \{x\} \le 1$ or $0 \le \{x\} \le 2$ or $x \in R$

Hence, domain is common values of (a), (b), (c) and (d),

i.e., $x \in \left(-\dfrac{1}{2}, \dfrac{1}{2}\right) - \{0\}$

6. $f(x) = \cos(\log_e \{x\})$.

For the given function to define,

$0 < \{x\} < 1$

or $-\infty < \log_e \{x\} < 0$

For these values of $\theta = (\log_e \{x\})$, $\cos \theta$ takes its all possible values.

Hence, the range is $[-1, 1]$.

7. $\log_{[x]} \dfrac{|x|}{x}$ is defined if

$\dfrac{|x|}{x} > 0$, $[x] > 0$, and $[x] \ne 1$

or $x > 0$, $x \in [1, \infty)$, and $x \notin [1, 2)$

or $x \in [2, \infty)$

For $x \in [2, \infty)$, we have $\log_{[x]} \dfrac{|x|}{x} = \log_{[x]} 1 = 0$.

Therefore, $f(x) = \cos^{-1} 0 = \pi/2$ for all $x \in [2, \infty)$.

Hence, domain (f) is $[2, \infty)$ and range (f) is $\{\pi/2\}$.

8. $f(x) = \log_{[x-1]} \sin x$, where $[.]$ denotes the greatest integer.

To get the range of $f(x)$, let us examine the values of x for which the function is defined.

$f(x)$ is defined if $\sin x > 0$ and $[x - 1] > 0$ and $[x - 1] \ne 1$, i.e.,

$0 < \sin x \le 1$ and $[x] > 2$

Now, for base of the logarithm is more than '2' and $\sin x \in (0, 1]$, clearly, $\log_{[x-1]} \sin x \in (-\infty, 0]$.

9. For $x \ge 2$, LHS is always non-negative and RHS is always negative.

Hence, for $x \ge 2$, there is no solution.

If $1 \le x < 2$, then $(x - 2) = (x - 1) - 1 = x - 2$, which is an identity.

For $0 \le x < 1$, LHS is 0 and RHS is $(-)$ve.

So, there is no solution.

For $x < 0$, LHS is $(+)$ve, RHS is $(-)$ve.

So, there is no solution.

Hence, $x \in [1, 2)$.

10. We have $f(x) = \dfrac{1}{8\{x\}^2 - 6\{x\} + 1}$

Clearly $f(x)$ is not defined if $8\{x\}^2 - 6\{x\} + 1 = 0$

$\Rightarrow$ $(4\{x\} - 1)(2\{x\} - 1) = 0$

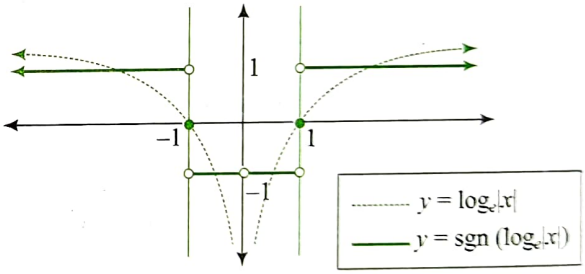
$\Rightarrow$ $\{x\} = 1/4$ or $1/2$

$\Rightarrow$ $x = n + 1/4$ or $x = n + 1/2$, where $n \in Z$

Hence domain of function is

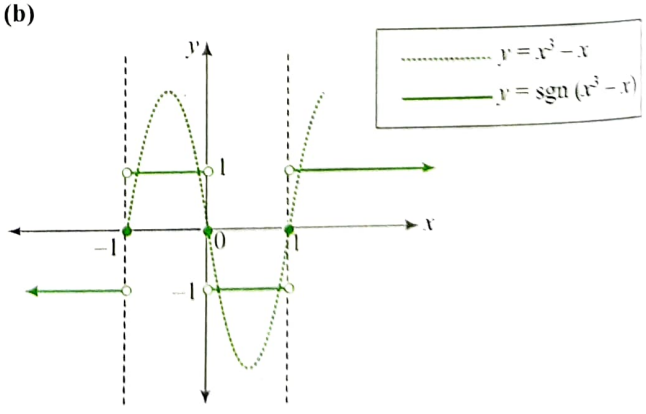
$R - \{x : x = n + 1/4, x = n + 1/2; n \in Z\}$

11. (a)



From the graph $f(x) = \begin{cases} 1, & |x| > 1 \\ -1, & 0 < |x| < 1 \\ 0, & |x| = 1 \end{cases}$

(b)



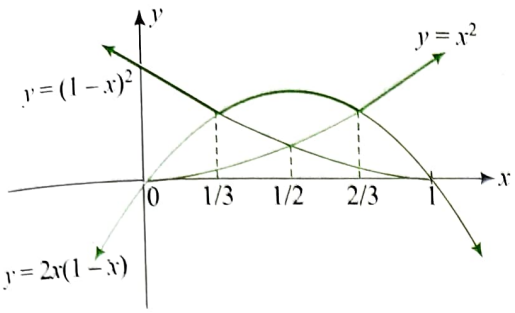
From the graph, $f(x) = \begin{cases} -1, & x < -1,\ 0 < x < 1 \\ 1, & -1 < x < 0,\ x > 1 \\ 0, & x = -1, 0, 1 \end{cases}$.

12. $f(3) = \text{max.}\{1, |3 - 1|, \text{min.}\{4, |9 - 1|\}$

$= \text{max.}\{1, 2, 4\}$

$= 4$

13. Here, for maximum, let us consider $f_1(x) = x^2$, $f_2(x) = (1-x)^2$, and $f_3(x) = 2x(1-x)$.
Now, graph for $f_1(x), f_2(x)$, and $f_3(x)$ are as follows.



Here, neglect the graph that is below the points of intersection.

$$f(x) = \begin{cases} (1-x)^2, & 0 \le x < \frac{1}{3} \\ 2x(1-x), & \frac{1}{3} \le x < \frac{2}{3} \\ x^2, & \frac{2}{3} \le x \le 1 \end{cases}$$

Exercise 1.10

1. $f(-x) = (g(-x) - g(x))^3 = -(g(x) - g(-x))^3 = -f(x)$
Hence, $f(x)$ is an odd function.

2. $\log\left(\frac{x^4+x^2+1}{x^2+x+1}\right) = \log(x^2 - x + 1)$,
which is neither odd nor even.

3. $f(-x) = (-x)g(-x) \cdot g(x) + \tan(\sin(-x))$
$= -(xg(x)\, g(-x) + \tan(\sin x)) = -f(x)$
Hence, $f(x)$ is an odd function.

4. $0 \le \left|\frac{\sin x}{2}\right| \le \frac{1}{2}$ or $\left[\left|\frac{\sin x}{2}\right|\right] = 0$
Therefore, $f(x) = \cos x$, which is even.

5. $f(-x) = \begin{cases} -x|-x|, & -x \le -1 \\ [-x+1]+[1+x], & -1 < -x < 1 \\ -(-x)|-x|, & -x \ge 1 \end{cases}$

$= \begin{cases} -x|x|, & x \ge 1 \\ [1-x]+[1+x], & -1 < x < 1 \\ x|x|, & x \le -1 \end{cases}$

$= f(x)$
Hence, the function is even.

6. Clearly $g(x)$ is the odd extension of the function $f(x)$ and $h(x)$ is the even extension.
Since x^2, $\cos x$, $\log(1 + |x|)$ are even functions and x, $\sin x$ and odd functions.
$g(x) = -x^2 + x + \sin x + \cos x - \log(1 + |x|)$
and $h(x) = x^2 - x - \sin x - \cos x + \log(1 + |x|)$
Clearly this function satisfies the restriction of the problem.

Exercise 1.11

1. **(b)** Since $\cos\sqrt{x}$ is not periodic, $\cos\sqrt{x} + \cos^2 x$ is not periodic though $\cos^2 x$ is periodic.

2. **(a)** $f(x) = \operatorname{sgn}(e^{-x}) = 1$ as $e^{-x} > 0\ \forall\ x \in R$.
Therefore, $f(x)$ is periodic function.

(b) $g(x) = \sin x + |\sin x|$ is periodic with period 2π as $f(x + 2\pi) = f(x)$.

(c) $h(x) = \min(\sin x, |x|) = \sin x$, which is periodic.

(d) $p(x) = \frac{x}{x} = 1, x \ne 0$, which is nonperiodic.

3. **(a)** Period of $|\sin 4x| + |\cos 4x|$ is $\frac{\pi}{8}$.

Period of $|\sin 4x - \cos 4x| + |\sin 4x + \cos 4x| = \frac{\pi}{8}$

So, the period of given function is $\frac{\pi}{8}$.

(b) $f(x) = \sin\frac{\pi x}{n!} - \cos\frac{\pi x}{(n+1)!}$

Period of $\sin\frac{\pi x}{n!}$ is $\frac{2\pi}{\left(\frac{\pi}{n!}\right)} = 2n!$ and period of $\cos\frac{\pi x}{(n+1)!}$ is

$\frac{2\pi}{\frac{\pi}{(n+1)!}} = 2(n+1)!$

Hence, period of $f(x)$ = L.C.M. of $\{2n!, 2(n+1)!\} = 2(n+1)!$

(c) $f(x) = \sin x + \tan\frac{x}{2} + \sin\frac{x}{2^2} + \tan\frac{x}{2^3} + \cdots + \sin\frac{x}{2^{n-1}} + \tan\frac{x}{2^n}$

Period of $\sin x$ is 2π.

Period of $\tan\frac{x}{2}$ is 2π.

Period of $\sin\frac{x}{2^2}$ is 8π.

Period of $\tan\frac{x}{2^3}$ is 8π.

Period of $\tan\frac{x}{2^n}$ is $2^n\pi$.

Hence, period of $f(x)$ = L.C.M. of $(2\pi, 8\pi, \ldots, 2^n\pi) = 2^n\pi$

4. **(p)** $f(x) = \sin^3 x + \cos^4 x$,
$\sin^3 x$ has period 2π and $\cos^4 x$ has period π, and L.C.M. of π and 2π is 2π. Hence, period is 2π.

(q) $f(x) = \sin^4 x + \cos^4 x$
Both $\sin^4 x$ and $\cos^4 x$ have the same period π, and L.C.M. of π and π is π.
But $f(x + \pi/2) = f(x)$. Then period is $\pi/2$.

(r) Both $\sin^3 x$ and $\cos^3 x$ have the same period 2π, and L.C.M. of 2π and 2π is 2π.
Hence, period is 2π, $[(f(x + \pi) \ne f(x)]$.

(s) $f(x) = \cos^4 x - \sin^4 x$
Both $\sin^4 x$ and $\cos^4 x$ have the same period π, and L.C.M. of π and π is π.
Hence, period is π, $[f(x + \pi/2) \ne f(x)]$.

5. Clearly, $f(x) = \tan(\sqrt{[n]}\, x)$ has period $\frac{\pi}{\sqrt{[n]}}$.

But it is given that $\tan(\sqrt{[n]}\, x)$ has period $\frac{\pi}{3}$. Therefore,

$\frac{\pi}{\sqrt{[n]}} = \frac{\pi}{3}$

or $[n] = 9$ or $n \in [9, 10)$.

6. Since the period of $|\sin x| + |\cos x|$ is $\pi/2$, it is possible when $\lambda = 1$.

7. $f(x) = \cos x \cos 2x \cos 3x$

$\cos x$, $\cos 2x$, and $\cos 3x$ are periodic with period 2π, π, and $\frac{2\pi}{3}$

L.C.M. of $\left(\pi, 2\pi, \frac{2\pi}{3}\right) = 2\pi$

But $f(x+\pi) = \cos(x+\pi)\cos(2x+2\pi)\cos(3x+3\pi)$

$= (-\cos x)(\cos 2x)(-\cos 3x)$

$= \cos x \cos 2x \cos 3x$

$= f(x)$

Hence, period is π.

Exercise 1.12

1. $f(x) = \begin{cases} 1+|x|, & x<-1 \\ [x], & x \geq -1 \end{cases}$

$f(-2.3) = 1 + |-2.3| = 1 + 2.3 = 3.3$

Now, $f(f(-2.3)) = f(3.3) = [3.3] = 3$

2. $f(x) = \log\left[\frac{1+x}{1-x}\right]$

or $f\left(\frac{2x}{1+x^2}\right) = \log\left[\frac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}\right] = \log\left[\frac{x^2+1+2x}{x^2+1-2x}\right]$

$= \log\left[\frac{1+x}{1-x}\right]^2 = 2\log\left[\frac{1+x}{1-x}\right] = 2f(x)$

3. $f(x) = \frac{\alpha x}{x+1}, \; x \neq -1$

Now $f(f(x)) = x$

$\Rightarrow \quad \frac{\alpha\left(\frac{\alpha x}{x+1}\right)}{\frac{\alpha x}{x+1}+1} = x$

$\Rightarrow \quad \frac{\alpha^2 x}{(\alpha+1)x+1} = x$

$\Rightarrow \quad (\alpha+1)x^2 + (1-\alpha^2)x = 0 \qquad \ldots(1)$

Now this is true for all real x.

$\Rightarrow \quad \alpha + 1 = 0$ and $1 - \alpha^2 = 0$

$\Rightarrow \quad \alpha = -1$ (common value)

4. Here, $f(x)$ is defined for $x \in [-3, 2]$.

For $g(x) = f(|[x]|)$ to be defined, we must have

$-3 \leq |[x]| \leq 2$

or $0 \leq |[x]| \leq 2$ [As $|x| \geq 0$ for all x]

or $-2 \leq [x] \leq 2$ [As $|x| \leq a \Rightarrow -a \leq x \leq a$]

or $-2 \leq x < 3$ [By the definition of greatest integral function]

Hence, domain of $g(x)$ is $[-2, 3)$.

5. g is meaningful if

$0 \leq 9x^2 - 1 \leq 2 \Leftrightarrow 1 \leq 9x^2 \leq 3$

i.e., $x \in \left[-\frac{1}{\sqrt{3}}, -\frac{1}{3}\right] \cup \left[\frac{1}{3}, \frac{1}{\sqrt{3}}\right]$

6. $g(x)$ is defined if $f(x+1)$ is defined.

Hence, the domain of g is all x such that $(x+1) \in [0, 2]$, i.e.,

$-2 \leq x \leq 1$.

Also, $f(x+1) \in [0, 1]$

$\therefore \quad -f(x+1) \in [-1, 0]$

$\therefore \quad 1 - f(x+1) \in [0, 1]$

Therefore, range of $g(x)$ is $[0, 1]$.

7. $g(f(x)) = \tan\left(x - \frac{\pi}{4}\right) = \frac{\tan x - 1}{\tan x + 1}$

or $g(x) = \frac{x-1}{x+1}$ or $f(g(x)) = \tan\left(\frac{x-1}{x+1}\right)$

8. We have $g(x) = 1 + x - [x] = 1 + \{x\}$, where $\{x\}$ represent fractional part function

and $f(x) = \begin{cases} -1, & x<0 \\ 0, & x=0 \\ 1, & x>0 \end{cases}$

$\Rightarrow \quad f(g(x)) = \begin{cases} -1, & 1+\{x\}<0 \\ 0, & 1+\{x\}=0 \\ 1, & 1+\{x\}>0 \end{cases}$

$\Rightarrow \quad f(g(x)) = 1, \; 1 + \{x\} > 0 \qquad (\because 0 \leq \{x\} < 1)$

$\Rightarrow \quad f(g(x)) = 1 \; \forall \, x \in R$

9. $f(x) = \begin{cases} \log_e x, & 0<x<1 \\ x^2-1, & x \geq 1 \end{cases}$ and $g(x) = \begin{cases} x+1, & x<2 \\ x^2-1, & x \geq 2 \end{cases}$

$g(f(x)) = \begin{cases} f(x)+1, & f(x)<2 \\ (f(x))^2-1, & f(x) \geq 2 \end{cases}$

$= \begin{cases} \log_e x + 1, & \log_e x < 2, \; 0<x<1 \\ x^2-1+1, & x^2-1<2, \; x \geq 1 \\ (\log_e x)^2 - 1, & \log_e x \geq 2, \; 0<x<1 \\ (x^2-1)^2-1, & x^2-1 \geq 2, \; x \geq 1 \end{cases}$

$= \begin{cases} \log_e x + 1, & x<e^2, \; 0<x<1 \\ x^2, & -\sqrt{3}<x<\sqrt{3}, \; x \geq 1 \\ (\log_e x)^2 - 1, & x \geq e^2, \; 0<x<1 \\ (x^2-1)^2-1, & x \leq -\sqrt{3} \text{ or } x \geq \sqrt{3}, \; x \geq 1 \end{cases}$

$= \begin{cases} \log_e x + 1, & 0<x<1 \\ x^2, & 1 \leq x < \sqrt{3} \\ (x^2-1)^2-1, & x \geq \sqrt{3} \end{cases}$

Exercise 1.13

1. $y = \frac{e^x - e^{-x}}{e^x + e^{-x}} + 2 = \frac{e^{2x}-1}{e^{2x}+1} + 2$

or $e^{2x} = \frac{1-y}{y-3} = \frac{y-1}{3-y}$

or $x = \frac{1}{2}\log_e\left(\frac{y-1}{3-y}\right)$

or $f^{-1}(y) = \log_e\left(\frac{y-1}{3-y}\right)^{1/2}$

or $f^{-1}(x) = \log_e\left(\frac{x-1}{3-x}\right)^{1/2}$

2. Let $y = 1 - 2^{-x}$

or $2^{-x} = 1 - y$

or $-x = \log_2 (1 - y)$

or $f^{-1}(x) = g(x) = -\log_2 (1 - x)$

3. Given $f: (2, 3) \to (0, 1)$ and $f(x) = x - [x]$.

$\therefore$ $f(x) = y = x - 2$

or $y + 2 = f^{-1}(y)$

or $f^{-1}(x) = x + 2$

4. Since the domain of the function is Z, we have $f(x) = x + 1$

or $f^{-1}(x) = x - 1$.

5. $f(x) = \begin{cases} x^3 - 1, & x < 2 \\ x^2 + 3, & x \geq 2 \end{cases}$

For $f(x) = x^3 - 1, x < 2$, $f^{-1}(x) = (x + 1)^{1/3}, x < 7$

(As $x < 2 \Rightarrow x^3 < 8 \Rightarrow x^3 - 1 < 7$)

For $f(x) = x^2 + 3, x \geq 2$, $f^{-1}(x) = (x - 3)^{1/2}, x \geq 7$

(As $x \geq 2 \Rightarrow x^2 \geq 4 \Rightarrow x^2 + 3 \geq 7$)

Hence, $f^{-1}(x) = \begin{cases} (x+1)^{1/3}, & x < 7 \\ (x-3)^{1/2}, & x \geq 7 \end{cases}$

6. $f: [-1, 1] \to [-1, 1]$ is defined by

$$f(x) = x|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$

or $f^{-1}(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ -\sqrt{-x}, & x < 0 \end{cases}$

7. $y = 2^{x(x-2)}$

or $x^2 - 2x = \log_2 y$

or $x^2 - 2x - \log_2 y = 0$

or $x = 1 \pm \sqrt{1 + \log_2 y}$

or $f^{-1}(x) = 1 - \sqrt{1 + \log_2 x}$

as $f^{-1}: \left[\frac{1}{2}, \infty\right) \to (-\infty, 1]$

Exercise 1.14

1. Given $f(x + y + 1) = (\sqrt{f(x)} + \sqrt{f(y)})^2$

Putting $x = y = 0$, we get

$f(1) = (\sqrt{f(0)} + \sqrt{f(0)})^2 = (1+1)^2 = 2^2$

Again putting $x = 0, y = 1$, we get

$f(2) = (\sqrt{f(0)} + \sqrt{f(1)})^2 = (1 + 2)^2 = 3^2$

and for $x = 1, y = 1$, we get

$f(3) = (\sqrt{f(1)} + \sqrt{f(1)})^2 = (2+2)^2 = 4^2$

Hence, $f(n) = (n + 1)^2$.

2. $g(a + b) = g(a) \cdot g(b)$

Put $a = b = 0$. Then,

$g(0) = g^2(0) = 1$ [as $g(0) \neq 0$]

Now, put $a = x, b = -x$. Then,

$g(0) = g(x)\, g(-x) = 1$

3. $f(x + 2a) = f(x - 2a)$

Replacing x by $x + 2a$, we get

$f(x) = f(x + 4a)$

Therefore, $f(x)$ is periodic with period $4a$.

4. $f(x + f(y)) = f(x) + y, f(0) = 1$

Putting $y = 0$, we get

$f(x + f(0)) = f(x) + 0$

or $f(x + 1) = f(x)\ \forall\ x \in R$

Thus, $f(x)$ is periodic with 1 as one of its period. Hence,

$f(7) = f(6) = f(5) = \ldots = f(1) = (0) = 1$

5. $f(x) + 3x\, f\left(\frac{1}{x}\right) = 2(x + 1)$ (1)

Replacing x by $\frac{1}{x}$, we get

$$f\left(\frac{1}{x}\right) + 3\frac{1}{x} f(x) = 2\left(\frac{1}{x} + 1\right)$$

or $x f\left(\frac{1}{x}\right) + 3f(x) = 2(x + 1)$ (2)

From (1) and (2), we have $f(x) = \frac{x+1}{2}$.

6. Obviously, f is a linear polynomial.

Let $f(x) = ax + b$. Hence,

$f(x^2 + x + 3) + 2f(x^2 - 3x + 5) \equiv 6x^2 - 10x + 17$

or $[a(x^2 + x + 3) + b] + 2[a(x^2 - 3x + 5) + b] \equiv 6x^2 - 10x + 17$

or $a + 2a = 6$ (1)

and $a - 6a = -10$ (2)

or $a = 2$ (Comparing coeff. of x^2 and coeff. of x both sides)

Again, $3a + b + 10a + 2b = 17$

or $6 + b + 20 + 2b = 17$

$\therefore$ $b = -3$

or $f(x) = 2x - 3$

7. $f(x) \cdot f(y) + f\left(\frac{3}{x}\right) f\left(\frac{3}{y}\right) = 2f(xy)$

Put $x = y = 1$. Then,

$f^2(1) + f^2(3) = 2f(1)$

or $(f(1) - 1)^2 = 0$ or $f(1) = 1$

Now, put $y = 1$. Then

$$f(x) f(1) + f\left(\frac{3}{x}\right) f(3) = 2f(x)$$

or $f(x) = f\left(\frac{3}{x}\right)\ \forall\ x > 0$ (1)

or $f(x) f\left(\frac{3}{x}\right) + f\left(\frac{3}{x}\right) f(x) = 2f(3)$

or $f(x) f\left(\frac{3}{x}\right) = 1$ (2)

Therefore, from (1) and (2),

$f^2(x) = 1\ \forall\ x > 0$

$\therefore$ $f(x) = 1\ \forall\ x > 0$ ($\because f(3) = 1$)

8. Given $f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$ (1)

Putting $x = f(y) = 0$, we get $f(0) = f(0) + 0 + f(0) - 1$, i.e.,

$f(0) = 1$ (2)

Again, putting $x = f(y) = \lambda$ in (1), we get

$f(0) = f(\lambda) + \lambda^2 + f(\lambda) - 1$

or $1 = 2f(\lambda) + \lambda^2 - 1$ [From (2)]

$$f(\lambda) = \frac{2 - \lambda^2}{2} = 1 - \frac{\lambda^2}{2}$$

Hence, $f(x) = 1 - \frac{x^2}{2}$.

9. Given $f^2(x+y) = f^2(x) + f^2(y)$.

Put $x = y = 0$. Therefore,

$f^2(0) = f^2(0) + f^2(0)$

$\therefore \quad f(0) = 0$

Now, replace y by $-x$. Therefore,

$f^2(0) = f^2(x) + f^2(-x)$

or $\quad f^2(x) + f^2(-x) = 0$

$\therefore \quad f(x) = 0$

10. $f(2x+3) + f(2x+7) = 2$ (1)

Replacing x by $x+2$, $f(2x+7) + f(2x+11) = 2$ (2)

From (1) − (2), we get

$f(2x+3) - f(2x+11) = 0$

or $\quad f(2x+3) = f(2x+11)$

Replace $2x+3$ by x. Then

$f(x) = f(x+8)$

Hence, fundamental period of $y = f(x)$ is 8.

11. $x^2 f(x) - 2f\left(\frac{1}{x}\right) = g(x)$ and $2f\left(\frac{1}{x}\right) - 4x^2 f(x) = 2x^2 g\left(\frac{1}{x}\right)$

or $\quad -3x^2 f(x) = g(x) + 2x^2 g\left(\frac{1}{x}\right)$

or $\quad f(x) = -\left(\frac{g(x) + 2x^2 g\left(\frac{1}{x}\right)}{3x^2}\right)$

Since $g(x)$ is odd, $f(-x) = -f(x)$.

But given that $f(x)$ is even.

$\therefore \quad f(x) = 0$

$\therefore \quad f(5) = 0$

12. $f(x) = f(b + (x-b))$

$= f(b - (x-b))$

$= f(2b - x)$

$= f(a + (2b - x - a))$

$= f(a - (2b - x - a))$

$= f(2a - 2b + x)$

Hence, $f(x)$ is periodic with period $2a - 2b$.

13. We have $f(x-y) = f(x)\,f(y) - f(a-x)\,f(a+y)$

Putting $x = a$ and $y = x - a$, we get

$f(a - (x-a)) = f(a)\,f(x-a) - f(0)\,f(x)$ (1)

Putting $x = 0$, $y = 0$, we get

$f(0) = f(0)\,(f(0)) - f(a)\,f(a)$

or $\quad f(0) = (f(0))^2 - (f(a))^2$

or $\quad 1 = (1)^2 - (f(a))^2$

or $\quad f(a) = 0$

$\therefore \quad f(2a - x) = -f(x)$ (from (1))

Replacing x by $a - x$, we get

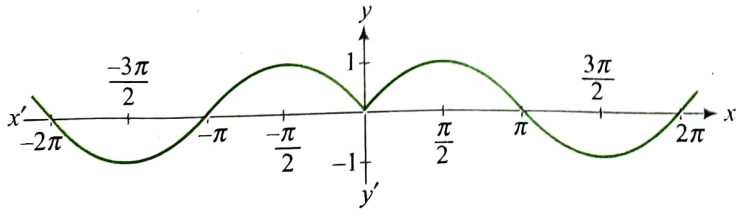
$f(2a - (a-x)) = -f(a-x)$

or $\quad f(a+x) = -f(a-x)$

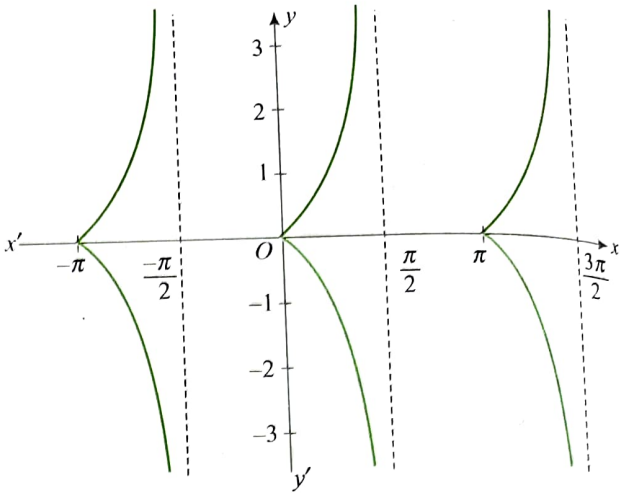
Therefore, $f(x)$ is symmetrical about point $(a, 0)$.

Exercise 1.15

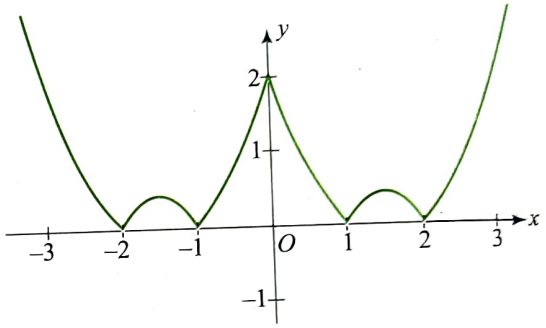
1.



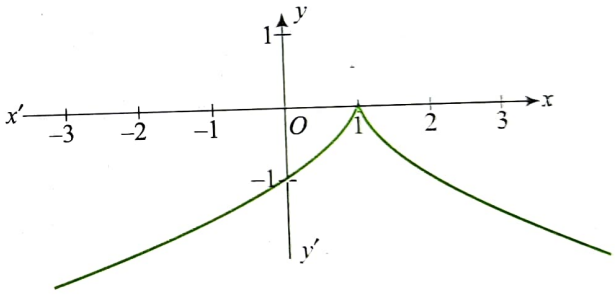
2.



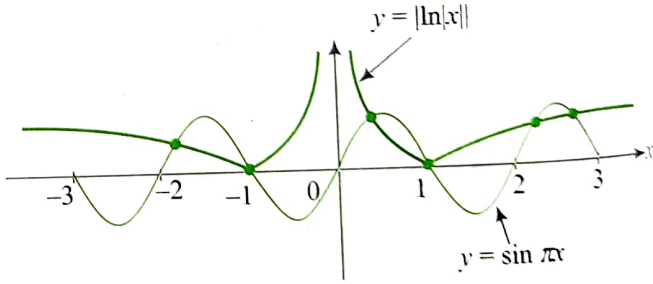
3.



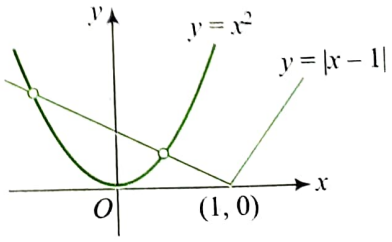
4.



5. There are exactly six solutions.



6. $\left|\frac{x^2}{x-1}\right| \le 1$ or $x^2 \le |x-1|, x \ne 1$

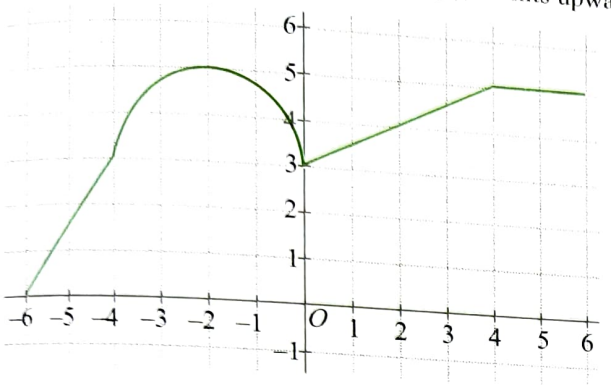


The figure represents the graph of $y = x^2$ and $y = |x-1|$.

Solving $x^2 = 1 - x$, we get $x = \dfrac{-1 \pm \sqrt{5}}{2}$.

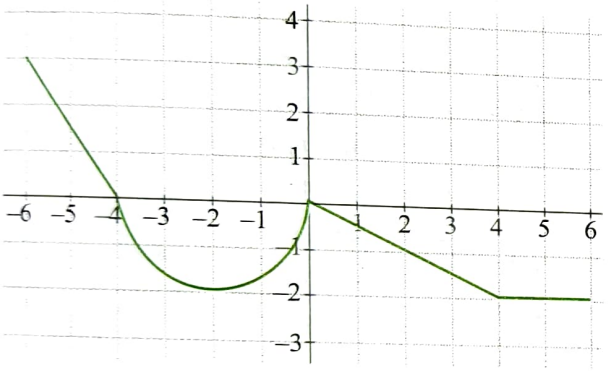
Thus, the solution is $\left[\dfrac{-1-\sqrt{5}}{2}, \dfrac{-1+\sqrt{5}}{2}\right]$.

7. **(a)** For $y = f(x) + 3$, shift the graph of $y = f(x)$, 3 units upward.

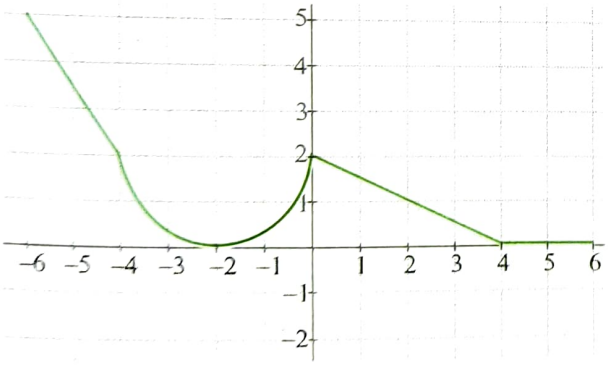


(b) $y = -f(x) + 2$

First flip the graph of $y = f(x)$ about x-axis to get $y = -f(x)$.

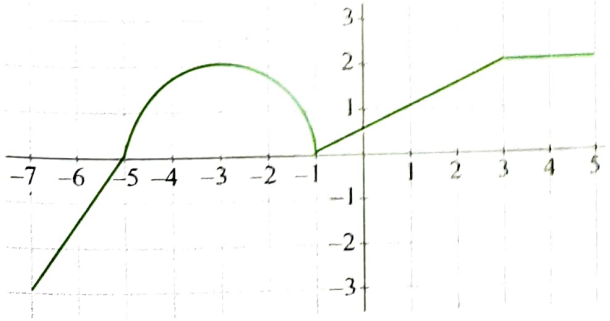


Now, shift the above graph 2 units upward to get $y = 2 - f(x)$.

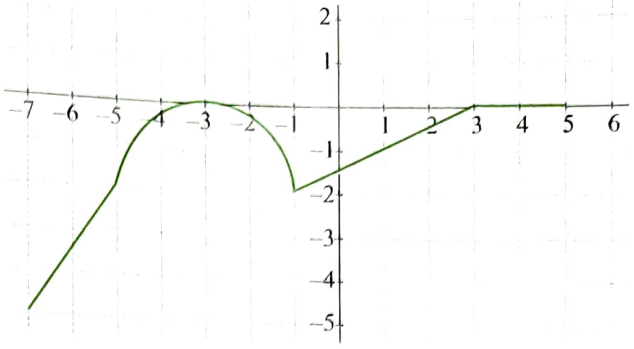


(c) $y = f(x + 1) - 2$

First shift the graph of $y = f(x)$, 1 units left to get $y = f(x + 1)$.

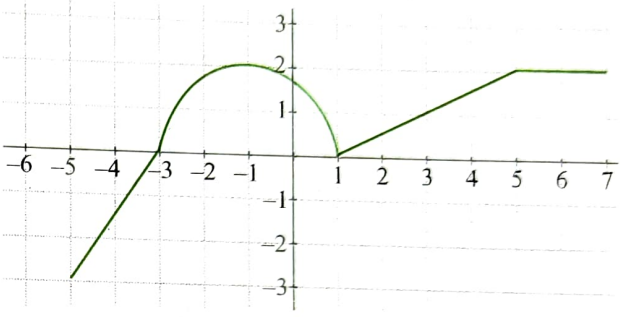


Now, shift the above graph 2 units downward to get $y = f(x + 1) - 2$.

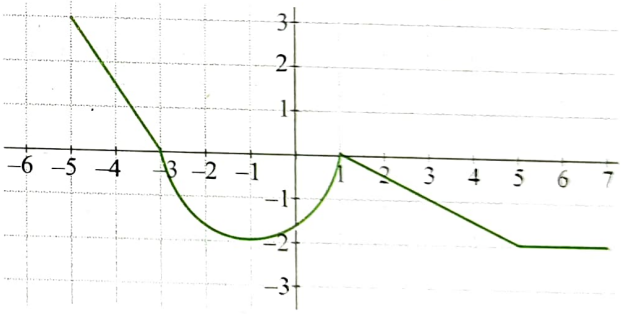


(d) $y = -f(x - 1)$

First shift the graph of $y = f(x)$, 1 unit right to get $y = f(x - 1)$.

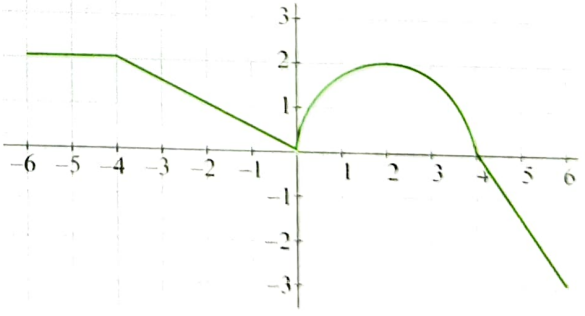


Now, flip the above graph about x-axis to get $y = -f(x - 1)$.



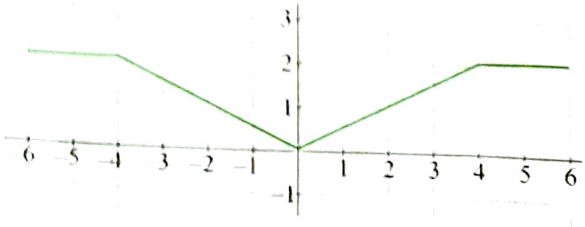
(e) $y = f(-x)$

Flip the graph about y-axis to get $y = f(-x)$.



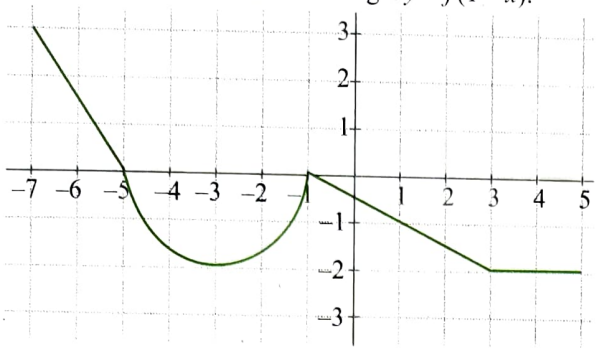
(f) $y = f(|x|)$

Neglect the graph of $y = f(x)$ for $x < 0$ and take the mirror image of $y = f(x)$ for $x > 0$ about y-axis, keeping $y = f(x)$ for $x > 0$.



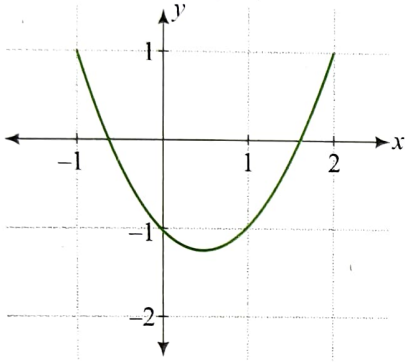
(g) $y = f(1 - x)$

First flip the graph to get $y = f(-x)$ as in (e). Then shift $y = f(-x)$, 1 unit left hand side to get $y = f(1 - x)$.

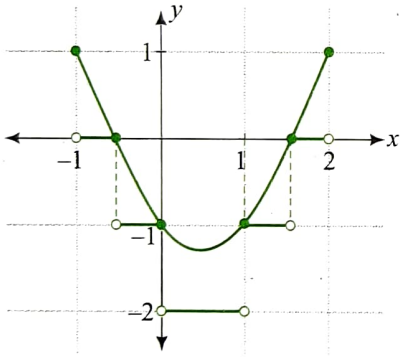


8. $f(x) = [x^2 - x - 1], x \in [-1, 2]$

Let us first draw the graph of $y = x^2 - x - 1$, which is upward parabola having vertex at (1/2, –5/4)



The graph of $f(x) = [x^2 - x - 1]$ can be drawn as shown in the following figure.



Exercises

Single Correct Answer Type

1. (1) As (1, 1); (2, 2); (3, 3) $\in R$, R is reflexive

As (1, 2) $\in R$ but (2, 1) $\notin R$, R is not symmetric

It can be easily seen that R is transitive

2. (2) Obviously, the relation is not reflexive and transitive but it is symmetric, because $x^2 + y^2 = 1 \Rightarrow y^2 + x^2 = 1$

3. (2) As R is an equivalence relation on set A, $(a, a) \in R$ for all $a \in A$. Hence, R has least n ordered pairs

4. (4) $Z_1 R Z_1 \Rightarrow \dfrac{Z_1 - Z_1}{Z_1 + Z_1} = 0$

$\therefore$ R is Reflexive because 0 is real.

$$Z_1 R Z_2 \Rightarrow \frac{Z_1 - Z_2}{Z_1 + Z_2} \text{ is real } \Rightarrow -\left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right) \text{ is real}$$

$\therefore$ R is symmetric

$Z_1 = a_1 + ib_1$; $Z_2 = a_2 + ib_2$.

$$Z_1 R Z_2 \Rightarrow \frac{Z_1 - Z_2}{Z_1 + Z_2} \text{ is real}$$

$$\Rightarrow (a_1 + a_2)(b_1 - b_2) - (a_1 - a_2)(b_1 + b_2) = 0$$

$$\Rightarrow 2a_1 b_1 - 2b_1 a_1 = 0$$

$$\Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2}$$

$$\therefore Z_2 R Z_3 \Rightarrow \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

$\therefore$ $Z_1 R Z_3 \Rightarrow R$ is transitive.

5. (1) $aR_1\, b \Leftrightarrow |a| = |b|$ satisfies.

6. (1) $|a - a| = 0 < 1 \therefore aRa\ \forall\ a \in R$

Therefore, R is reflexive.

Again $aRb \Rightarrow |a - b| \le 1$ and $|b - a| \le 1 \Rightarrow bRa$

Therefore, R is symmetric.

Therefore, R is not anti-symmetric.

Further, $1R2$ and $2R3$ but $1 \not R 3$, $[\because |1 - 3| = 2 > 1]$

Therefore, R is not transitive.

7. (2) $f: N \to N, f(n) = 2n + 3$.

Here, the range of the function is $\{5, 6, 7, \ldots\}$ or $N - \{1, 2, 3, 4\}$, which is a subset of N (co-domain).

Hence, function is into.

Also, it is clear that $f(n)$ is one-one or injective.

Hence, $f(n)$ is injective only.

8. (3) $f(x) = \begin{cases} x - 1, & x \text{ is even} \\ x + 1, & x \text{ is odd} \end{cases}$, which is clearly one-one and onto.

9. (2) Two triangles may have equal areas

Therefore, f is not one-one.

Since each positive real number can represent area of a triangle. f is onto.

10. (4) $f(x) = e^{x^3 - 3x + 2}$

Let $g(x) = x^3 - 3x + 2$;

$g'(x) = 3x^2 - 3 = 3(x^2 - 1)$

≥ 0 for $x \in (-\infty, -1]$

Therefore, $f(x)$ is increasing function. Hence, $f(x)$ is one-one.

Now, the range of $f(x)$ is $(0, e^4]$.

But co-domain is $(0, e^5]$.

Hence, $f(x)$ is an into function.

11. (3) Let $x, y \in N$ such that $f(x) = f(y)$. Then

$f(x) = f(y)$

or $x^2 + x + 1 = y^2 + y + 1$

or $(x - y)(x + y + 1) = 0$

i.e., $x = y$ or $x = (-y - 1) \notin N$

Therefore, f is one-one.

Also, $f(x)$ does not take all positive integral values. Hence f is into.

12. (4) Image b_1 is assigned to any three of the six pre-images in ${}^6C_3$ ways.

Rest two images can be assigned to remaining three pre-images in $2^3 - 2$ ways (as function is onto).

Hence, number of functions are ${}^6C_3 \times (2^3 - 2) = 20 \times 6 = 120$.

13. (4) $f(x)\ \dfrac{2^x + 1}{4^x - 1} = \dfrac{1}{2^x - 1}$ is one-one function.

14. (1) We have

$$f(x+y)+f(x-y)=\frac{1}{2}\left[a^{x+y}+a^{-x-y}+a^{x-y}+a^{-x+y}\right]$$

$$=\frac{1}{2}\left[a^{x}(a^{y}+a^{-y})+a^{-x}(a^{y}+a^{-y})\right]$$

$$=\frac{1}{2}(a^{x}+a^{-x})(a^{y}+a^{-y})=2f(x)f(y)$$

15. (4) $f(x)=\cos(\log x)$

or $f(x)f(y)-\frac{1}{2}\left[f\left(\frac{x}{y}\right)+f(xy)\right]$

$=\cos(\log x)\cos(\log y)-\frac{1}{2}\left[\cos(\log x-\log y)\right.$

$\left.+\cos(\log x+\log y)\right]$

$=\cos(\log x)\cos(\log y)-\frac{1}{2}[2\cos(\log x)\cos(\log y)]$

$=0$

16. (4) Given function is defined if ${}^{10}C_{x-1}>3\,{}^{10}C_{x}$

or $\frac{1}{11-x}>\frac{3}{x}$ or $4x>33$

or $x\geq 9$

But $x\leq 10$

$\therefore \quad x=9, 10$

17. (2) $f(x)=\frac{\sin^{-1}(3-x)}{\log(|x|-2)}$

Let $g(x)=\sin^{-1}(3-x)$

or $-1\leq 3-x\leq 1$

The domain of $g(x)$ is $[2, 4]$.

Let $h(x)=\log(|x|-2)$

i.e., $|x|-2>0$ or $|x|>2$

i.e., $x<-2$ or $x>2$

$\therefore$ Domain $(-\infty,-2)\cup(2,\infty)$

We know that

$$(f/g)(x)=\frac{f(x)}{g(x)}\ \forall\, x\in D_1\cap D_2-\{x\in R: g(x)=0\}$$

Therefore, the domain of $f(x)$ is $(2, 4]-\{3\}=(2, 3)\cup(3, 4]$.

18. (4) Here, $x+3>0$ and $x^2+3x+2\neq 0$.

Therefore, $x>-3$ and $(x+1)(x+2)\neq 0$, i.e., $x\neq -1,-2$.

Therefore, the domain is $(-3,\infty)-\{-1,-2\}$.

19. (4) $x^2-[x]^2\geq 0$ or $x^2\geq[x]^2$

This is true for all non-negative values of x and all negative integers x.

20. (3) $f(x)$ is to be defined when $x^2-1>0$ and $3+x>0$

and $3+x\neq 1$, i.e.,

$x^2>1$ and $x>-3$ and $x\neq -2$,

i.e., $x<-1$ or $x>1$ and $x>-3$ and $x\neq -2$

$\therefore \quad D_f=(-3,-2)\cup(-2,-1)\cup(1,\infty)$

21. (2) We have $f(x)=\left[\log_{10}\left(\frac{5x-x^2}{4}\right)\right]^{1/2}$ (1)

From (1), clearly, $f(x)$ is defined for those values of x for which

$$\log_{10}\left[\frac{5x-x^2}{4}\right]\geq 0$$

or $\left(\frac{5x-x^2}{4}\right)\geq 10^{0}$

or $\left(\frac{5x-x^2}{4}\right)\geq 1$

or $x^2-5x+4\leq 0$

or $(x-1)(x-4)\leq 0$

Hence, the domain of the function is $[1, 4]$.

22. (3) $f(x)=\log|\log x|$. $f(x)$ is defined if $|\log x|>0$ and $x>0$, i.e., if $x>0$ and $x\neq 1$ $(\because |\log x|>0$ if $x\neq 1)$

or $x\in(0,1)\cup(1,\infty)$

23. (2) $x^3 f(x)=\sqrt{1+\cos 2x}+|f(x)|=\sqrt{2}\,|\cos x|+|f(x)|$

Since R.H.S is positive, L.H.S must be positive.

So, $x^3 f(x)>0$

$\Rightarrow f(x)<0$

$\Rightarrow f(x)<0 \qquad \left(\because \frac{-3\pi}{4}<x<\frac{-\pi}{2}\Rightarrow x^3<0\right)$

$\therefore x^3 f(x)=-\sqrt{2}\cos x-f(x)$

$\Rightarrow f(x)=\frac{-\sqrt{2}\cos x}{1+x^3}$

24. (2) The function $\sec^{-1}x$ is defined for all $x\in R-(-1, 1)$

and the function $\frac{1}{\sqrt{x-[x]}}$ is defined for all $x\in R-Z$.

So, the given function is defined for all $x\in R-\{(-1, 1)\cup\{n\mid n\in Z\}\}$.

25. (4) It is given that $2^x+2^y=2\ \forall\, x, y\in R$

or $2^y=2-2^x$

or $y=\log_2(2-2^x)$

Therefore, function is defined only when $2-2^x>0$ or $2^x<2$ or $x<1$.

26. (2) $\cos^{-1}\left(\frac{2-|x|}{4}\right)$ exists if

$$-1\leq\frac{2-|x|}{4}\leq 1$$

or $-6\leq -|x|\leq 2$

or $-2\leq|x|\leq 6$

or $|x|\leq 6$

or $-6\leq x\leq 6$

The function $[\log(3-x)]^{-1}=\frac{1}{\log(3-x)}$ is defined if $3-x>0$ and $x\neq 2$, i.e., if $x\neq 2$ and $x<3$.

Thus, the domain of the given function is

$\{x\mid -6\leq x\leq 6\}\cap\{x\mid x\neq 2, x<3\}=[-6, 2)\cup(2, 3)$

27. (2) $f(x)$ is defined for

$$\log\left(\frac{1}{|\sin x|}\right)\geq 0$$

or $\frac{1}{|\sin x|}\geq 1$ and $|\sin x|\neq 0$

or $|\sin x|\neq 0$ $\left[\because \frac{1}{|\sin x|}\geq 1 \text{ for all } x\right]$

or $x\neq n\pi, n\in Z$

Hence, the domain of $f(x)$ is $R-\{n\pi: n\in Z\}$.

28. (1) $f(x)$ is defined if

$$-\log_{1/2}\left(1+\frac{1}{x^{1/4}}\right)-1>0$$

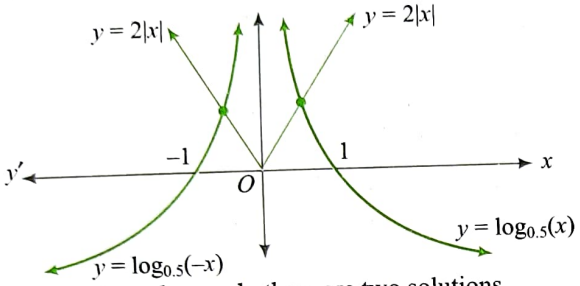
or $\log_{1/2}\left(1+\frac{1}{x^{1/4}}\right)<-1$

or $1+\frac{1}{x^{1/4}}>\left(\frac{1}{2}\right)^{-1}$

or $\frac{1}{x^{1/4}}>1$

or $0<x<1$

29. (2) Draw the graph of $y=\log_{0.5}|x|$ and $y=2|x|$.



Clearly, from the graph, there are two solutions.

30. (3) Since co-domain $=\left[0,\frac{\pi}{2}\right)$, for f to be onto,

Range $=\left[0,\frac{\pi}{2}\right)$

This is possible only when x^2+x+a is perfect square.

$\therefore\ 1-4a=0$ or $a=\frac{1}{4}$

31. (3) **Case I:**

$0<|x|-1<1$ or $1<|x|<2$

Then $x^2+4x+4\le 1$

or $x^2+4x+3\le 0$

or $-3\le x\le -1$

So, $x\in(-2,-1)$ (1)

Case II:

$|x|-1>1$ or $|x|>2$

Then $x^2+4x+4\ge 1$

or $x^2+4x+3\ge 0$

or $x\ge -1$ or $x\le -3$

So, $x\in(-\infty,-3]\cup(2,\infty)$ (2)

From (1) and (2), $x\in(-\infty,-3]\cup(-2,-1)\cup(2,\infty)$.

32. (1) We must have $ax^3+(a+b)x^2+(b+c)x+c>0$

or $ax^2(x+1)+bx(x+1)+c(x+1)>0$

or $(x+1)(ax^2+bx+c)>0$

or $a(x+1)\left(x+\frac{b}{2a}\right)^2>0$ as $b^2=4ac$

or $x>-1$ and $x\ne -\frac{b}{2a}$

33. (4) $f(x)=\frac{1}{\sqrt{4x-|x^2-10x+9|}}$

For $f(x)$ to be defined, $|x^2-10x+9|<4x$

or $x^2-10x+9<4x$ and $x^2-10x+9>-4x$

or $x^2-14x+9<0$ and $x^2-6x+9>0$

or $x\in(7-\sqrt{40},7+\sqrt{40})$ and $x\in R-\{3\}$

or $x\in(7-\sqrt{40},3)\cup(3,7+\sqrt{40})$

34. (4) $|\cos x|+\cos x=\begin{cases}0, & \cos x\le 0\\ 2\cos x, & \cos x>0\end{cases}$

For $f(x)$ to be defined, $\cos x>0$

or $x\in\left(\frac{(4n-1)\pi}{2},\frac{(4n+1)\pi}{2}\right)$, $n\in Z$ (1st and 4th quadrant).

35. (4) $f(x)=\sqrt{x^{12}-x^9+x^4-x+1}$

We must have $x^{12}-x^9+x^4-x+1\ge 0$ (1)

Obviously, (1) is satisfied by $x\in(-\infty,0]$.

Also, $x^9(x^3-1)+x(x^3-1)+1\ge 0\ \forall\ x\in[1,\infty)$.

Further, $x^{12}-x^9+x^4-x+1=(1-x)+x^4(1-x^5)+x^{12}$ is also satisfied by $x\in(0,1)$.

Hence, the domain is R.

36. (2) We have $f(x)=\sqrt{\sin x+\cos x}+\sqrt{7x-x^2-6}$

$$=\sqrt{\sqrt{2}\sin\left(\frac{\pi}{4}+x\right)}+\sqrt{7x-x^2-6}$$

$f(x)$ is defined if

(i) $7x-x^2-6\ge 0$

$\Rightarrow\ 1\le x\le 6$ (1)

(ii) $\sin\left(\frac{\pi}{4}+x\right)\ge 0$

$$x+\frac{\pi}{4}\in\ldots[-2\pi,-\pi]\cup[0,\pi]\cup[2\pi,3\pi]\ldots$$

$\Rightarrow\ x\in\ldots\left[-\frac{\pi}{4},\frac{3\pi}{4}\right]\cup\left[\frac{7\pi}{4},\frac{11\pi}{4}\right]\ldots$ (2)

From (1) and (2), we get

$$x\in\left[1,\frac{3\pi}{4}\right]\cup\left[\frac{7\pi}{4},6\right]$$

37. (3) Given, $y=x^{\log_x\pi}=\pi$

Domain is $x\in(0,1)\cup(1,\infty)$

Range is $\{\pi\}$

38. (3) $\frac{x^2+14x+9}{x^2+2x+3}=y$

or $x^2+14x+9=x^2y+2xy+3y$

or $x^2(y-1)+2x(y-7)+(3y-9)=0$

Since x is real, we have

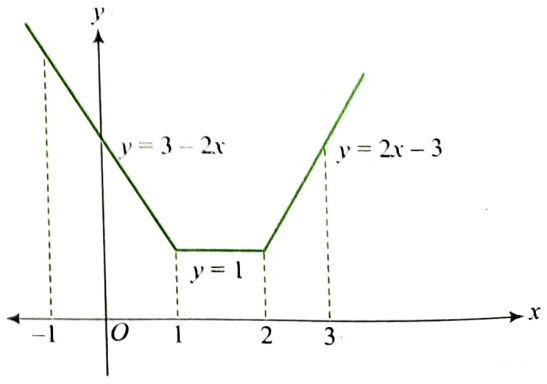
$4(y-7)^2-4(3y-9)(y-1)\ge 0$

or $4(y^2+49-14y)-4(3y^2+9-12y)\ge 0$

or $(y+5)(y-4)\le 0$

$\therefore\ y\in[-5,4]$

39. (2)



Clearly, from the graph, the range is $[1, f(-1)] \equiv [1, 5]$.

If $x < 1, f(x) = -(x-1) - (x-2) = -2x + 3$.

In this interval, $f(x)$ is decreasing.

If $1 \le x < 2, f(x) = x - 1 - (x-2) = 1$.

In this interval, $f(x)$ is constant.

If $2 \le x \le 3, f(x) = x - 1 + x - 2 = 2x - 3$.

In this interval, $f(x)$ is increasing.

$\therefore$ max $f(x)$ = the greatest among $f(-1)$ and $f(3) = 5$,

min $f(x) = f(1) = 1$

So, range $= [1, 5]$

3) $y = f(x) = \cos^2 x + \sin^4 x$

$= \cos^2 x + \sin^2 x(1 - \cos^2 x)$

$= \cos^2 x + \sin^2 x - \sin^2 x \cos^2 x$

$= 1 - \sin^2 x \cos^2 x$

$= 1 - \frac{1}{4} \sin^2 2x$

$\therefore \quad \frac{3}{4} \le f(x) \le 1 \qquad (\because 0 \le \sin^2 2x \le 1)$

$\therefore \quad f(x) \in [3/4, 1]$

3) $y = |\sin x| + |\cos x|$

or $y^2 = 1 + |\sin 2x|$

or $1 \le y^2 \le 2$

or $y \in [1, \sqrt{2}]$

or $f(x) = 1 \ \forall \ x \in R$

1) We have $f(x) = {}^{7-x}P_{x-3} = \frac{(7-x)!}{(10-2x)!}$

We must have $7 - x > 0$, $x \ge 3$, and $7 - x \ge x - 3$. Therefore,

$x < 7, x \ge 3$, and $x \le 5$

or $3 \le x \le 5$

or $x = 3, 4, 5$

Now, $f(3) = \frac{4!}{4!} = 1, f(4) = \frac{3!}{2!} = 3, f(5) = \frac{2!}{0!} = 2$.

Hence, $R_f = \{1, 2, 3\}$.

3) Here, $\frac{x^2+1}{x^2+2} = 1 - \frac{1}{x^2+2}$

Now, $2 \le x^2 + 2 < \infty$ for all $x \in R$

or $\frac{1}{2} \ge \frac{1}{x^2+2} > 0$

or $-\frac{1}{2} \le \frac{-1}{x^2+2} < 0$

or $\frac{1}{2} \le 1 - \frac{1}{x^2+2} < 1$

or $\frac{\pi}{6} \le \sin^{-1}\left(1 - \frac{1}{x^2+2}\right) < \frac{\pi}{2}$

3) $f(x) = \frac{e^x - e^{|x|}}{e^x + e^{|x|}} = \begin{cases} 0, & x \ge 0 \\ \frac{e^x - e^{-x}}{e^x + e^{-x}}, & x < 0 \end{cases}$

Clearly, $f(x)$ is identically zero if $x \ge 0$ (1)

If $x < 0$, let $y = f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ or $e^{2x} = \frac{1+y}{1-y}$

$\because \quad x < 0$

$e^{2x} < 1$ or $0 < e^{2x} < 1$

$\therefore \quad 0 < \frac{1+y}{1-y} < 1$

or $\frac{1+y}{1-y} > 0$ and $\frac{1+y}{1-y} < 1$

or $(y+1)(y-1) < 0$ and $\frac{2y}{1-y} < 0$

i.e., $-1 < y < 1$ and $y < 0$ or $y > 1$

or $\quad -1 < y < 0$ (2)

Combining (1) and (2), we get $-1 < y \le 0$ or Range $= (-1, 0]$.

45. (1) When $[x] = 0$, we have $\sin^{-1}(\cos^{-1}0) = \sin^{-1}(\pi/2)$, not defined.

When $[x] = -1$, we have $\sin^{-1}(\cos^{-1}(-1)) = \sin^{-1}(\pi)$, not defined.

When $[x] = 1$, we have $\sin^{-1}(\cos^{-1}1) = \sin^{-1}(0) = 0$.

Hence, $x \in [1, 2)$ and the range of function is $\{0\}$.

46. (4) Since $\{x\} \in [0, 1)$, $\sin\{x\} \in (0, \sin 1)$ as $f(x)$ is defined if $\sin\{x\} \ne 0$, i.e.,

$$\frac{1}{\sin\{x\}} \in \left(\frac{1}{\sin 1}, \infty\right) \quad \text{or} \quad \left[\frac{1}{\sin\{x\}}\right] \in \{1, 2, 3, \ldots\}$$

Note that $1 < \frac{\pi}{3}$ or $\sin 1 < \sin\frac{\pi}{3} = 0.866$ or $\frac{1}{\sin 1} > 1.155$.

47. (4) $y = \cos(K \sin x)$

$$\frac{\cos^{-1} y}{K} = \sin x$$

$\Rightarrow \quad -1 \le \frac{\cos^{-1} y}{K} \le 1$

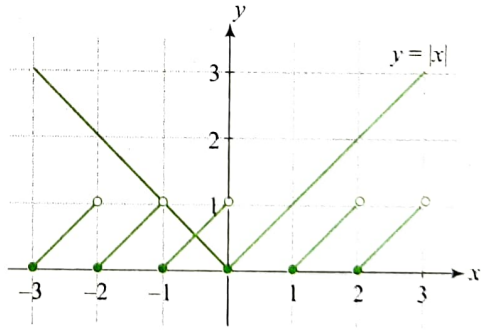
$\Rightarrow \quad -K \le \cos^{-1} y \le K$

Now $\cos^{-1} y \in [0, \pi]$

$\Rightarrow \quad K = 4$

48. (3) $f(x) = \sqrt{|x| - \{x\}}$ is defined if $|x| \ge \{x\}$

or $\quad x \in \left(-\infty, -\frac{1}{2}\right] \cup [0, \infty)$ or $Y \in [0, \infty)$.



49. (4) $\cos^{-1}\left(\frac{1+x^2}{2x}\right)$ is defined if $\left|\frac{1+x^2}{2x}\right| \le 1$ and $x \ne 0$

or $\quad 1 + x^2 - 2|x| \le 0$

or $\quad (|x| - 1)^2 \le 0$

or $\quad x = 1, -1$

Thus, the domain of $f(x)$ is $\{1, -1\}$. Hence, the range is $\{1, 1 + \pi\}$.

50. (3) Given $f(x) = \sqrt{(1-\cos x)\sqrt{(1-\cos x)\sqrt{(1-\cos x)\sqrt{\cdots\infty}}}}$

$= (1-\cos x)^{\frac{1}{2}}(1-\cos x)^{\frac{1}{4}}(1-\cos x)^{\frac{1}{8}}\cdots\infty$

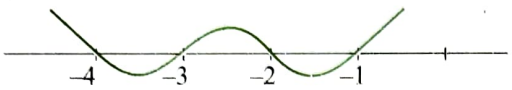
$= (1-\cos x)^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\infty}$

$$= (1-\cos x)^{\frac{1/2}{1-(1/2)}}$$
$$= 1-\cos x$$

Thus, the range of $f(x)$ is $[0, 2)$.

51. (1) Let $g(x) = (x+1)(x+2)(x+3)(x+4)$

The rough graph of $g(x)$ is given as follows:



$\therefore \quad g(x) = (x+1)(x+2)(x+3)(x+4)$
$= (x+1)(x+4)(x+2)(x+3)$
$= (x^2+5x+4)(x^2+5x+6)$
$= t(t+2) = (t+1)^2 - 1 \qquad$ where $t = x^2+5x$

Now, $g_{min} = -1$, for which $x^2+5x = -1$ has real roots in $[-6, 6]$.

Also, $g(6) = 7 \times 8 \times 9 \times 10 = 5040$.

Hence, the range of $g(x)$ is $[-1, 5040]$ for $x \in [-6, 6]$.

Then, the range of $f(x)$ is $[4, 5045]$.

52. (1) $f(x) = \sec^{-1}(\log_3 \tan x + \log_{\tan x} 3)$.

$$= \sec^{-1}\left(\log_3 \tan x + \frac{1}{\log_3 \tan x}\right)$$

Now, for $\log_3 \tan x$ to get defined, $\tan x \in (0, \infty)$

or $\quad \log_3 \tan x \in (-\infty, \infty)$ or $\log_3 \tan x \in R$

Also, $x + \frac{1}{x} \le -2$ or $x + \frac{1}{x} \ge 2$

i.e., $\quad \log_3 \tan x + \frac{1}{\log_3 \tan x} \le -2$

or $\quad \log_3 \tan x + \frac{1}{\log_3 \tan x} \ge 2$

i.e., $\quad \sec^{-1}\left(\log_3 \tan x + \frac{1}{\log_3 \tan x}\right) \le \sec^{-1}(-2)$

or $\quad \sec^{-1}\left(\log_3 \tan x + \frac{1}{\log_3 \tan x}\right) \ge \sec^{-1} 2$

i.e., $\quad f(x) \le \frac{2\pi}{3}$ or $f(x) \ge \frac{\pi}{3}$

i.e., $\quad f(x) \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{2\pi}{3}\right]$

53. (3) $f(x) = \{x\}^{\{x\}} + [x]^{[x]}$

Here, base $\{x\}$ and $[x]$ should not be zero.

Hence, $x \notin I$ and $x \notin [0, 1)$.

Thus, domain is $R - \{I \cup (0, 1)\}$.

54. (2) **Case (i)**

$x^2 - 2x + a < 0 \ \forall \ x \in R$ which not possible

$x^2 - 2x + a \ge 1$

$x^2 - 2x + a - 1 \ge 0$

Disc ≤ 0

$4 - 4(a-1) \le 0$

$2 - a \le 0$

$a \ge 2$

$a \in [2, \infty)$

55. (3) $\sum_{r=1}^{2000} \frac{\{x+r\}}{2000} = \sum_{r=1}^{2000} \frac{\{x\}}{2000} = 2000 \frac{\{x\}}{2000} = \{x\}$.

56. (3) $f(x) = \frac{\sin[x]\pi}{x^2+x+1}$

Let $[x] = n \simeq$ integer

$\therefore \quad \sin[x]\pi = 0$

or $\quad f(x) = 0$

Hence, $f(x)$ is constant function.

57. (4) $f(x) = \alpha x^3 - \beta x - (\tan x)\ \text{sgn}\ x$

$f(-x) = f(x)$

or $\quad -\alpha x^3 + \beta x - \tan x\ \text{sgn}\ x = \alpha x^3 - \beta x - (\tan x)(\text{sgn}\ x)$

or $\quad 2(-\alpha x^2 - \beta)x = 0 \ \forall \ x \in R$

or $\quad \alpha = 0$ and $\beta = 0$

$\therefore \quad [a]^2 - 5[a] + 4 = 0$ and $6\{a\}^2 - 5\{a\} + 1 = 0$

or $\quad [x] = 1, 4$ and $\{x\} = \frac{1}{2}, \frac{1}{3}$

$\therefore \quad a = 1 + \frac{1}{3}, 1 + \frac{1}{2}, 4 + \frac{1}{3}, 4 + \frac{1}{2}$

Sum of values of $a = \frac{35}{3}$

58. (4) $[x]\{x\} = 1$

Obviously, x cannot be negative.

Further, $[x] \ne 0$.

For $[x] = 1$, $\{x\} = 1$, which is not possible.

Hence, $x \notin [0, 2)$.

For $x \in [2, 3)$, $[x] = 2$ or $\{x\} = 1/2$ or $x = 5/2 = 2 + 1/2$.

For $x \in [3, 4)$, $[x] = 3$ or $\{x\} = 1/3$ or $x = 3 + 1/3$.

For $x \in [4, 5)$, $[x] = 4$ or $\{x\} = 1/4$ or $x = 4 + 1/4$.

Thus, solution is $x = m + \frac{1}{m}$, where $m \in N - \{1\}$.

59. (2) $[\sqrt{n^2+1}] = n$

or $\quad [\sqrt{n^2+\lambda}] = n + 2$

or $\quad n + 2 \le \sqrt{n^2+\lambda} < n + 3$

or $\quad n^2 + 4n + 4 \le n^2 + \lambda < n^2 + 6n + 9$

or $\quad 4n + 4 \le \lambda < 6n + 9$

$\therefore \quad 2n + 5$ values

60. (3) Given equation is $x^2 - 2 = [\sin x]$.

There are three possibilities:

$[\sin x] = -1, [\sin x] = 0, [\sin x] = 1$

Case I: If $[\sin x] = -1$,

$x^2 = 1$

or $\quad x = \pm 1$

When $x = 1$, $[\sin x] = 0$.

But $\quad x = -1 \Rightarrow [\sin x] = -1$

$x = -1$ is a solution.

Case II: If $[\sin x] = 0$, the equation is

$x^2 = 2$

or $\quad x = \pm\sqrt{2}$

$[\sin \sqrt{2}] = 0$

But $[\sin(-\sqrt{2})] = -1$.

So, $x = \sqrt{2}$ is a solution.

Case III: If $[\sin x] = 1$

$$x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$$

Clearly, these values do not satisfy the original equation.

Thus, number of solutions = 2

61. (4) We must have $-1 \le [2x^2 - 3] \le 1$

or $\quad -1 \le 2x^2 - 3 < 2$ or $1 \le x^2 < \frac{5}{2}$

or $\quad x \in \left(-\sqrt{\frac{5}{2}}, -1\right] \cup \left[1, \sqrt{\frac{5}{2}}\right)$

62. (2) We must have

$2\{x\}^2 - 3\{x\} + 1 \ge 0$, i.e., $\{x\} \ge 1$ or $\{x\} \le 1/2$.

Thus, we have $0 \le \{x\} \le 1/2$ or $x \in \left[n, n + \frac{1}{2}\right], n \in I$.

63. (2) $\left[x^2 + \frac{1}{2}\right] = \left[x^2 - \frac{1}{2} + 1\right] = 1 + \left[x^2 - \frac{1}{2}\right]$.

Thus, from domain point of view,

$$\left[x^2 - \frac{1}{2}\right] = 0, -1 \text{ or } \left[x^2 + \frac{1}{2}\right] = 1, 0$$

i.e., $f(x) = \sin^{-1}(1) + \cos^{-1}(0)$ or $\sin^{-1}(0) + \cos^{-1}(-1)$

or $\quad f(x) = \{\pi\}$

64. (1) $h(x) = \log(f(x) \cdot g(x)) = \log e^{\{y\}+[y]} = \{y\} + [y] = e^{|x|} \operatorname{sgn} x$

$$= \begin{cases} e^x, & x > 0 \\ 0, & x = 0 \\ -e^{-x}, & x < 0 \end{cases}$$

$$\therefore \quad h(-x) = \begin{cases} e^{-x}, & x < 0 \\ 0, & x = 0 \\ -e^x, & x > 0 \end{cases}$$

$\therefore \quad h(x) + h(-x) = 0$ for all x

65. (4) $[y - [y]] = 2 \cos x$

or $\quad [y] + [y] = 2 \cos x \qquad (\because [x + n] = [x] + n \text{ if } n \in I)$

or $\quad 2[y] = 2 \cos x$ or $[y] = \cos x \qquad (1)$

Also, $y = \frac{1}{3}[\sin x + [\sin x + [\sin x]]]$

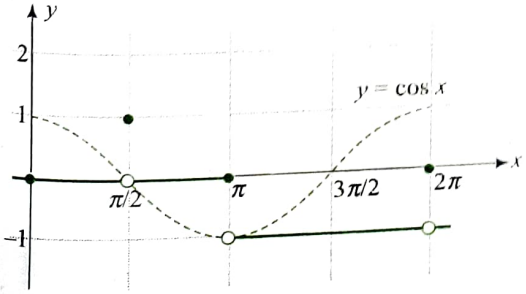
$$= \frac{1}{3}(3[\sin x])$$

$$= [\sin x] \qquad (2)$$

From (1) and (2),

$[[\sin x]] = \cos x$

or $\quad [\sin x] = \cos x$



The number of solutions is 0.

66. (2) $f(x) = \sin\left(\log(x + \sqrt{1 + x^2})\right)$

or $\quad f(-x) = \sin[\log(-x + \sqrt{1 + x^2})]$

$$= \sin \log\left((\sqrt{1 + x^2} - x)\frac{(\sqrt{1 + x^2} + x)}{(\sqrt{1 + x^2} + x)}\right)$$

$$= \sin \log\left[\frac{1}{(x + \sqrt{1 + x^2})}\right]$$

$$= \sin\left[-\log(x + \sqrt{1 + x^2})\right]$$

$$= -\sin\left[\log(x + \sqrt{1 + x^2})\right]$$

$\therefore \quad f(-x) = -f(x)$

Therefore, $f(x)$ is an odd function.

67. (1) Given $f(x) = \sqrt[n]{x^m}$, $n \in N$, is an even function where $m \in I$. So,

$f(x) = f(-x)$

or $\quad \sqrt[n]{x^m} = \sqrt[n]{(-x)^m}$

or $\quad x^m = (-x)^m$

i.e., m is an even integer

or $\quad m = 2k, k \in I$

68. (2) $f(-x) = \begin{cases} (-x)^2 \sin\frac{\pi(-x)}{2}, & |-x| < 1 \\ (-x)|-x|, & |-x| \ge 1 \end{cases}$

$$= \begin{cases} -x^2 \sin\frac{\pi x}{2}, & |x| < 1 \\ -x|x|, & |x| \ge 1 \end{cases}$$

$$= -f(x)$$

69. (4) $f(x) = \frac{a^x - 1}{x^n(a^x + 1)}$

$f(x)$ is symmetrical about y-axis. Thus,

$f(x) = f(-x)$

or $\quad \frac{a^x - 1}{x^n(a^x + 1)} = \frac{a^{-x} - 1}{(-x)^n(a^{-x} + 1)}$

or $\quad \frac{a^x - 1}{x^n(a^x + 1)} = \frac{1 - a^x}{(-x)^n(1 + a^x)}$

or $\quad x^n = -(-x)^n$

Hence, the value of n which satisfies this relation is $-\frac{1}{3}$.

70. (1) Since $f(x)$ and $f^{-1}(x)$ are symmetric about the line $y = -x$, if (α, β) lies on $y = f(x)$, then $(-\beta, -\alpha)$ lies on $y = f^{-1}(x)$.

Therefore, $(-\alpha, -\beta)$ lies on $y = f(x)$.

Hence, $y = f(x)$ is odd.

71. (1) $f(7) + f(-7) = -10$

or $\quad f(7) = -17$

or $\quad f(7) + 17 \cos x = -17 + 17 \cos x$

which has the range $[-34, 0]$.

72. (3) $g(x) = x^3 + \tan x + \left[\frac{x^2 + 1}{P}\right]$

or $\quad g(-x) = (-x)^3 + \tan(-x) + \left[\frac{(-x)^2 + 1}{P}\right]$

$$= -x^3 - \tan x + \left[\frac{x^2+1}{P}\right]$$

Since $g(x)$ is odd,

$$g(x) + g(-x) = 0$$

$$\therefore \quad \left(-x^3 - \tan x + \left[\frac{x^2+1}{P}\right]\right) + \left(\begin{matrix} -x^3 - \tan x \\ + \left[\frac{x^2+1}{P}\right] \end{matrix}\right) = 0$$

or $\quad 2\left[\frac{(x^2+1)}{P}\right] = 0$ or $0 \le \frac{x^2+1}{P} < 1$

Now, $x \in [-2, 2]$

$\therefore \quad x^2 + 1 \in [1, 5]$

$\therefore \quad P > 5$

73. (4) Since $f(x)$ is an odd function,

$$\left[\frac{x^2}{a}\right] = 0 \text{ for all } x \in [-10, 10]$$

or $\quad 0 \le \frac{x^2}{a} < 1$ for all $x \in [-10, 10]$ or $a > 100$

74. (1) $f(-x) = \dfrac{\cos(-x)}{\left[-\frac{2x}{\pi}\right] + \frac{1}{2}} = \dfrac{\cos x}{-1 - \left[\frac{2x}{\pi}\right] + \frac{1}{2}}$ (As x is not an integral multiple of π)

$$= -\frac{\cos x}{\left[\frac{2x}{\pi}\right] + \frac{1}{2}} = -f(x)$$

Therefore, $f(x)$ is an odd function.

75. (2) For odd function,

$$f(x) = -f(-x)$$

$$= -\begin{cases} \sin(-x) + \cos(-x) & 0 < -x < \pi/2 \\ a, & -x = \pi/2 \\ \tan^2(-x) + \operatorname{cosec}(-x), & \pi/2 < -x < \pi \end{cases}$$

$$= \begin{cases} \sin x - \cos x, & -\pi/2 < x < 0 \\ -a, & x = -\pi/2 \\ -\tan^2 x + \operatorname{cosec} x, & -\pi < x < -\pi/2 \end{cases}$$

76. (2) $f(x) = \left|\sin^3\frac{x}{2}\right| + \left|\cos^5\frac{x}{5}\right|$

The period of $\sin^3 x$ is 2π.

So, the period of $\sin^3\frac{x}{2}$ is $\frac{2\pi}{1/2} = 4\pi$.

So, the period of $\left|\sin^3\frac{x}{2}\right|$ is 2π.

The period of $\cos^5 x$ is 2π.

So, the period of $\cos^5\frac{x}{5}$ is $\frac{2\pi}{\left(\frac{1}{5}\right)} = 10\pi$.

So, the period of $\left|\cos^5\frac{x}{2}\right|$ is 5π.

Now, period of $f(x)$ = LCM of $\{2\pi, 5\pi\} = 10\pi$

77. (3) From the given data, $g(x)$ must be linear function.

Hence, $g(x) = ax + b$.

Also, $g(2) = 2a + b = 3$ and $g(4) = 4a + b = 7$.

Solving, we get $a = 2$ and $b = -1$.

Hence, $g(x) = 2x - 1$.

Then, $g(6) = 11$.

78. (1) The period of $\sin \pi x$ and $\cos 2\pi x$ is 2 and 1, respectively.

The period of $2^{\{x\}}$ is 1.

The period of $3^{\{x/2\}}$ is 2.

Hence, the period of $f(x)$ is LCM of 1 and 2, i.e., 2.

79. (3) $f(x) = [6x + 7] + \cos \pi x - 6x$

$= [6x] + 7 + \cos \pi x - 6x$

$= 7 + \cos \pi x - \{6x\}$

$\{6x\}$ has period 1/6 and $\cos \pi x$ has period 2. Then,

Period of $f(x)$ = LCM of 2 and 1/6 = 2

Hence, the period is 2.

80. (4) The period of $f(x)$ is 7. So, the period of $f\left(\frac{x}{3}\right)$ is $\frac{7}{1/3} = 21$.

The period of $g(x)$ is 11. So, the period of $g\left(\frac{x}{5}\right)$ is $\frac{11}{1/5} = 55$.

Hence, T_1 = period of $f(x)g\left(\frac{x}{5}\right) = 7 \times 55 = 385$

and $\quad T_2$ = period of $g(x)f\left(\frac{x}{3}\right) = 11 \times 21 = 231$

$\therefore$ Period of $F(x)$ = LCM $\{T_1, T_2\}$

= LCM $\{385, 231\}$

$= 7 \times 11 \times 3 \times 5$

$= 1155$

81. (4) $\sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)$

$$= \sin^2 x + \left(\frac{\sin x}{2} + \frac{\sqrt{3}\cos x}{2}\right)^2 + \cos x\left(\frac{\cos x}{2} - \frac{\sqrt{3}\sin x}{2}\right)$$

$$= \sin^2 x + \frac{\sin^2 x}{4} + \frac{3\cos^2 x}{4} + \frac{\cos^2 x}{2}$$

$$= \frac{5\sin^2 x}{4} + \frac{5\cos^2 x}{4}$$

$$= \frac{5}{4}$$

Hence, $f(x) = c^{5/4}$ = constant, which is periodic but has no fixed period.

82. (2) $f(x + 4) = \{(0.1)\, 3^{[x+4]}\}$

$= \{(0.1) \times 81 \times 3^{[x]}\}$

$= \{8 \times 3^{[x]} + (0.1)\, 3^{[x]}\}$

$= \{(0.1)3^{[x]}\} = f(x)$

83. (3) The period of $\cos(\sin nx)$ is $\frac{\pi}{n}$ and the period of $\tan\left(\frac{x}{n}\right)$ is πn.

Thus, $6\pi = \text{LCM}\left(\frac{\pi}{n}, \pi n\right)$

or $\quad 6\pi = \frac{\pi}{n}\lambda_1$ or $n = \frac{\lambda_1}{6}$,

and $\quad 6\pi = \lambda_2 \pi n$ or $n = \frac{6}{\lambda_2}$, $\lambda_1, \lambda_2 \in I^+$

From $n = \frac{6}{\lambda_2}$, $n = 6, 3, 2, 1$.

Clearly, for $n = 6$, we get the period of $f(x)$ to be 6π.

84. (2) $f(x) = [x] + [2x] + [3x] + \cdots + [nx] - (x + 2x + 3x + \cdots + nx)$
$= -(\{x\} + \{2x\} + \{3x\} + \cdots + \{nx\})$

Period of $f(x) = \text{LCM}\left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}\right) = 1$.

85. (3) Clearly, $f(x + \pi) = f(x)$, $g(x + \pi) = g(x)$,

and $\phi\left(x + \frac{\pi}{2}\right) = \{(-1) f(x)\}\ \{(-1)\, g(x)\} = \phi(x)$

86. (3) $f(x) = \frac{1}{x}$, $g(x) = \frac{1}{x^2}$, and $h(x) = x^2$

$f(g(x)) = x^2, x \neq 0$

$h(g(x)) = \frac{1}{x^4} = (g(x))^2, x \neq 0$

87. (4) $f(f(x)) = \begin{cases} (f(x))^2, & \text{for } f(x) \geq 0 \\ f(x), & \text{for } f(x) < 0 \end{cases}$

$= \begin{cases} (x^2)^2, & x^2 \geq 0, x \geq 0 \\ x^2, & x \geq 0, x < 0 \\ x^2, & x^2 < 0, x \geq 0 \\ x, & x < 0, x < 0 \end{cases}$

$= \begin{cases} x^4, & x \geq 0 \\ x, & x < 0 \end{cases}$

(4) We have $fog(x) = f(g(x)) = \sin(\log_e |x|)$.

$\log_e |x|$ has range R, for which $\sin(\log_e |x|) \in [-1, 1]$.

Therefore, $R_1 = \{u: -1 \leq u \leq 1\}$.

Also, $gof(x) = g(f(x)) = \log_e |\sin x|$.

$\because$ $0 \leq |\sin x| \leq 1$

$-\infty < \log_e |\sin x| \leq 0$

or $R_2 = \{v: -\infty < v \leq 0\}$

(1) $f(f(x)) = \begin{cases} f(x), & f(x) \text{ is rational} \\ 1 - f(x), & f(x) \text{ is irrational} \end{cases}$

$= \begin{cases} x, & x \text{ is rational} \\ 1 - (1 - x) = x, & x \text{ is irrational} \end{cases}$

(3) (a) $f(x) = \sin x$ and $g(x) = \cos x$, $x \in [0, \pi/2]$

Here, both $f(x)$ and $g(x)$ are one-one functions.

But $h(x) = f(x) + g(x) = \sin x + \cos x$ is many-one

as $h(0) = h(\pi/2) = 1$.

(b) $h(x) = f(x)\, g(x) = \sin x \cos x = \frac{\sin 2x}{2}$ is many-one, as $h(0) = h(\pi/2) = 0$.

(c) It is a fundamental property.

(3) $f(x)$ is defined for $x \in (0, 1)$

So, $f(e^x) + f(\ln|x|)$ is defined for

$0 < e^x < 1$ and $0 < \ln|x| < 1$

or $-\infty < x < 0$ and $1 < |x| < e$

or $x \in (-\infty, 0)$ and $x \in (-e, -1) \cup (1, e)$

or $x \in (-e, -1)$

(2) $-5 \leq |kx + 5| \leq 7$

or $-12 \leq kx \leq 2$ where $-6 \leq x \leq 1$

or $-6 \leq \frac{k}{2} x \leq 1$ where $-6 \leq x \leq 1$

$\therefore$ $k = 2$ [$\because$ range of $h(x)$ = domain of $f(x)$]

93. (2) $f(x) = \sin x + \cos x$, $g(x) = x^2 - 1$

$\Rightarrow$ $g(f(x)) = (\sin x + \cos x)^2 - 1 = \sin 2x$

Clearly $g(f(x))$ is invertible in $-\frac{\pi}{2} \leq 2x \leq \frac{\pi}{2}$

[$\because$ $\sin\theta$ is invertible when $-\pi/2 \leq \theta \leq \pi/2$]

$\Rightarrow$ $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$

94. (2) $y = 2^{x(x-1)}$ or $x^2 - x - \log_2 y = 0$

or $x = \frac{1}{2}\left(1 \pm \sqrt{1 + 4\log_2 y}\right)$

Since $x \in [1, \infty)$, we choose $x = \frac{1}{2}\left(1 + \sqrt{1 + 4\log_2 y}\right)$

or $f^{-1}(x) = \frac{1}{2}\left(1 + \sqrt{1 + 4\log_2 x}\right)$.

95. (3) Since $f(x) = (x + 1)^2 - 1$ is continuous function, solution of $f(x) = f^{-1}(x)$ lies on the line $y = x$. Therefore,

$f(x) = f^{-1}(x) = x$

or $(x + 1)^2 - 1 = x$

or $x^2 + x = 0$

i.e., $x = 0$ or -1

Therefore, the required set is $\{0, -1\}$.

96. (1) $f: [1, \infty) \to [2, \infty)$

$f(x) = x + \frac{1}{x} = y$

or $x^2 - yx + 1 = 0$

or $x = \frac{y \pm \sqrt{y^2 - 4}}{2}$

But given $f: [1, \infty) \to [2, \infty)$

$\therefore$ $x = \frac{y + \sqrt{y^2 - 4}}{2}$

or $f^{-1}(x) = \frac{x + \sqrt{x^2 - 4}}{2}$

97. (4) Given that $f(x) = (x + 1)^2$, $x \geq -1$.

Now, if $g(x)$ is the reflection of $f(x)$ in the line $y = x$, then $g(x)$ is an inverse function of $y = f(x)$.

Given $y = (x + 1)^2$ $(x \geq -1$ and $y \geq 0)$

or $x = \pm\sqrt{y} - 1$

or $g(x) = f^{-1}(x) = \sqrt{x} - 1$, $x \geq 0$

98. (2) $y = f(x) = \sqrt{3}\sin x - \cos x + 2 = 2\sin\left(x - \frac{\pi}{6}\right) + 2$ (1)

Since $f(x)$ is one-one and onto, f is invertible.

From (1), $\sin\left(x - \frac{\pi}{6}\right) = \frac{y - 2}{2}$

or $x = \sin^{-1}\frac{y - 2}{2} + \frac{\pi}{6}$

or $f^{-1}(x) = \sin^{-1}\left(\frac{x - 2}{2}\right) + \frac{\pi}{6}$

99. (1) By checking for different functions, we find that for

$$f(x)=\frac{1-x}{1+x},\ f^{-1}(x)=f(x).$$

100. (3) $\frac{1}{2}(gof)(x)=2x^2-5x+2$

or $\frac{1}{2}g[f(x)]=2x^2-5x+2$

$\therefore$ $[\{f(x)\}^2+\{f(x)\}-2]=2[2x^2-5x+2]$

or $f(x)^2+f(x)-(4x^2-10x+6)=0$

$\therefore$ $f(x)=\frac{-1\pm\sqrt{1+4(4x^2-10x+6)}}{2}$

$=\frac{-1\pm\sqrt{(16x^2-40x+25)}}{2}=\frac{-1\pm(4x-5)}{2}$

$=2x-3$ or $-2x+2$

101. (3) $f(x)=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)+2\sqrt{2}$

or $f(x)=\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)+2\sqrt{2}$

i.e., $Y=[\sqrt{2},3\sqrt{2}]$ and $X=\left[-\frac{3\pi}{4},\frac{\pi}{4}\right]$ or $\left[\frac{\pi}{4},\frac{5\pi}{4}\right]$

102. (4) Replacing x by $g^{-1}(x)$, we get $x=2f(g^{-1}(x))+5$

$\therefore$ $f(g^{-1}(x))=\frac{x-5}{2}$

$\therefore$ $g^{-1}(x)=f^{-1}\left(\frac{x-5}{2}\right)$

103. (2) $y=[x]+\sqrt{\{x\}}$

$\Rightarrow [y]+\{y\}=[x]+\sqrt{\{x\}}$

$\Rightarrow \{y\}=\sqrt{\{x\}}$ ... $(\because [y]=[x])$

$\Rightarrow \{x\}=\{y\}^2$

$\Rightarrow \{x\}+[x]=[y]+\{y\}^2$

$\Rightarrow x=[y]+\{y\}^2$

$\Rightarrow f^{-1}(x)=[x]+\{x\}^2$

104. (2) $x=0 \Rightarrow f(2)=2f(0)-f(1)=2\times2-3=1$

$x=1 \Rightarrow f(3)=6-1=5$

$x=2 \Rightarrow f(4)=2f(2)-f(3)=2\times1-5=-3$

$x=3 \Rightarrow f(5)=2f(3)-f(4)=2(5)-(-3)=13$

105. (2) $x^2F(x)+F(1-x)=2x-x^4$ (1)

Replacing x by $1-x$, we get

$(1-x)^2F(1-x)+F(x)=2(1-x)-(1-x)^4$ (2)

Eliminating $F(1-x)$ from (1) and (2), we get $F(x)=1-x^2$.

106. (2) $f(x)=x^n+1$

or $f(3)=3^n+1=28$

or $3^n=27$

$\therefore$ $n=3$

or $f(4)=4^3+1=65$

107. (4) Let $2x+\frac{y}{8}=\alpha$ and $2x-\frac{y}{8}=\beta$.

Then $x=\frac{\alpha+\beta}{4}$ and $y=4(\alpha-\beta)$.

Given $f\left(2x+\frac{y}{8},2x-\frac{y}{8}\right)=xy$

$\therefore$ $f(\alpha,\beta)=\alpha^2-\beta^2$

or $f(m,n)+f(n,m)=m^2-n^2+n^2-m^2=0$ for all m,n

108. (1) $f(xy+1)=f(yx+1)$

$f(x)f(y)-f(y)-x+2=f(y)f(x)-f(x)-y+2$

$f(x)-f(y)=x-y$

Putting $y=0$

$f(x)-1=x-0$

$f(x)=x+1$

109. (2) Given $f(x+y)=f(x)+f(y)-xy-1\ \forall\ x,y\in R$

$f(1)=1, f(2)=f(1+1)=f(1)+f(1)-1-1=0$

$f(3)=f(2+1)=f(2)+f(1)-2\times1-1=-2$

$f(n+1)=f(n)+f(1)-n-1=f(n)-n<f(n)$

Thus, $f(1)>f(2)>f(3)>\cdots$, and $f(1)=1$.

Therefore, $f(1)=1$ and $f(n)<1$, for $n>1$.

Hence, $f(n)=n$, $n\in N$, has only one solution $n=1$.

110. (2) $3f(x)+2f\left(\frac{x+59}{x-1}\right)=10x+30$

For $x=7$, $3f(7)+2f(11)=70+30=100$.

For $x=11$, $3f(11)+2f(7)=140$.

$\frac{f(7)}{-20}=\frac{f(11)}{-220}=\frac{-1}{9-4}$ or $f(7)=4$

111. (2) $\left(f\left(\frac{16+y^2}{y^2}\right)\right)^{\frac{4}{\sqrt{y}}}=\left(f\left(\frac{16}{y^2}+1\right)\right)^{\frac{4}{\sqrt{y}}}$

$=\left(f\left(\left(\frac{4}{y}\right)^2+1\right)\right)^{\frac{4}{\sqrt{y}}}$

$=\left[\left(f\left(\left(\frac{4}{y}\right)^2+1\right)\right)^{\frac{2}{\sqrt{y}}}\right]^2$

$=5^2$

112. (4) $f(x)-1+f(1-x)-1=0$

So, $g(x)+g(1-x)=0$.

Replacing x by $x+\frac{1}{2}$, we get $g\left(\frac{1}{2}+x\right)+g\left(\frac{1}{2}-x\right)=0$.

So, it is symmetrical about $\left(\frac{1}{2},0\right)$.

113. (1) Putting $x=1$, $f(2)+f(0)=2f(1)$ or $f(2)=2f(1)$.

Putting $x=2$, $f(3)+f(1)=2f(2)$

or $f(3)=2\times2f(1)-f(1)=3f(1)$ and so on.

$\therefore$ $f(n)=nf(1)$, for $n=1,2,\ldots,n$

$f(n+1)+f(n-1)=2f(n)$

or $f(n+1)+(n-1)f(1)=2nf(1)$

or $f(n+1)=(n+1)f(1)$

114. (3) $f\left(x+\frac{1}{2}\right)+f\left(x-\frac{1}{2}\right)=f(x)$ (1)

Replacing x by $\left(x+\frac{1}{2}\right)$, we get

$f(x+1)+f(x)=f\left(x+\frac{1}{2}\right)$ (2)

or $f(x+1)+f\left(x-\frac{1}{2}\right)=0$ (from (1) and (2))

or $f\left(x+\frac{3}{2}\right)=-f(x)$ (Replacing x by $\left(x+\frac{1}{2}\right)$)

or $f(x+3)=-f\left(x+\frac{3}{2}\right)=f(x)$ (Replacing x by $\left(x+\frac{3}{2}\right)$)

Therefore, $f(x)$ is periodic with period 3.

115. (1) $af(x+1)+bf\left(\frac{1}{x+1}\right)=(x+1)-1$ (1)

Replacing $x+1$ by $\frac{1}{x+1}$, we get

$$af\left(\frac{1}{x+1}\right)+bf(x+1)=\frac{1}{x+1}-1 \quad (2)$$

Eq. (1) $\times a$ – Eq. (2) $\times b$

$$\Rightarrow \quad (a^2-b^2)f(x+1)=a(x+1)-a-\frac{b}{x+1}+b$$

Putting $x=1$, $(a^2-b^2)f(2)=2a-a-\frac{b}{2}+b=a+\frac{b}{2}=\frac{2a+b}{2}$

116. (4) $f(3x+2)+f(3x+29)=0$ (1)

Replacing x by $x+9$, we get

$f(3(x+9)+2)+f(3(x+9)+29)=0$

or $f(3x+29)+f(3x+56)=0$ (2)

From (1) and (2), we get

$f(3x+2)=f(3x+56)$

or $f(3x+2)=f(3(x+18)+2)$

Therefore, $f(x)$ is periodic with period 54.

117. (3) From the given data,

$f(1-x)=f(1+x)$ (1)

and $f(2-x)=f(2+x)$ (2)

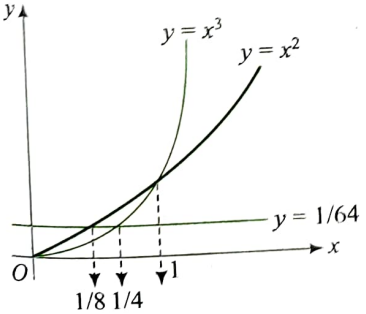
In (2), replacing x by $1+x$, we have

$f(1-x)=f(3+x)$

or $f(1+x)=f(3+x)$ [From (1)]

or $f(x)=f(2+x)$

118. (3)



Clearly, from the graph, $f(x)=\begin{cases}\frac{1}{64}, & 0\le x\le\frac{1}{8}\\ x^2, & \frac{1}{8}<x\le 1\\ x^3, & x>1\end{cases}$.

119. (1) $|x-2|+a=\pm 4$

or $|x-2|=\pm 4-a$

For four real roots,

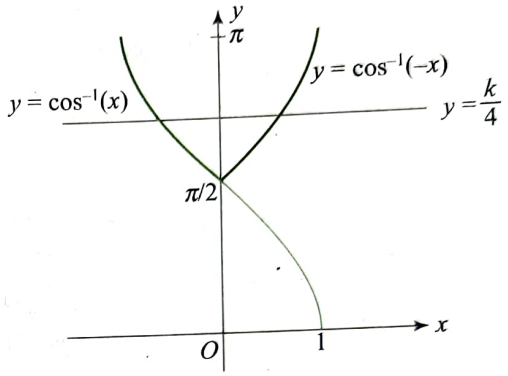
$4-a>0$ and $-4-a>0$

or $a\in(-\infty,-4)$

120. (3) $4\cos^{-1}(-|x|)=k$ (1)

$\Rightarrow \quad \cos^{-1}(-|x|)=\frac{k}{4}$

The graph of $y=\cos^{-1}(-|x|)$ is as shown in the following figure



Clearly from the figure equation (1) has two solutions if line $y=\frac{k}{4}$ intersect the graph between $\frac{\pi}{2}$ and π.

$\therefore \quad \frac{\pi}{2}<\frac{k}{4}\le\pi$

$\therefore \quad 2\pi<k\le 4\pi$

Therefore, integral values of k are 7, 8, 9, 10, 11, 12.

121. (3) $f\left(\frac{\pi}{2}-x\right)=\ln(1-\cos x)$

and $f\left(\frac{\pi}{2}+x\right)=\ln(1-\cos x)$

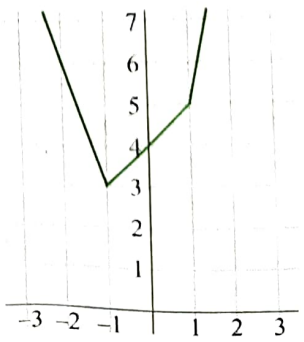
Thus, $f\left(\frac{\pi}{2}+x\right)=f\left(\frac{\pi}{2}-x\right)$

Thus, $f(x)$ is symmetrical about line $x=\frac{\pi}{2}$.

122. (1) Let $f(x)=x+2|x+1|+2|x-1|$

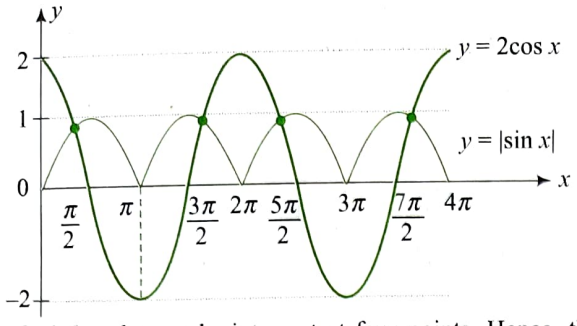
$$=\begin{cases}x-2(x+1)-2(x-1), & x<-1\\ x+2(x+1)-2(x-1), & -1\le x\le 1\\ x+2(x+1)+2(x-1), & x>1\end{cases}$$

$$=\begin{cases}-3x, & x<-1\\ x+4, & -1\le x\le 1\\ 5x, & x>1\end{cases}$$



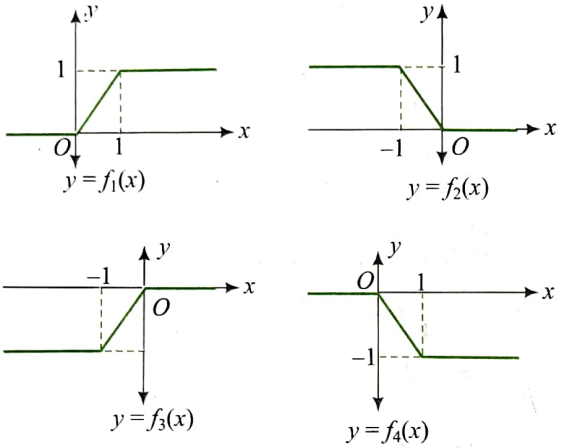
Graph of $y=f(x)$ is as shown.
Clearly, $y=k$ can intersect $y=f(x)$ at exactly one point only if $k=3$.

123. (3) See the graph of $y = 2\cos x$ and $y = |\sin x|$. Their points of intersection represent the solution of the given equation.



We find that the graphs intersect at four points. Hence, the equation has four solutions.

124. (2)



125. (3) $\log_4\left(\dfrac{2f(x)}{1-f(x)}\right) = x$

$$\Rightarrow \quad \frac{2f(x)}{1-f(x)} = 4^x \qquad (1)$$

Replacing x by $1 - x$, we get

$$\frac{2f(1-x)}{1-f(1-x)} = 4^{1-x} \qquad (2)$$

Multiplying (1) and (2), we get

$$\frac{2f(x)}{1-f(x)}\left(\frac{2f(1-x)}{1-f(1-x)}\right) = 4^{1-x}$$

$$\Rightarrow \quad f(x)f(1-x) = 1 - f(x) - f(1-x) + f(x)f(1-x)$$

$$\Rightarrow \quad f(x) + f(1-x) = 1$$

Putting $x = 2010$, we get

$$f(2010) + f(-2009) = 1$$

Multiple Correct Answers Type

1. (1), (4)

$f(x) = \sec^{-1}[1 + \cos^2 x]$

$f(x)$ is defined if $[1 + \cos^2 x] \le -1$ or $[1 + \cos^2 x] \ge 1$

i.e., $[\cos^2 x] \le -2$ (not possible) or $[\cos^2 x] \ge 0$

i.e., $\cos^2 \ge 0$ or $x \in R$

Now, $0 \le \cos^2 x \le 1$ or $1 \le 1 + \cos^2 x \le 2$

or $[1 + \cos^2 x] = 1, 2$

or $\sec^{-1}[1 + \cos^2 x] = \sec^{-1}1, \sec^{-1}2$

Hence, the range is $\{\sec^{-1}1, \sec^{-1}2\}$.

2. (2), (4)

The period of $g(x) = |\sin 2x| + |\cos 2x|$ is $\pi/4$.

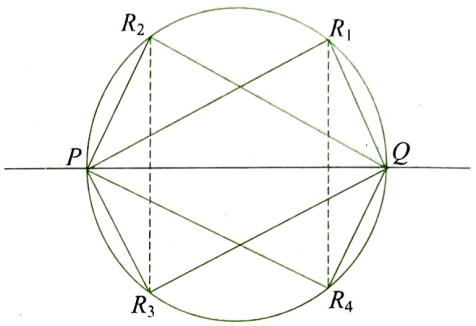
Also, $1 \le g(x) \le \sqrt{2}$.

Hence $[g(x)] = 1$ $f(x)$ is a many-one and into function.

3. (1), (2), (3), (4)

Hence $\angle PRQ = \pi/2$ and points P, Q, R lie on the circle with PQ as diameter.

Also, $PQ = 5$.



Now, the maximum area of the triangle is

$$\Delta_{max} = \frac{1}{2}(5)\left(\frac{5}{2}\right) = 6.25$$

For area = 5, we have four symmetrical positions of point R (shown as R_1, R_2, R_3, R_4).

For area = 6.25, we have exactly two points.

For area = 7, no such points exist.

4. (1), (2), (3)

$f(x)$ is defined if $\log_{|\sin x|}(x^2 - 8x + 23) - \dfrac{3}{\log_2|\sin x|} > 0$

or $\log_{|\sin x|}\left(\dfrac{x^2 - 8x + 23}{8}\right) > 0$

This is true if $|\sin x| \ne 0, 1$ and $\dfrac{x^2 - 8x + 23}{8} < 1$.

Now, $\dfrac{x^2 - 8x + 23}{8} < 1$ or $x^2 - 8x + 15 < 0$

or $x \in (3, 5) - \left\{\pi, \dfrac{3\pi}{2}\right\}$

Domain $= (3, \pi) \cup \left(\pi, \dfrac{3\pi}{2}\right) \cup \left(\dfrac{3\pi}{2}, 5\right)$.

5. (1), (2), (3), (4)

$f(x) = \text{sgn}(\cot^{-1}x) + \tan\left(\dfrac{\pi}{2}[x]\right)$

$\text{sgn}(\cot^{-1}x)$ is defined when $\cot^{-1}x$ is defined, which is $\forall\, x \in R$.

$\tan\left(\dfrac{\pi}{2}[x]\right)$ is defined when $\dfrac{\pi}{2}[x] \ne \dfrac{(2n+1)}{2}\pi$, where $n \in Z$,

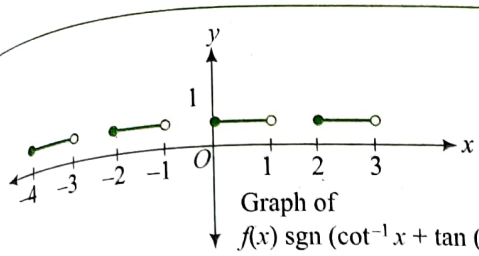
or $[x] \ne 2n + 1$ or $x \notin [2n + 1, 2n + 2)$.

Hence, domain of $f(x)$ is $\bigcup_{n \in Z}[2n, 2n + 1)$.

Also, $\cot^{-1}x > 0\ \forall\, x \in R$. Then

$$f(x) = 1 + \tan\left(\frac{\pi}{2}[x]\right) = 1$$

$\therefore \quad f(x) = 1, x \in D_f$



Graph of
$f(x)$ sgn $(\cot^{-1} x + \tan(\pi/2[x]))$

From the graph, $f(x)$ is periodic with period 2.

6. **(1), (2), (4)**

$f(x) = |\cos x| + |\sec x|$.

7. **(2), (3), (4)**

$f(x) = \sin(\sin^{-1} x) = x \; \forall \; x \in [-1, 1]$, which is one-one and onto.

$$f(x) = \frac{2}{\pi}\sin^{-1}(\sin x) = \frac{2}{\pi}x$$

The range of the function for $x \in [-1, 1]$ is $\left[-\frac{2}{\pi}, \frac{2}{\pi}\right]$,

which is a subset of $[-1, 1]$.

Hence, the function is one-one but not onto and, hence, is not bijective.

$$f(x) = (\text{sgn}(x)) \ln(e^x) = (\text{sgn}(x))\, x = \begin{cases} x, & x > 0 \\ -x, & x < 0 \\ 0, & x = 0 \end{cases}.$$

This function has range $[0, 1]$ which is a subset of $[-1, 1]$.

Hence, the function is into. Also, the function is many-one.

$$f(x) = x^3 \text{ sgn}(x) = \begin{cases} x^3, & x > 0 \\ -x^3, & x < 0 \\ 0, & x = 0 \end{cases}$$

which is many-one and into.

8. **(1), (2), (3)**

$(f-g)(3.5) = f(3.5) + g(3.5) = (-0.5) + 0.5 = 0$

$f(g(3)) = f(0) = 3$

$(fg)(2) = f(2)g(2) = (-1) \times (-1) = 1$

$(f-g)(4) = f(4) - g(4) = 0 - 26 = -26$

9. **(1), (2), (4)**

$f(0) = \max\{1 + \sin 0, 1, 1 - \cos 0\} = 1$

$g(0) = \max\{1, |0 - 1|\} = 1$

$f(1) = \max\{1 + \sin 1, 1, 1 - \cos 1\} = 1 + \sin 1$

$g(f(0)) = g(1) = \max\{1, |1 - 1|\} = 1$

$f(g(0)) = f(1) = 1 + \sin 1$

$g(f(1)) = g(1 + \sin 1) = \max\{1, |1 + \sin 1 - 1|\} = 1$

10. **(2), (4)**

$$f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2}$$

or $$f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$$

or $$f(y) = y^2 - 2$$

Now, $y = x + \frac{1}{x} \geq 2$ or ≤ -2

Hence, the domain of the function is $(-\infty, -2] \cup [2, \infty)$.

Also, for these values of y, $y^2 \geq 4$ or $y^2 - 2 \geq 2$.

Hence, the range of the function is $[2, \infty)$.

11. **(2), (3)**

Given $2f(\sin x) + f(\cos x) = x$ (1)

Replacing x by $\frac{\pi}{2} - x$, we get

$$2f(\cos x) + f(\sin x) = \frac{\pi}{2} - x \quad (2)$$

Eliminating $f(\cos x)$ from (1) and (2), we get

$$3f(\sin x) = 3x - \frac{\pi}{2}$$

or $$f(\sin x) = x - \frac{\pi}{6}$$

or $$f(x) = \sin^{-1} x - \frac{\pi}{6}$$

$f(x)$ has the domain $[-1, 1]$.

Also, $\sin^{-1} x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ or $\sin^{-1} x - \frac{\pi}{6} \in \left[-\frac{2\pi}{3}, \frac{\pi}{3}\right]$.

12. **(2), (3)**

$f(x)$ must be a linear function. Let $f(x) = ax + b$. Then

$f(ax + b) = 6x - ax - b$

or $a(ax + b) + b = 6x - ax - b$

or $a^2 = 6 - a$ and $ab + b = -b$

i.e., $a = 2$ or -3

$\therefore \quad b = 0$

$\therefore \quad f(x) = 2x$ or $-3x$

$\therefore \quad f(17) = 34$ or -51

13. **(2), (4)**

$f(x) = x^2 - 2ax + a(a + 1)$

$f(x) = (x - a)^2 + a, x \in [a, \infty)$

Let $y = (x - a)^2 + a$. Clearly, $y \geq a$. Thus,

$(x - a)^2 = y - a$

or $x = a + \sqrt{y - a}$

$\therefore \quad f^{-1}(x) = a + \sqrt{x - a}$

Now, $f(x) = f^{-1}(x)$

or $(x - a)^2 + a = a + \sqrt{x - a}$

$(x - a)^2 = \sqrt{x - a}$

or $(x - a)^4 = (x - a)$

i.e., $x = a$ or $(x - a)^3 = 1$

i.e., $x = a$ or $a + 1$

If $a = 5049$, then $a + 1 = 5050$.

If $a + 1 = 5049$, then $a = 5048$.

14. **(1), (2), (3), (4)**

$$f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$$

or $$f(x + k) = \begin{cases} 1, & x + k \text{ is rational} \\ 0, & x + k \text{ is irrational} \end{cases}$$ where k is any rational number

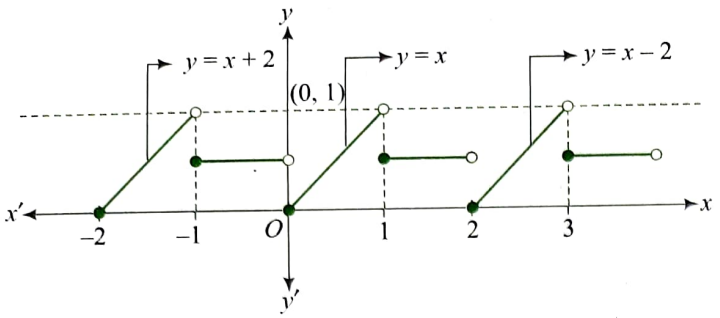
$$= \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$$

$= f(x)$

Therefore, $f(x)$ is periodic function, but its fundamental period cannot be determined. Thus,

$$f(x) = \begin{cases} x - [x], & 2n \leq x < 2n + 1 \\ 1/2, & 2n + 1 \leq x < 2n + 2 \end{cases}.$$

Draw the graph from which it can be verified that period is 2.



$$f(x) = (-1)^{\left[\frac{2x}{\pi}\right]}$$

or $f(x+\pi) = (-1)^{\left[\frac{2(\pi+x)}{\pi}\right]} = (-1)^{\left[\frac{2x}{\pi}\right]+2} = (-1)^{\left[\frac{2x}{\pi}\right]}$.

Hence, the period is π.

$$f(x) = x - [x+3] + \tan\left(\frac{\pi x}{2}\right) = \{x\} - 3 + \tan\left(\frac{\pi x}{2}\right)$$

$\{x\}$ is periodic with period 1; $\tan\left(\frac{\pi x}{2}\right)$ x is periodic with period 2.

Now, the LCM of 1 and 2 is 2. Hence, the period of $f(x)$ is 2.

15. (1), (3), (4)

$$f^2(x) = f\left(\frac{3}{4}x+1\right) = \frac{3}{4}\left(\frac{3}{4}x+1\right)+1 = \left(\frac{3}{4}\right)^2 x + \frac{3}{4} + 1 \quad (1)$$

$$f^3(x) = f\left\{f^2(x)\right\} = \frac{3}{4}\left\{f^2(x)+1\right\}$$

$$= \frac{3}{4}\left\{\left(\frac{3}{4}\right)^2 x + \frac{3}{4} + 1\right\} + 1$$

$$= \left(\frac{3}{4}\right)^3 x + \left(\frac{3}{4}\right)^2 + \frac{3}{4} + 1$$

$$\therefore \quad f^n(x) = \left(\frac{3}{4}\right)^n x + \left(\frac{3}{4}\right)^{n-1} + \left(\frac{3}{4}\right)^{n-2} + \cdots + \left(\frac{3}{4}\right) + 1$$

$$= \left(\frac{3}{4}\right)^n x + \frac{1-\left(\frac{3}{4}\right)^n}{1-\frac{3}{4}}$$

$$\therefore \quad \lambda = \lim_{n\to\infty} f^n(x) = 0 + 4 = 4$$

16. (1), (2), (3), (4)

Period of $\sin x$ is 2π and period of $\cos\left(\sqrt{4-a^2}\,x\right)$ is $\frac{2\pi}{\sqrt{4-a^2}}$.

$$\Rightarrow \quad LCM\left(2\pi, \frac{2\pi}{\sqrt{4-a^2}}\right) = 4\pi \text{ (given)}$$

i.e., $\sqrt{4-a^2} = \frac{p}{2}$ where $p = 1, 3$.

Hence $a^2 = \frac{15}{4}, \frac{7}{4}$, $a = \pm\frac{\sqrt{15}}{2}, \pm\frac{\sqrt{7}}{2}$

17. (1), (3)

We have $f(x) = \sin^{-1}[e^x] + \sin^{-1}[e^{-x}]$

We must have

$0 < e^x < 2$ and $0 < e^{-x} < 2$

$\Rightarrow \quad x \in (-\log_e 2, \log_e 2)$

$$\Rightarrow \quad f(x) = \begin{cases} \pi, & x = 0 \\ \frac{\pi}{2}, & x \in (-\log_e 2, 0) \cup (0, \log_e 2) \end{cases}$$

18. (3), (4)

$$[2x] = [x] + \left[x + \frac{1}{2}\right]$$

$$\therefore \quad [2x] - 2[x] = \left[x + \frac{1}{2}\right] - [x]$$

$$= \left[[x] + \{x\} + \frac{1}{2}\right] - [x]$$

$$= \left[\{x\} + \frac{1}{2}\right]$$

If $\{x\} \geq \frac{1}{2} \Rightarrow \{x\} + \frac{1}{2} \geq 1 \Rightarrow \left[\{x\} + \frac{1}{2}\right] = 1 \Rightarrow \lambda = 1$

If $\{x\} < \frac{1}{2} \Rightarrow \{x\} + \frac{1}{2} < 1 \Rightarrow \left[\{x\} + \frac{1}{2}\right] = 0 \Rightarrow \lambda = 0$

19. (1), (2)

$\{x\} = [x]\,x$

$\{x\} = x(x - \{x\}) = x^2 - x\{x\}$

$$\Rightarrow \quad \{x\} = \frac{x^2}{1+x}$$

Now $0 \leq \{x\} < 1$

$$\Rightarrow \quad \frac{x^2}{1+x} \geq 0$$

$\Rightarrow \quad 1 + x > 0$

$\Rightarrow \quad x > -1$

$$\frac{x^2}{1+x} < 1$$

$\Rightarrow \quad x^2 < 1 + x$

$\Rightarrow \quad x^2 - x - 1 < 0$

$$\Rightarrow \quad \frac{1-\sqrt{5}}{2} < x < \frac{1+\sqrt{5}}{2}$$

Given equation again can be written as

$x - [x] = x[x]$

$$\frac{x}{1+x} = [x] = \lambda$$

$$x = \frac{\lambda}{1-\lambda}$$

$$\frac{1-\sqrt{5}}{2} < \frac{\lambda}{1-\lambda} < \frac{1+\sqrt{5}}{2}$$

$$\Rightarrow \quad \frac{-1-\sqrt{5}}{2} < \lambda < \frac{\sqrt{5}-1}{2}$$

$\lambda = 0 \Rightarrow x = 0$

$\lambda = -1 \Rightarrow x = \frac{-1}{2}$

20. (1), (2)

$$g(x) = \sin(\sin^{-1}\sqrt{\{x\}}) + \cos\left(\sin^{-1}\sqrt{\{x\}}\right) - 1$$

$$= \sqrt{\{x\}} + \cos\left(\cos^{-1}\sqrt{1-\{x\}}\right) - 1$$

$$= \sqrt{\{x\}} + \sqrt{1-\{x\}} - 1$$

If $x \in I$ then $\{x\} = 0$ then $g(x) = 0$

If $x \notin I$ then $\{-x\} = 1 - \{x\}$

$\Rightarrow$ $g(-x) = \sqrt{1-\{x\}} + \sqrt{\{x\}} - 1 = g(x)$

$\Rightarrow$ $g(-x) = g(x)$

$\Rightarrow$ $g(x)$ is even function.

Also $g(x) = \begin{cases} 0, & x \in I \\ g(-x), & x \notin I \end{cases}$

Thus $g(x)$ is periodic function

21. (1), (3)

$$f(x+y) + f(x-y) = 2f(x) \cdot f(y) \quad (1)$$

Put $x = 0$. Then $f(y) + f(-y) = 2f(0)f(y)$ (2)

Put $x = y = 0$. Then $f(0) + f(0) = 2f(0)f(0)$

$\therefore$ $f(0) = 1$ [as $f(0) \neq 0$]

$\therefore$ $f(-y) = f(y)$ [From (2)]

Hence, the function is even. Then $f(-2) = f(2) = a$.

22. (1), (4)

Given $f(x) + f(y) = \left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$ (1)

Replace y by x. Then $2f(x) = f\left(2x\sqrt{1-x^2}\right)$

$$3f(x) = f(x) + 2f(x)$$

$$= f(x) + f\left(2x\sqrt{1-x^2}\right)$$

$$= f\left(x\sqrt{1-4x^2(1-x^2)} + 2x\sqrt{1-x^2}\sqrt{1-x^2}\right)$$

$$= f\left(x\sqrt{(2x^2-1)^2} + 2x(1-x^2)\right)$$

$$= f(x|2x^2-1| + 2x - 2x^3)$$

$$= f(2x^3 - x + 2x - 2x^3) \text{ or } f(x - 2x^3 + 2x - 2x^3)$$

$$= f(x) \text{ or } f(3x - 4x^3)$$

$\therefore$ $f(x) = 0$ or $3f(x) = f(3x - 4x^3)$

Consider $3f(x) = f(3x - 4x^3)$.

Replacing x by $-x$, we get

$3f(-x) = f(4x^3 - 3x)$ (2)

Also, from (1), $f(x) + f(-x) = f(0)$.

Putting $x = y = 0$ in (1), we have $f(0) = 0$ or $f(x) + f(-x) = 0$.

Thus, $f(x)$ is an odd function.

Now, from (2), $-3f(x) = f(4x^3 - 3x)$

or $f(4x^3 - 3x) + 3f(x) = 0$

23. (1), (2), (3), (4)

$$f(x+1) = \frac{f(x)-5}{f(x)-3} \quad (1)$$

or $f(x)f(x+1) - 3f(x+1) = f(x) - 5$

or $f(x) = \dfrac{3f(x+1)-5}{f(x+1)-1}$

Replacing x by $(x-1)$, we get

$$f(x-1) = \frac{3f(x)-5}{f(x)-1} \quad (2)$$

Using (1), $f(x+2) = \dfrac{f(x+1)-5}{f(x+1)-3} = \dfrac{\frac{f(x)-5}{f(x)-3} - 5}{\frac{f(x)-5}{f(x)-3} - 3}$

$$= \frac{2f(x)-5}{f(x)-2} \quad (3)$$

Using (2), $f(x-2) = \dfrac{3f(x-1)-5}{f(x-1)-1} = \dfrac{3\left(\frac{3f(x)-5}{f(x)-1}\right)-5}{\frac{3f(x)-5}{f(x)-1} - 1}$

$$= \frac{2f(x)-5}{f(x)-2} \quad (4)$$

Using (3) and (4), we have $f(x+2) = f(x-2)$.

Therefore, $f(x+4) = f(x)$, i.e., $f(x)$ is periodic with period 4.

24. (1), (2), (3), (4)

$$\left(f\left(\frac{1}{x}\right) - 1\right) = \frac{1}{(f(x)-1)} \text{ i.e., } \left(f\left(\frac{1}{x}\right) - 1\right)$$

is reciprocal of $(f(x) - 1)$.

Now, for $f(x) = \dfrac{\frac{\pi}{2}}{\tan^{-1}|x|}$

$$f(x) - 1 = \frac{\cot^{-1}|x|}{\tan^{-1}|x|}, \; f\left(\frac{1}{x}\right) - 1 = \frac{\tan^{-1}|x|}{\cot^{-1}|x|}$$

Also for $f(x) = \dfrac{2}{1 + k\ln|x|}$

$$f(x) - 1 = \frac{1 - k\ln|x|}{1 + k\ln|x|}, \; f\left(\frac{1}{x}\right) - 1 = \frac{1 + k\ln|x|}{1 - k\ln|x|}$$

25. (2), (3), (4)

We have $f(x) = f(2 - x)$

Replacing x by $f(1 + x)$

$f(1 + x) = f(1 - x)$

Hence graph of $f(x)$ is symmetrical about the line $x = 1$

Also $g(x) = f(1 + x) = f(1 - x) = g(-x)$

$\therefore$ $g(x)$ is an even function.

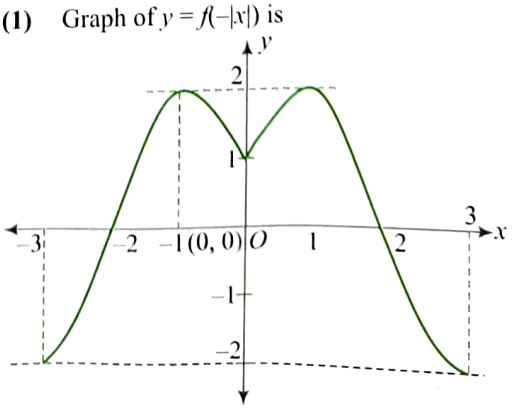
Further $f'(1 + x) = -f'(1 - x)$

$\therefore$ $f'(1) = -f'(1)$

$\therefore$ $f'(1) = 0$

26. (1), (2), (3), (4)

(1) Graph of $y = f(-|x|)$ is



$\therefore$ range is $[-2, 2]$

(2) Domain of $y = f(|x|)$ is

$-3 \le |x| \le 2$

$\Rightarrow \quad -2 \le x \le 2$

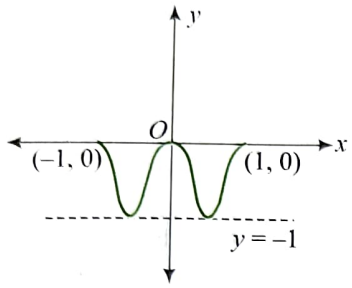
(3) Domain of $y = f(|x| + 1)$ is

$-3 \le |x| + 1 \le 2$

$\Rightarrow \quad -4 \le |x| \le 1$

$\Rightarrow \quad x \in [-1, 1]$

(4) Graph of $y = f(|x| + 1)$ is



$\therefore$ Range is $[-1, 0]$

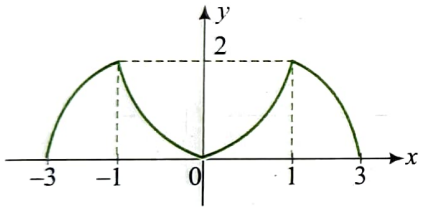
27. (1), (2), (3)

(1) $-1 \le |x| - 1 \le 4$

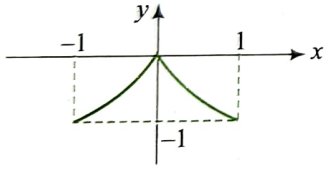
$\Rightarrow \quad 0 \le |x| \le 5 \quad \Rightarrow -5 \le x \le 5$

(2) $f(|x| + 1)$

range of $f(|x| + 1)$ is $[0, 2]$



(3) range of $f(-|x|)$ is $[-1, 0]$



(4) $-1 \le |x| \le 4$

$\Rightarrow \quad x \in [-4, 4]$

ked Comprehension Type

Problems 1–3

$$f(x) = \begin{cases} x+1, & x \le 1 \\ 2x+1, & 1 < x \le 2 \end{cases}$$

$$g(x) = \begin{cases} x^2, & -1 \le x < 2 \\ x+2, & 2 \le x \le 3 \end{cases}.$$

$$\therefore \quad f(x) = \begin{cases} g(x)+1, & g(x) \le 1 \\ 2g(x)+1, & 1 < g(x) \le 2 \end{cases}$$

or $$f(g(x)) = \begin{cases} x^2+1, & x^2 \le 1, \quad -1 \le x < 2 \\ x+2+1, & x+2 \le 1, \quad 2 \le x \le 3 \\ 2x^2+1, & 1 < x^2 \le 2, \quad -1 \le x < 2 \\ 2(x+2)+1, & 1 < x+2 \le 2, \quad 2 \le x \le 3 \end{cases}$$

or $$f(g(x)) = \begin{cases} x^2+1, & -1 \le x \le 1 \\ 2x^2+1, & 1 < x \le \sqrt{2} \end{cases}$$

1. (3) Hence, the domain of $f(x)$ is $[-1, \sqrt{2}]$.

2. (3) For $-1 \le x \le 1$, we have $x^2 \in [0, 1]$ or $x^2 + 1 \in [1, 2]$.

For $1 < x \le \sqrt{2}$, we have $x^2 \in (1, 2]$ or $2x^2 + 1 \in (3, 5]$.

Hence, the range is $[1, 2] \cup (3, 5]$.

3. (2) For $f(g(x)) = 2$, $x^2 + 1 = 2$ and $2x^2 + 1 = 2$, i.e.,

$$x = \pm 1 \text{ or } x = \pm \frac{1}{\sqrt{2}}.$$

Therefore, $x = \pm 1$ only. Hence, two roots.

For Problems 4–6

$$f(x) + f\left(\frac{x-1}{x}\right) = 1 + x \qquad (1)$$

In (1), replace x by $\frac{x-1}{x}$. Then

$$f\left(\frac{x-1}{x}\right) + f\left(\frac{\frac{x-1}{x} - 1}{\frac{x-1}{x}}\right) = 1 + \frac{x-1}{x}$$

or $$f\left(\frac{x-1}{x}\right) + f\left(\frac{1}{1-x}\right) = 1 + \frac{x-1}{x} \qquad (2)$$

Now, from (1) – (2), we have

$$f(x) - f\left(\frac{1}{1-x}\right) = x - \frac{x-1}{x} \qquad (3)$$

In (3), replace x by $\frac{1}{1-x}$. Then

$$f\left(\frac{1}{1-x}\right) - f\left(\frac{x-1}{x}\right) = \frac{1}{1-x} - \frac{\frac{1}{1-x} - 1}{\frac{1}{1-x}}$$

or $$f\left(\frac{1}{1-x}\right) - f\left(\frac{x-1}{x}\right) = \frac{1}{1-x} - x \qquad (4)$$

Now, from (1) + (3) + (4), we have

$$2f(x) = 1 + x + x - \frac{x-1}{x} + \frac{1}{1-x} - x$$

or $$f(x) = \frac{x^3 - x^2 - 1}{2x(x-1)}$$

4. (2) $g(x) = \dfrac{x^3 - x^2 - 1}{x(x-1)} - x + 1 = \dfrac{x^2 - x - 1}{x(x-1)}$

Now, for $y = \sqrt{g(x)}$, we must have $\dfrac{x^2 - x - 1}{x(x-1)} \ge 0$

or $$\frac{\left(x - \frac{1-\sqrt{5}}{2}\right)\left(x - \frac{1+\sqrt{5}}{2}\right)}{x(x-1)} \ge 0$$

or $$x \in \left(-\infty, \frac{1-\sqrt{5}}{2}\right] \cup (0, 1) \cup \left[\frac{1+\sqrt{5}}{2}, \infty\right)$$

5. (3) $y = g(x) = \dfrac{x^2 - x - 1}{x(x-1)}$ or $(y-1)x^2 + (1-y)x + 1 = 0$

Now, x is real, Therefore, $D \ge 0$ or $(1-y)^2 - 4(y-1) \ge 0$

or $\quad (y-1)(y-5) \ge 0$

or $\quad y \in (-\infty, 1) \cup [5, \infty)$

6. (4) $g(x) = 1$, or $\dfrac{x^2 - x - 1}{x(x-1)} = 1$ or $-x - 1 = -x$, which has no solution.

Problems 7–9

(1), 8. (2), 9. (3)

$$\left(f(x)\right)^2 f\left(\frac{1-x}{1+x}\right) = 64x \qquad (1)$$

Putting $\frac{1-x}{1+x} = y$ or $x = \frac{1-y}{1+y}$, we get

$$\left\{f\left(\frac{1-y}{1+y}\right)\right\}^2 \cdot f(y) = 64\left(\frac{1-y}{1+y}\right)$$

or $$f(x)\cdot\left\{f\left(\frac{1-x}{1+x}\right)\right\}^2 = 64\left(\frac{1-x}{1+x}\right) \qquad (2)$$

Squaring (1) and dividing by (2),

$$\frac{f(x)^4\left\{f\left(\frac{1-x}{1+x}\right)\right\}^2}{f(x)\left\{f\left(\frac{1-x}{1+x}\right)\right\}^2} = \frac{(64x)^2}{64\left(\frac{1-x}{1+x}\right)}$$

or $$\{f(x)\}^3 = 64x^2\left(\frac{1+x}{1-x}\right)$$

or $$f(x) = 4x^{2/3}\left(\frac{1+x}{1-x}\right)^{1/3}$$

$\therefore$ $f(9/7) = -4(9/7)^{2/3}(2)$

For Problems 10–12

$|g(x)| = |\sin x|,\ x \in R$

$$f(|g(x)|) = \begin{cases} |\sin x| - 1, & -1 \le |\sin x| < 0 \\ (|\sin x|)^2, & 0 \le (|\sin x|) \le 1 \end{cases} = \sin^2 x,\ x \in R$$

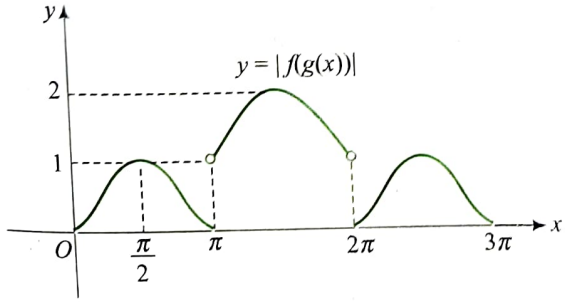
$$f(g(x)) = \begin{cases} \sin x - 1, & -1 \le \sin x < 0 \\ \sin^2 x, & 0 \le \sin x \le 1 \end{cases}$$

$$= \begin{cases} \sin x - 1, & (2n-1)\pi < x < 2n\pi \\ \sin^2 x, & 2n\pi \le x \le (2n+1)\pi \end{cases},\ n \in Z.$$

or $$|f(g(x))| = \begin{cases} 1 - \sin x, & (2n-1)\pi < x < 2n\pi \\ \sin^2 x, & 2n\pi \le x \le (2n+1)\pi \end{cases},\ n \in Z$$

10. (4) Clearly, $h_1(x) = f(|g(x)|) = \sin^2 x$ has period π, range $[0, 1]$, and domain R.

11. (3) $h_2(x) = |f(g(x))|$ has domain R.



Also, from the graph, it is periodic with period 2π and has range $[0, 2]$.

12. (3) For $h_1(x) \equiv h_2(x) = \sin^2 x,\ x \in [2n\pi, (2n+1)\pi],\ n \in Z$, and has range $[0, 1]$ for the common domain.

Also, the period is 2π (from the graph).

For Problems 13–15

Given $a_{n+1} = f(a_n)$

Now, $a_1 = f(a_0) = f(x)$

or $a_2 = f(a_1) = f(f(a_0)) = fof(x)$

or $$a_n = \frac{fofofof \ldots f(x)}{n \text{ times}}$$

13. (4) $a_1 = f(x) = (a - x^m)^{1/m}$

or $a_2 = f(f(x)) = [a - \{(a - x^m)^{1/m}\}^m]^{1/m} = x$

or $a_3 = f(f(f(x))) = f(x)$

Obviously, the inverse does not exist when m is even and n is odd.

14. (4) Now, if $f(x) = \frac{1}{1-x}$, $fof(x) = \frac{1}{1-\frac{1}{1-x}} = \frac{x-1}{x}$

or $$fofof(x) = \frac{\frac{1}{1-x} - 1}{\frac{1}{1-x}} = x$$

or $$a_n = \frac{fofof \ldots of(x)}{n \text{ times}} = \frac{1}{1-x} \text{ if } n = 3k+1$$

$$= \frac{x-1}{x} \text{ if } n = 3k+2$$

$$= x \text{ if } n = 3k$$

15. (3) Since $a_1 = g(x) = 3 + 4x$, we have

$a_2 = g\{g(x)\} = g(3+4x) = 3 + 4(3+4x) = (4^2 - 1) + 4^2 x$

$a_3 = g\{g^2(x)\} = g(15 + 4^2 x) = 3 + 4(15 + 4^2 x) = 63 + 4^3 x$

$= (4^3 - 1) + 4^3 x$

Similarly, we get $a_n = (4^n - 1) + 4^n x$

or $A = 4^n - 1$ and $B = 4^n$

or $A + B + 1 = 2^{2n+1}$, $|A - B| = 1$, and

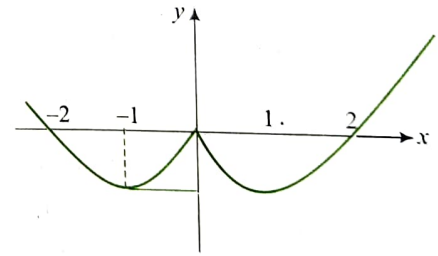
$$\lim_{n\to\infty} \frac{4^n - 1}{4^n} = \lim_{n\to\infty}\left(1 - \frac{1}{4^n}\right) = 1$$

For Problems 16–18

16. (1) $f_1(x) = x^2$ and $f_2(x) = |x|$

or $f(x) = f_1(x) - 2f_2(x) = x^2 - 2|x|$

Graph of $f(x)$



$$g(x) = \begin{cases} f(x), & -3 \le x < -1 \\ -1, & -1 \le x < 0 \\ 0, & 0 \le x \le 2 \\ f(x), & 2 < x \le 3 \end{cases} = \begin{cases} x^2 + 2x, & -3 \le x < -1 \\ -1, & -1 \le x < 0 \\ 0, & 0 \le x \le 2 \\ x^2 - 2x, & 2 < x \le 3 \end{cases}$$

The range of $g(x)$ for $[-3, -1]$ is $[-1, 3]$.

17. (2) For $x \in (-1, 0)$, $f(x) + g(x) = x^2 + 2x - 1$.

18. (1) Obviously, the graph is broken at $x = 0$.

For Problems 19–21

19. (1) $g(f(x))$ is not defined if

(i) $-2 + a > 8$ and **(ii)** $b + 3 > 8$

or $a > 10$ and $b > 5$

20. (4) $x \in [-1, 2]$

or $-1 \le x \le 2$

or $-2 \le 2x \le 4$

or $-2 + a \le 2x + a \le 4 + a$

or $-2 + a \le -2$ and $4 + a \le 4$, i.e., $a = 0$

b can take any value.

21. (3) If $a = 2$, $b = 3$,

$$f(x) = \begin{cases} 2x + 2, & x \ge -1 \\ 3x^2 + 3, & x < -1 \end{cases}$$

The range of $f(x)$ is $[0, \infty)$.

$\therefore$ $g(f(x)) = 2x + 6$, $-1 \le x \le 1$

So, range of $g(f(x))$ is $[4, 8]$.

For Problems 22–24

$$f(2 - x) = f(2 + x) \quad (1)$$

Replace x by $2 - x$. Then $f(x) = f(4 - x)$ (2)

Also, given $f(20 - x) = f(x)$ (3)

From (1) and (2), $f(4 - x) = f(20 - x)$.

Replace x by $4 - x$. Then $f(x) = f(x + 16)$.

Hence, the period of $f(x)$ is 16.

22. (3) Given $f(0) = 5$.

$\therefore$ $f(0) = f(16) = f(32) = \ldots = f(160) = 5$

23. (3) $f(2 - x) = f(2 + x)$

Hence, $y = f(x)$ is symmetrical about $x = 2$.

Also, $f(20 - x) = f(x)$

or $f(20 - (10 + x)) = f(10 + x)$

or $f(10 - x) = f(10 + x)$

Hence, $y = f(x)$ is symmetrical about $x = 10$.

24. (3) If 1 is a period, then $f(x) = f(x + 1)\ \forall\ x \in R$

or $f(2) = f(3) = f(4) = f(5) = f(6)$

which contradicts the given hypotheses that $f(2) \ne f(6)$.

Therefore, 1 cannot be period of $f(x)$.

For Problems 25–27

25. (3), 26. (3), 27. (2)

$$g(f(x)) = \begin{cases} [f(x)] & -\pi \le f(x) < 0 \\ |\sin f(x), & 0 \le f(x) \le \pi \end{cases}$$

$$= \begin{cases} [[x]], & -\pi \le [x] < 0, \quad -2 \le x \le -1 \\ [|x| + 1], & -\pi \le |x| + 1 < 0, \quad -1 < x \le 2 \\ \sin[x], & 0 \le [x] \le \pi, \quad -2 \le x \le -1 \\ \sin(|x| + 1), & 0 \le |x| + 1 \le \pi, \quad -1 < x \le 2 \end{cases}$$

$$= \begin{cases} [x], & -2 \le x \le -1 \\ \sin(|x| + 1), & -1 < x \le 2 \end{cases}$$

Hence, the domain is $[-2, 2]$.

Also, for $-2 \le x \le -1$, $[x] = -2, -1$,

and for $-1 < x \le 2$, $|x| + 1 \in [1, 3]$

$\therefore$ $\sin(|x| + 1) \in [\sin 3, 1]$

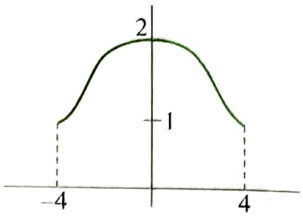
Hence, the range is $\{-2, -1\} \cup [\sin 3, 1]$.

Also, for $y \in [\sin 3, 1]$, $[y] = 0, 1$.

Hence, the number of integral points in the range is 4.

For Problems 28 and 29

28. (1) $-3 \le |x| \le 4$



$\Rightarrow$ $0 \le |x| \le 4$

$\Rightarrow$ $-4 \le x \le 4$

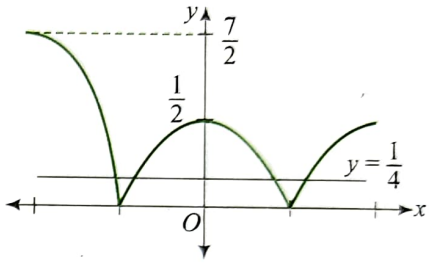
$\Rightarrow$ Domain of $g = [-4, 4]$

$\therefore$ $(b - a) = 8$

Also range of $g(x)$ = is $[1, 2]$, $(c + d) = 3$

$\therefore$ $(b - a + c + d) = 11$.

29. (4)



Range of $h(x) = \left[0, \frac{7}{2}\right]$

Number of solutions of $h(x) = \frac{1}{4}$ is $n = 4$.

$\therefore$ $(n + e + 2f) = 4 + 0 + 7 = 11$.

For Problems 30–32

30. (1), 31. (3), 32. (2)

$$f(x + y) = 2^x f(y) + 4^y f(x) \quad \ldots(i)$$

Interchanging x and y, we get

$$f(x + y) = 2^y f(x) + 4^x f(y) \quad \ldots(ii)$$

$\Rightarrow$ $2^x f(y) + 4^y f(x) = 2^y f(x) + 4^x f(y)$

$\Rightarrow$ $\frac{f(x)}{4^x - 2^x} = \frac{f(y)}{4^y - 2^y} = k$

$\Rightarrow$ $f(x) = k(4^x - 2^x)$

$f(1) = 2$

$\Rightarrow$ $k = 1$.

Hence, $f(x) = 4^x - 2^x$.

$f(4) = 4^4 - 2^4 = 240$

$f(x) = (2^x)^2 - 2^x = (2^x - 1/2)^2 - 1/4$

Thus $f(x)$ has least value $-1/4$

Also $4^x - 2^x = 2$

$\Rightarrow$ $(2^x)^2 - 2^x - 2 = 0$

$\Rightarrow$ $(2^x - 2)(2^x + 1) = 0$

$\Rightarrow$ $2^x - 2 = 0$

$\Rightarrow$ $x = 1$

Matrix Match Type

1. a → s; b → r; c → p; d → q.

$f(\tan x)$ is defined if $0 \le \tan x \le 1$

or $x \in \left[n\pi, n\pi + \frac{\pi}{4}\right]$, $n \in Z$

$f(\sin x)$ if defined if $0 \le \sin x \le 1$

or $x \in [2n\pi, (2n+1)\pi], n \in Z$

$f(\cos x)$ is defined if $0 \le \cos x \le 1$

or $x \in \left[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right], n \in Z$

$f(2\sin x)$ is defined if $0 \le 2\sin x \le 1$ or $0 \le \sin x \le 1/2$

or $\left[2n\pi, 2n\pi + \frac{\pi}{6}\right] \cup \left[2n\pi + \frac{5\pi}{6}, (2n+1)\pi\right], n \in Z$

2. **a → p; b → q; c → q, s; d → r, s.**

a. $f(x) = \{(\text{sgn } x)^{\text{sgn } x}\}^n$

$$= \begin{cases} \left[(1)^1\right]^n, & x > 0 \\ \left[(-1)^{-1}\right]^n, & x < 0 \end{cases} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

Hence, $f(x)$ is an odd function.

b. $f(x) = \frac{x}{e^x - 1} + \frac{x}{2} + 1$

or $f(-x) = \frac{-x}{e^{-x} - 1} - \frac{x}{2} + 1$

$$= \frac{xe^x}{e^x - 1} - \frac{x}{2} + 1$$

$$= \frac{xe^x - x + x}{e^x - 1} - \frac{x}{2} + 1$$

$$= x + \frac{x}{e^x - 1} - \frac{x}{2} + 1$$

$$= \frac{x}{e^x - 1} + \frac{x}{2} + 1$$

$$= f(x)$$

c. $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$

or $f(-x) = \begin{cases} 0, & \text{If } -x \text{ is rational} \\ 1, & \text{If } -x \text{ is irrational} \end{cases}$

$$= \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

$$= f(x)$$

d. $f(x) = \max\{\tan x, \cot x\}$

From the graph of function, it can be verified that $f(x)$ is neither odd nor even.

Also, $f(x + \pi) = \max\{\tan(x + \pi), \cot(x + \pi)\}$

$$= \max\{\tan x, \cot x\}$$

Hence, $f(x)$ is periodic with period π.

3. **a → p, r , s; b → q, s; c → q, s; d → p, r, s.**

We have $f(x) = \frac{x^2 - 6x + 5}{x^2 - 5x + 6} = \frac{(x-5)(x-1)}{(x-2)(x-3)}$

a → p, r, s.

If $-1 < x < 1$, then $f(x) = \frac{(-\text{ve})(-\text{ve})}{(-\text{ve})(-\text{ve})} = +\text{ve}$

$\therefore \quad f(x) > 0$

Also, $f(x) - 1 = \frac{-x-1}{x^2 - 5x + 6} = -\frac{(x+1)}{(x-2)(x-3)}$

For $-1 < x < 1$, $f(x) - 1 = \frac{-(+\text{ve})}{(-\text{ve})(-\text{ve})} = -\text{ve}$

or $f(x) - 1 < 0$ or $f(x) < 1$

$\therefore \quad 0 < f(x) < 1$

b → q, s.

If $1 < x < 2$, then $f(x) = \frac{(-\text{ve})(+\text{ve})}{(-\text{ve})(-\text{ve})} = -\text{ve}$

Therefore, $f(x) < 0$ and, so, $f(x) < 1$.

c → q, s.

If $3 < x < 5$, then

$$f(x) = \frac{(-\text{ve})(+\text{ve})}{(+\text{ve})(+\text{ve})} = -\text{ve}$$

Therefore, $f(x) < 0$ and, so, $f(x) < 1$.

d → p, r, s.

For $x > 5$, $f(x) > 0$. Also,

$$f(x) - 1 = \frac{-(x+1)}{(x-2)(x-5)} < 0 \text{ for } x > 5$$

or $f(x) < 1$,

$\therefore \quad 0 < f(x) < 1$

4. **a → r, s; b → p, q, r, s; c → s. d → p.**

a. $\tan^{-1}\left(\frac{2x}{1-x^2}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

or $2\tan^{-1} x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

or $\tan^{-1} x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

or $\tan^{-1} x \in (-1, 1)$

b. $f(x) = \sin^{-1}(\sin x)$ and $g(x) = \sin(\sin^{-1} x)$

$f(x)$ is defined if $\sin x \in [-1, 1]$ which is true for all $x \in R$.

But $g(x)$ is defined for only $x \in [-1, 1]$.

Hence, $f(x)$ and $g(x)$ are identical if $x \in [-1, 1]$.

c. $f(x) = \log_{x^2} 25$ and $g(x) = \log_x 5$

$f(x)$ is defined $\forall\, x \in R - \{0, 1\}$ and $g(x)$ is defined for $(0, \infty) - \{1\}$.

Hence, $f(x)$ and $g(x)$ are identical if $x \in (0, 1) \cup (1, \infty)$.

d. $f(x) = \sec^{-1} x + \text{cosec}^{-1} x$, $g(x) = \sin^{-1} x + \cos^{-1} x$

$f(x)$ has domain $R - (-1, 1)$ and $g(x)$ has domain $[-1, 1]$

Hence, both the functions are identical only if $x = -1, 1$.

5. **a → r, s; b → r, s; c → p, q; d → p, s.**

a. $f(x) = \cot^{-1}(2x - x^2 - 2)$

$$= \cot^{-1}(-1 - (x-1)^2)$$

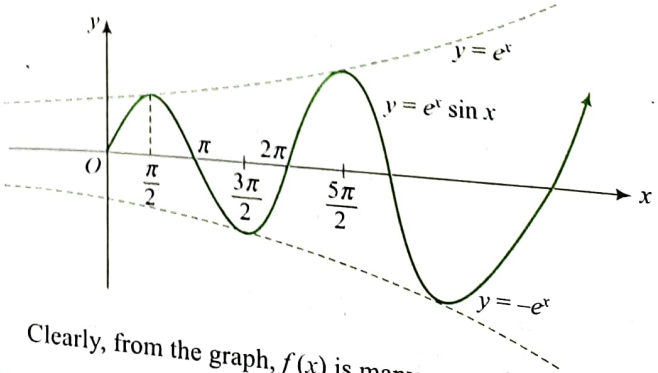
Now $-\infty < -1 - (x-1)^2 < -1$

$\therefore \quad \cot^{-1}(-1 - (x-1)^2) \in \left[\frac{3\pi}{4}, \pi\right)$

Also, $f(0) = f(2)$

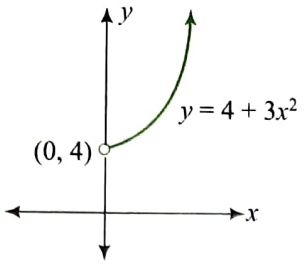
So, $f(x)$ is onto and many-one.

b.



Clearly, from the graph, $f(x)$ is many-one and onto.

c.



d. Clearly, $f(x)$ is inverse to itself.
So, $f(x)$ is bijective.

6. **a → q; b → q; c → s; d → p.**

a. $f(x + \pi/2) = \cos(|\sin(x + \pi/2)| - |\cos(x + \pi/2)|)$
$= \cos(|\cos x| - |-\sin x|)$
$= \cos(|\cos x| - |\sin x|)$
$= \cos(|\sin x| - |\cos x|)$
$= f(x)$

b. $f(x + \pi/2) = \cos[\tan(x + \pi/2) + \cot(x + \pi/2)] \cdot \cos[\tan(x + \pi/2) - \cot(x + \pi/2)]$
$= \cos[-\cot x - \tan x] \cdot \cos[-\cot x + \tan x]$
$= \cos(\tan x + \cot x) \cdot \cos(\tan x - \cot x)$
$= f(x)$

c. The period of $\sin^{-1}(\sin x)$ is 2π. The period of $e^{\tan x}$ is π.
Thus, the period of $f(x)$ is LCM $(2\pi, \pi) = 2\pi$.

d. The given function is
$f(x) = \sin^3 x \sin 3x$

$$= \left(\frac{3\sin x - \sin 3x}{4}\right)\sin 3x$$

$$= \frac{3}{8}(\cos 2x - \cos 4x) - \frac{1}{8}(1 - \cos 6x)$$

Thus, the period of $f(x)$ is π.

7. **a → s; b → r; c → s; d → p.**

a. $f(x) = e^{\cos^4 \pi x + x - [x] + \cos^2 \pi x}$

$\cos^2 \pi x$ and $\cos^4 \pi x$ have period 1.

$x - [x] = \{x\}$ has period 1

Then the period of $f(x)$ is 1.

b. $f(x) = \cos 2\pi\{2x\} + \sin 2\pi\{2x\}$

The period of $\{2x\}$ is 1/2. Then the period of $f(x)$ is 1/2.

c. Clearly, $\tan \pi[x] = 0 \forall x \in R$ and the period of $\sin 3\pi\{x\}$ is equal to 1.

d. $f(x) = 3x - [3x + a] - b = 3x + a - [3x + a] - (a + b)$
$= \{3x + a\} - (a + b)$

Thus, the period of $f(x)$ is 1/3.

8. **(2)**

a. $f(x) = \log_3(5 + 4x - x^2)$
$= \log_3(9 - (x - 2)^2)$

Now, $-\infty < 9 - (x - 2)^2 \le 9$

But for $f(x)$ to get defined, $0 < 9 - (x - 2)^2 \le 9$

or $-\infty < \log_3(9 - (x - 2)^2) \le \log_3 9$

or $-\infty < \log_3(9 - (x - 2)^2) \le 2$

Hence, the range is $(-\infty, 2]$.

b. $f(x) = \log_3(x^2 - 4x - 5)$
$= \log((x - 2)^2 - 9)$

For $f(x)$ to get defined, $0 < (x - 2)^2 - 9 < \infty$

or $\lim_{x \to 0} \log x < \log_e(x - 2)^2 - 9 < \lim_{x \to \infty} \log x$

or $-\infty < f(x) < \infty$

Hence, the range is R.

c. $f(x) = \log_3(x^2 - 4x + 5)$
$= \log_3((x - 2)^2 + 1)$
$(x - 2)^2 + 1 \in [1, \infty)$

or $\log_3(x^2 - 4x + 5) \in [0, \infty)$

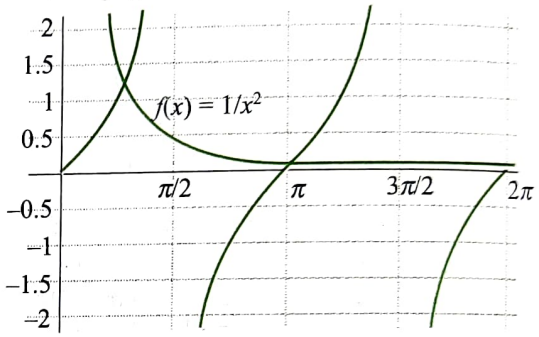
d. $f(x) = \log_3(4x - 5 - x^2)$
$= \log_3(-5 - (x^2 - 4x))$
$= \log_3(-1 - (x - 2)^2)$

Now, $-1 - (x - 2)^2 < 0$ for all x.
Hence, the function is not defined.

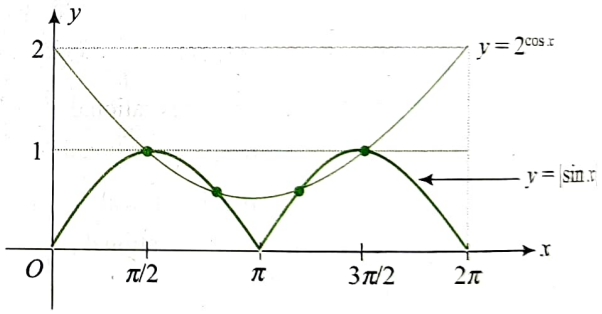
9. **(3)**

p. $y = \tan x = \frac{1}{x^2}$

From the graph, it is clear that it will have two real roots.



q. See the graphs of $y = 2^{\cos x}$ and $y = |\sin x|$. Two curves meet at four points for $x \in [0, 2\pi]$.



So, the equation $2^{\cos x} = |\sin x|$ has four solutions.

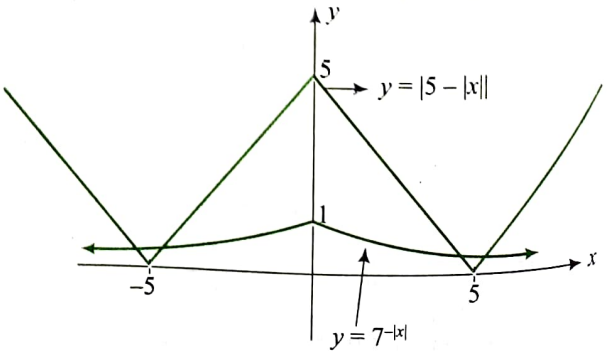
r. Given that $f(|x|) = 0$ has 8 real roots or $f(x) = 0$ has four positive roots.

Since $f(x)$ is a polynomial of degree 5, $f(x)$ cannot have even number of real roots.

Hence, $f(x)$ has all the five roots real and one root is negative.

s. $7^{|x|}(|5 - |x||) = 1$.
or $|5 - |x|| = 7^{-|x|}$

Draw the graph of $y = 7^{-|x|}$ and $y = |5 - |x||$.



From the graph, the number of roots is 4.

Numerical Value Type

1. **(3)** We have $f\left(\frac{2x-3}{x-2}\right) = 5x-2$

or $f^{-1}(5x-2) = \frac{2x-3}{x-2}$

Let $5x-2 = 13$. Then $x = 3$.

Hence, $f^{-1}(13) = \frac{2(3)-3}{3-2} = 3$.

2. **(5)** $f(x)$ and $f^{-1}(x)$ can only intersect on the line $y = x$ and, therefore, $y = x$ must be tangent at the common point of tangency. Hence,

$3x^2 - 7x + c = x$

or $3x^2 - 8x + c = 0$ (1)

This equation must have equal roots. Therefore,

$64 - 12c = 0$

or $c = \frac{64}{12} = \frac{16}{3}$

3. **(6)** $\log_{|x^2-1|}|x-3|$ is defined when $x - 3 \neq 0$

and $|x^2-1| > 0$ and $|x^2-1| \neq 1$.

$\Rightarrow$ $x \neq 3$ $x^2 - 1 \neq 0$ i.e, $x \neq 1$ or -1

and $|x^2-1| \neq 1$

$\Rightarrow$ $x \neq -\sqrt{2}$ or $\sqrt{2}$ and $x \neq 0$

$\therefore$ the points at which the function is not defined are $x = 0$, $1, -1$ $\sqrt{2}, -\sqrt{2}, 3$.

4. **(5)** $x! - (x-1)! \neq 0$ or $x \in I^+ - \{1\}$

$2^{\frac{\pi}{\tan^{-1}x}} > 4$ as $\tan^{-1}x < \frac{\pi}{2}$

or $\frac{(x-4)(x-10)}{(x-1)!(x-1)} < 0$

or $x \in \{5, 6, \ldots, 9\}$

5. **(0)** For $x \neq 0$, $f(x) = x\cos x + \log_e\left(\frac{1-x}{1+x}\right)$

$\therefore$ $f(-x) = (-x)\cos(-x) + \log_e\left(\frac{1+x}{1-x}\right)$

$= -x\cos x - \log_e\left(\frac{1-x}{1+x}\right)$

$= -f(x)$

For $x = 0$, we must have $f(-x) = -f(x)$

$\therefore$ $a = -a$

$\therefore$ $a = 0$

6. **(5)** $f(x) = \left|4\frac{(\sqrt{\cos x} - \sqrt{\sin x})(\sqrt{\cos x} + \sqrt{\sin x})}{(\cos x + \sin x)}\right|$ is defined only if

$\cos x > 0$, $\sin x > 0$

Therefore, x lies in first quadrant only.

$$f(x) = \left|4\frac{(\cos x - \sin x)}{(\cos x + \sin x)}\right| = \left|\ 4\tan\left(\frac{\pi}{4} - x\right)\ \right|$$

$$= \left|\ 4\tan\left(x - \frac{\pi}{4}\right)\ \right|$$

Now, $0 \le x \le \frac{\pi}{2}$

or $\frac{-\pi}{4} \le x - \frac{\pi}{4} \le \frac{\pi}{4}$

or $-1 \le \tan\left(x - \frac{\pi}{4}\right) \le 1$

or $-4 \le 4\tan\left(x - \frac{\pi}{4}\right) \le 4$

$0 \le \left|4\tan\left(x - \frac{\pi}{4}\right)\right| \le 4$

7. **(2)** $f(x) + f\left(\frac{1}{x}\right) = x^2 + \frac{1}{x}$

Replacing x by $\frac{1}{x}$, we get $f\left(\frac{1}{x}\right) + f(x) = \frac{1}{x^2} + x$

or $\frac{1}{x^2} + x = x^2 + \frac{1}{x}$

or $x - \frac{1}{x} = x^2 - \frac{1}{x^2}$

or $\left(x - \frac{1}{x}\right) = \left(x - \frac{1}{x}\right)\left(x + \frac{1}{x}\right)$

or $\left(x - \frac{1}{x}\right)\left(x + \frac{1}{x} - 1\right) = 0$

$x = \frac{1}{x}$; $x + \frac{1}{x} = 1$ (rejected)

Hence, $x = 1$ or -1.

8. **(5)** $f(x)$ must be linear function. Therefore,

$f(x) = px + q$

$\therefore$ $f(f(x)) = pf(x) + q = p(px + q) + q = p^2x + pq + q$

$\therefore$ $f(f(f(x))) = p(p^2x + pq + q) + q$

$= p^3x + p^2q + pq + q$

$= 8x + 21$ (Given)

$\therefore$ $p^3 = 8$ or $p = 2$

and $p^2q + pq + q = 21$ or $q = 3$

$\therefore$ $p + q = 5$

9. **(9)** Given $f(x + 2) = f(x) + f(2)$

Put $x = -1$. Then

$f(1) = f(-1) + f(2)$

or $f(1) = -f(1) + f(2)$ [as $f(x)$ is an odd function]

or $f(2) = 2f(1) = 6$

Now, put $x = 1$.

We have $f(3) = f(1) + f(2) = 3 + 6 = 9$.

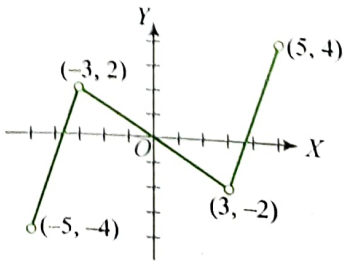
10. **(3)** $f(x) + f(-x) = 0$

Therefore, $f(x)$ is an odd function.

Since points $(-3, 2)$ and $(5, 4)$ lie on the curve,

$\therefore$ $(3, -2)$ and $(-5, -4)$ will also lie on the curve.

For minimum number of roots, graph of continuous function $f(x)$ is as follows:



From the above graph of $f(x)$, it is clear that equation $f(x) = 0$ has at least three real roots.

11. (3) $f(x) = \sqrt{\sin x + \cos x} + \sqrt{7x - x^2 - 6}$

$$= \sqrt{\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)} + \sqrt{(x-6)(1-x)}$$

Now, $f(x)$ is defined if $\sin\left(x + \frac{\pi}{4}\right) \geq 0$ and $(x-6)(1-x) \geq 0$

i.e., $0 \leq x + \frac{\pi}{4} \leq \pi$ or $2\pi \leq x + \frac{\pi}{4} \leq 3\pi$ and $1 \leq x \leq 6$

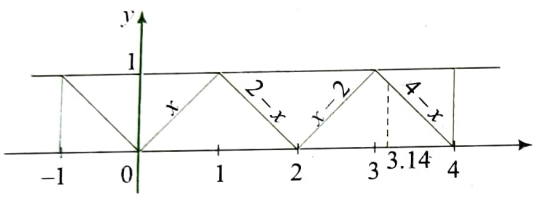
i.e., $-\frac{\pi}{4} \leq x \leq \frac{3\pi}{4}$ or $\frac{7\pi}{4} \leq x \leq \frac{11\pi}{4}$ and $1 \leq x \leq 6$

or $x \in \left[1, \frac{3\pi}{4}\right] \cup \left[\frac{7\pi}{4}, 6\right]$

Integral values of x are $x = 1$, 2, and 6.

12. (0.86) Since f is periodic with period 2 and $f(x) = x \ \forall x \in [0, 1]$, and $f(x)$ is even, there is symmetry about y-axis.

Therefore, graph of $f(x)$ is as shown.



$f(3.14) = 4 - 3.14 = 0.86$

13. (6) Let $x^2 = 4\cos^2\theta + \sin^2\theta$. Then

$(4 - x^2) = 3\sin^2\theta$ and $(x^2 - 1) = 3\cos^2\theta$

$\therefore \quad f(x) = \sqrt{3}\,|\sin\theta| + \sqrt{3}|\cos\theta|$

or $y_{\min} = \sqrt{3}$ and $y_{\max} = \sqrt{3}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = \sqrt{6}$

Hence, range of $f(x)$ is $[\sqrt{3}, \sqrt{6}]$.

Hence, maximum value of $(f(x))^2$ is 6.

14. (0) $g(x) = \frac{f(x) + f(-x)}{2}$

$$= \frac{1}{2}\left[\frac{x+1}{x^3+1} + \frac{1-x}{1-x^3}\right]$$

$$= \frac{1}{2}\left[\frac{1}{x^2 - x + 1} + \frac{1}{1 + x + x^2}\right]$$

$$= \frac{1}{2}\left[\frac{2(x^2+1)}{(x^2+1)^2 - x^2}\right]$$

$$= \frac{x^2+1}{x^4 + x^2 + 1}$$

$\therefore \quad g(0) = 1$

15. (0.25) $f(x) = [8x + 7] + |\tan 2\pi x + \cot 2\pi x| - 8x$

$= [8x] - 8x - 7 + |\tan 2\pi x + \cot 2\pi x|$

$= -\{8x\} + |\tan 2\pi x + \cot 2\pi x| + 7$

Period of $\{8x\}$ is 1/8.

Also, $|\tan 2\pi x + \cot 2\pi x| = \left|\frac{\sin 2\pi x}{\cos 2\pi x} + \frac{\cos 2\pi x}{\sin 2\pi x}\right|$

$$= \left|\frac{1}{\sin 2\pi x \cos 2\pi x}\right| = |2\operatorname{cosec} 4\pi x|$$

Now, period of $2\operatorname{cosec} 4\pi x$ is 1/2. Then period of $|2\operatorname{cosec} 4\pi x|$ is 1/4.

Therefore, period is L.C.M. of $\frac{1}{8}$ and $\frac{1}{4}$ which is $\frac{1}{4}$.

16. (–15) We have $f(2x) - f(2x)\ f\left(\frac{1}{2x}\right) + f(16x^2y) = f(-2) - f(4xy)$

Replacing y by $\frac{1}{8x^2}$, we get

$$f(2x) - f(2x)\left(\frac{1}{2x}\right) + f(2) = f(-2) - f\left(\frac{1}{2x}\right)$$

$$\therefore \quad f(2x) + f\left(\frac{1}{2x}\right) = f(2x)\ f\left(\frac{1}{2x}\right) \qquad [\text{As } f(x) \text{ is even}]$$

$\therefore \quad f(2x) = 1 \pm (2x)^n$

or $f(x) = 1 \pm x^n$

Now, $f(4) = 1 \pm 4^n = -255$ (Given)

Taking negative sign, we get $256 = 4^n$ or $n = 4$.

Hence, $f(x) = 1 - x^4$, which is an even function.

Therefore, $f(2) = -15$.

17. (1) $f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)$

$$= \sin^2 x + \frac{1}{4}\left(\sin x + \sqrt{3}\cos x\right)^2 + \frac{1}{2}\cos x\left(\cos x - \sqrt{3}\sin x\right)$$

$$= \frac{5}{4}\left(\sin^2 x + \cos^2 x\right) = \frac{5}{4}$$

$(gof)\,x = g[f(x)] = g(5/4) = 1$

18. (14) From E to F, we can define, in all, $2 \times 2 \times 2 \times 2 = 16$ functions (2 options for each element of E) out of which 2 are into, when all the elements of E either map to 1 or to 2. Therefore,

Number of onto functions $= 16 - 2 = 14$

19. (1) Given $f(f(x)) = -x + 1$

Replacing x by $f(x)$, we get

$f(f(f(x))) = -f(x) + 1$

$f(1 - x) = -f(x) + 1$

$f(x) + f(1 - x) = 1$

or $f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) = 1$

20. (105) $\because k \in$ odd

$f(k) = k + 3$

$f(f(k)) = \frac{k+3}{2}$

If $\frac{k+3}{2}$ is odd, then $27 = \frac{k+3}{2} + 3$ or $k = 45$ which is not possible.

Therefore, $\frac{k+3}{2}$ is even. Thus,

$$27 = f(f(f(k))) = f\left(\frac{k+3}{2}\right) = \frac{k+3}{4}$$

$\therefore \quad k = 105$

Verifying, $f(f(f(105))) = f(f(108)) = f(54) = 27$

$\therefore \quad k = 105$

21. (3) Clearly, fundamental period is $\frac{4\pi}{3}$. Then z lies in the third quadrant.

22. (2) $\log_a(x^2 - x + 2) > \log_a(-x^2 + 2x + 3)$

Putting $x = \frac{4}{9}$, $\log_a\left(\frac{142}{81}\right) > \log_a\left(\frac{299}{81}\right)$

$\because \frac{142}{81} < \frac{299}{81}$, we have $0 < a < 1$

$\therefore \log_a(x^2 - x + 2) > \log_a(-x^2 + 2x + 3)$

gives $0 < x^2 - x + 2 < -x^2 + 2x + 3$

or $x^2 - x + 2 > 0$ and $2x^2 - 3x - 1 < 0$

or $\frac{3 - \sqrt{17}}{4} < x < \frac{3 + \sqrt{17}}{4}$

4.(4) $(2^{2x} - 4 \cdot 2^x + 4) + 1 + ||b - 1| - 3| = |\sin y|$

or $(2^x - 2)^2 + 1 + ||b - 1| - 3| = |\sin y|$

LHS ≥ 1 and RHS ≤ 1

$\therefore 2^x = 2, |b - 1| - 3 = 0$

or $(b - 1) = \pm 3$

or $b = 4, -2$

5.(3) $f(3n) = f(f(f(n))) = 3f(n) \ \forall \ n \in N$

Putting $n = 1, f(3) = 3f(1)$

If $f(1) = 1$, then $f(f(1)) = f(1) = 1$. But $f(f(n)) = 3n$

or $f(f(1)) = 3$ giving $1 = 3$ which is absurd.

Therefore, $f(1) \neq 1$.

Thus, $3 = f(f(1)) > f(1) > 1$

So, $f(1) = 2$

$f(2) = f(f(1)) = 3$

6.(3) $\log_{1/3} \log_7(\sin x + a) > 0$

or $0 < \log_7(\sin x + a) < 1$

$1 < (\sin x + a) < 7 \quad \forall \ x \in R$ [a should be less than the minimum value of $7 - \sin x$ and a must be greater than the maximum value of $1 - \sin x$]

$\Rightarrow 1 - \sin x < a < 7 - \sin x \ \forall \ x \in R$

$2 < a < 6$

7.(9) $g(x) = \frac{1}{2}\tan^{-1}|x| + 1$ or $\text{sgn}(g(x)) = 1$

or $\sin^{23} x - \cos^{22} x = 1$

or $\sin^{23} x = 1 + \cos^{22} x$

which is possible if $\sin x = 1$ and $\cos x = 0$. Therefore,

$$x = 2n\pi + \frac{\pi}{2}, n \in Z$$

Hence, $-10\pi \leq 2n\pi + \frac{\pi}{2} \leq 8\pi$ or $-\frac{21}{4} \leq n \leq \frac{15}{4}$

or $-5 \leq n \leq 3$

Hence, number of values of $x = 9$

8.(28) $f(x) = \frac{ax^6 + bx^5 + cx^4 + dx^3 + 15x + 1}{x}$

$$= \underbrace{ax^7 + bx^5 + cx^3 + dx + \frac{1}{x}}_{\text{odd function}} + 15$$

Now, $f(x) + f(-x) = 30$

or $f(-5) = 30 - f(5) = 28$

9.(2) **Case-I:** When $\frac{p-2}{2} \leq 2$

$\Rightarrow p \leq 6$

$f(2) = 8$

$\Rightarrow 4 - 2(p - 2) + 3p - 2 = 8$

$\Rightarrow p = 2$

Case-II : When $\frac{p-2}{2} > 2 \Rightarrow p > 6$

$\frac{-D}{4a} = 8 \Rightarrow -\frac{(p-2)^2 - 4(3p-2)}{4} = 8$

$\Rightarrow p^2 - 4p + 4 - 12p + 8 = -32$

$\Rightarrow p^2 - 16p + 44 = 0$

$p = \frac{16 \pm \sqrt{256 - 176}}{2} = \frac{16 \pm 4\sqrt{5}}{2} = 8 \pm 2\sqrt{5}$

$\therefore p = 8 + 2\sqrt{5}$

29.(2) $f(x) = \sin\left(\frac{x}{2}\right)\cos\left[\frac{x}{2}\right] - \cos\left(\frac{x}{2}\right)\sin\left[\frac{x}{2}\right]$

$= \sin\left(\frac{x}{2} - \left[\frac{x}{2}\right]\right)$

$= \sin\left(\left\{\frac{x}{2}\right\}\right)$

$\therefore$ period of $f(x)$, $T = 2$

30.(5) We know that

$$[x] + [-x] = \begin{cases} -1, & x \notin I \\ 0, & x \in I \end{cases}$$

and RHS is either + 1 or –1

$\therefore$ above equation valid only when

$$\frac{\log_3(x-2)}{|\log_3(x-2)|} = -1$$

$\Rightarrow \log_3(x - 2) =$ negative

$\Rightarrow 0 < x - 2 < 1$

$\Rightarrow 2 < x < 3$

$\Rightarrow x \in (2, 3)$

$\Rightarrow a = 2, b = 3$

31. (5) Since $f(|x|) = 0$ has 8 roots

$\Rightarrow$ $f(x) = 0$ has 4 real positive roots but since $f(x)$ is polynomial of degree 5

$\Rightarrow$ fifth root is also real.

$\therefore$ Number of real roots of $f(x) = 0$ is 5

Archives

JEE Main

Single Correct Answer Type

1. (3) Given $f(x) = x^3 + 5x + 1$

Now $f'(x) = 3x^2 + 5 > 0, \forall \ x \in R$

Therefore, $f(x)$ is a strictly increasing function, and so it is one-one.

Clearly, $f(x)$ is a continuous function and also increasing on R.

$\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$

Hence, $f(x)$ takes every value between $-\infty$ and ∞.

Thus, $f(x)$ is an onto function.

2. (3) There is no information about co-domain therefore $f(x)$ is not necessarily onto.

Therefore statement 2 is false.

However for $x \geq -1$, $f(x) = (x + 1)^2 - 1$ is one-one.

Assume that $f(x)$ is onto.

Now roots of $f(x) = f^{-1}(x)$ lies on line $y = x$.

So we have to find roots of $f(x) = x$ or $(x+1)^2 - 1 = x$

$\therefore \quad x^2 + x = 0$

$\therefore \quad x = 0, -1$

Hence, statement 1 is true.

3. (4) xRy need not imply yRx.

$\frac{m}{n} S \frac{p}{q} \Leftrightarrow qm = pn$

$\frac{m}{n} S \frac{m}{n} \Rightarrow S$ is reflexive

$\frac{m}{n} S \frac{p}{q}$

$\Rightarrow \quad \frac{p}{q} S \frac{m}{n} \Rightarrow S$ is symmetric

$\frac{m}{n} S \frac{p}{q}, \frac{p}{q} S \frac{r}{s}$

$\Rightarrow \quad qm = pn, ps = rq \Rightarrow ms = rn$

$\Rightarrow \quad \frac{m}{n} S \frac{r}{s} \Rightarrow R$ is transitive.

Hence, S is an equivalence relation.

4. (4) Statement 1 is true.

We observe that

R is reflexive xRx as $x - x = 0$ is an integer $\forall\, x \in A$

Let $(x, y) \in A$

$\Rightarrow \quad y - x$ is an integer

$\Rightarrow \quad x - y$ is also an integer

$\Rightarrow \quad R$ is Symmetric

Let $(x, y) \in A$ and $(y, z) \in A$

$\Rightarrow \quad y - x$ is an integer and $z - y$ is an integer

$\Rightarrow \quad y - x + z - y$ is also integer

$\Rightarrow \quad z - x$ is an integer

$\Rightarrow \quad (x, z) \in A$

$\Rightarrow \quad R$ is Transitive

Because of the above properties A is an equivalence relation over R

Statement 2 is false as 'B' is not symmetric on R we observe that

$0\ B\ x$ as $0 = 0\ \forall\, x \in R$ but $(x, 0) \notin B$

5. (4) The given function f is well defined only when $|x| - x > 0$

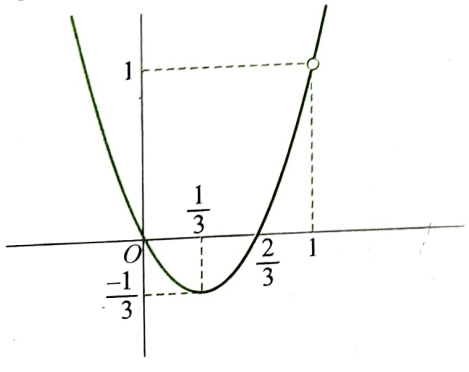
$\Rightarrow \quad x < 0$

$\therefore \quad$ Required domain is $(-\infty, 0)$

6. (1) $a^2 = 3t^2 - 2t$, where $t = \{x\} \in [0, 1)$

$\therefore \quad a^2 = 3t(t - 2/3)$

Graph of $f(t) = 3t(t - 2/3)$ is as shown in the following figure.



Clearly, from graph

$0 < a^2 < 1$

or $\quad a \in (-1, 0) \cup (0, 1)$

7. (2) $f(x) + 2f\left(\frac{1}{x}\right) = 3x \qquad (1)$

Replacing x by $\frac{1}{x}$, we get

$$f\left(\frac{1}{x}\right) + 2f(x) = \frac{3}{x} \qquad (2)$$

Solving (1) and (2) for $f(x)$, we get

$$-3f(x) = 3x - \frac{6}{x} \quad \text{or} \quad f(x) = \frac{2}{x} - x$$

Now given $f(x) = f(-x)$

$\therefore \quad \frac{2}{x} - x = -\frac{2}{x} + x$

$\therefore \quad \frac{4}{x} = 2x$

$\therefore \quad \frac{2}{x} = x$

$\therefore \quad x = \pm\sqrt{2}$

Hence set S contains two elements.

8. (4) $f : R \to \left[-\frac{1}{2}, \frac{1}{2}\right], \quad f(x) = \frac{x}{1+x^2} \quad \forall x \in R$

For range of the function, $y = \frac{x}{1+x^2}$

$\Rightarrow \quad yx^2 - x + y = 0$

Since x is real, $D \geq 0$

$\Rightarrow \quad 1 - 4y^2 \geq 0$

$\Rightarrow \quad y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

Also, for $y = \frac{1}{4}, \frac{1}{4} = \frac{x}{1+x^2}$

or $x^2 - 4x + 1 = 0$, which has two distinct roots.

Thus, $f(x)$ is onto but not injective.

9. (2) $f(x) = ax^2 + bx + c$

$f(1) = a + b + c = 3$ (given)

Now $f(x + y) = f(x) + f(y) + xy$

Putting $y = 1$, we get

$f(x + 1) = f(x) + f(1) + x = f(x) + x + 3$

Putting $x = 1$, we get $f(2) = f(1) + 1 + 3 = 7$

Putting $x = 2$, we get $f(3) = f(2) + 2 + 3 = 12$ and so on.

Now $S_{10} = 3 + 7 + 12 + \ldots + t_n$ ...(i)

$S_{10} = \quad 3 + \quad 7 + \ldots + t_{n-1} + t_n$ ...(ii)

Subtracting (ii) from (i), we get

$t_n = 3 + 4 + 5 + \ldots$ upto n terms

$$\therefore \quad t_n = \frac{(n+2)(n+3)}{2} - 3 = \frac{n^2 + 5n}{2}$$

$$S_n = \sum t_n = \sum \frac{n^2 + 5n}{2}$$

$$= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{5n(n+1)}{2}\right]$$

$$= \frac{n(n+1)(n+8)}{6}$$

$\therefore \quad S_{10} = 330$

IIT ADVANCED
Multiple Correct Answers Type

1. **(1), (2), (3)**

$f(x) = (\log(\sec x + \tan x))^3 \ \forall\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore\ f(-x) = (\log(\sec x - \tan x))^3$

$$= \left(\log \frac{1}{\sec x + \tan x}\right)^3$$

$$= \left(-\log(\sec x + \tan x)\right)^3$$

$$= -\left(\log(\sec x + \tan x)\right)^3$$

$$= -f(x)$$

Hence, $f(x)$ is odd function.

Let $g(x) = \sec x + \tan x \ \forall\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

So, $g'(x) = \sec x\,(\sec x + \tan x) > 0 \ \forall\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow$ $g(x)$ is one-one function

Hence, $(\log_e(g(x)))^3$ is also one-one function.

And $g(x) \in (0, \infty) \ \forall\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$g(x) = \sec x + \tan x$$

$$= \frac{1+\sin x}{\cos x} = \frac{1-\cos\left(\frac{\pi}{2}+x\right)}{\sin\left(\frac{\pi}{2}+x\right)} = \tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$$

Now

$$x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

$$\Rightarrow \frac{\pi}{4}+\frac{x}{2} \in \left(0, \frac{\pi}{2}\right)$$

$$\Rightarrow \tan\left(\frac{\pi}{4}+\frac{x}{2}\right) \in (0, \infty)$$

$\Rightarrow$ $\log(g(x)) \in R$

Hence, $f(x)$ is an onto function.

2. **(1), (2), (3)**

$$f(x) = \sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)$$

We know that $-1 \le \sin x \le 1$

$$\Rightarrow -\frac{\pi}{2} \le \frac{\pi}{2}\sin x \le \frac{\pi}{2}$$

$$\Rightarrow -1 \le \sin\left(\frac{\pi}{2}\sin x\right) \le 1$$

$$\Rightarrow -\frac{\pi}{6} \le \frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right) \le \frac{\pi}{6}$$

$$\Rightarrow -\frac{1}{2} \le \sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right) \le \frac{1}{2}$$

Now, $fog(x) = \sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin\left(\frac{\pi}{2}\sin x\right)\right)\right)$

$$-1 \le \sin\left(\frac{\pi}{2}\sin x\right) \le 1$$

$$\Rightarrow \frac{-\pi}{2} \le \frac{\pi}{2}\left(\sin\left(\frac{\pi}{2}\sin x\right)\right) \le \frac{\pi}{2}$$

$$\Rightarrow -1 \le \sin\left(\frac{\pi}{2}\left(\sin\left(\frac{\pi}{2}\sin x\right)\right)\right) \le 1$$

$$\Rightarrow \frac{-\pi}{6} \le \frac{\pi}{6}\sin\left(\frac{\pi}{2}\left(\sin\left(\frac{\pi}{2}\sin x\right)\right)\right) \le \frac{\pi}{6}$$

$$\Rightarrow \frac{-1}{2} \le f(x) \le \frac{1}{2}$$

Thus, range of fog is also $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

Now, $\lim\limits_{x\to 0} \dfrac{\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)}{\frac{\pi}{2}\sin x}$

$$= \lim_{x\to 0} \frac{2}{\pi} \frac{\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)}{\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)} \times \frac{\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)}{\frac{\sin x}{x} \times x}$$

$$= \lim_{x\to 0} \frac{2}{\pi} \times \frac{\pi}{6} \times \frac{\sin\left(\frac{\pi}{2}\sin x\right)}{\frac{\pi}{2}\sin x} \times \frac{\frac{\pi}{2}\sin x}{x}$$

$$= \frac{1}{3} \times \frac{\pi}{2} = \frac{\pi}{6}$$

$gof(x) \in \left[-\frac{\pi}{2}\sin\left(\frac{1}{2}\right), \frac{\pi}{2}\sin\left(\frac{1}{2}\right)\right]$

$\Rightarrow$ $gof(x) \ne 1$

Matrix Match Type

1. **(1)** $E_1 : \dfrac{x}{x-1} > 0$

$\Rightarrow$ $x \in (-\infty, 0) \cup (1, \infty)$

Also, $E_2 : -1 \le \log_e\left(\frac{x}{x-1}\right) \le 1$

$$\Rightarrow \frac{1}{e} \le \frac{x}{x-1} \le e$$

$$\Rightarrow \frac{1}{e} \le 1 + \frac{1}{x-1} \le e$$

$$\Rightarrow \frac{1}{e} - 1 \le \frac{1}{x-1} \le e - 1$$

$$\Rightarrow (x-1) \in \left(-\infty, \frac{e}{1-e}\right] \cup \left[\frac{1}{e-1}, \infty\right)$$

$$\Rightarrow x \in \left(-\infty, \frac{1}{1-e}\right] \cup \left[\frac{e}{e-1}, \infty\right)$$

Now, $\dfrac{x}{x-1} \in (0, \infty) - \{1\} \ \forall\ x \in E_1$

$$\Rightarrow \log_e\left(\frac{x}{x-1}\right) \in (-\infty, \infty) - \{0\}$$

$$\Rightarrow \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$$

$$\left(\text{considering } \log_e\left(\frac{x}{x-1}\right) \in [-1, 1] - \{0\}\right)$$

Concept Application Exercises

Exercise 2.1

1. $L = \lim\limits_{x\to -2^+} \dfrac{x^2-1}{2x+4}$

$= \lim\limits_{h\to 0} \dfrac{(-2+h)^2-1}{2(-2+h)+4}$

$= \lim\limits_{h\to 0} \dfrac{3-4h+h^2}{2h}$

For $h \to 0$, Numerator $\to 3$ and Denominator $\to 0$

$\therefore \quad L \to \infty$

2. $L = \lim\limits_{x\to 2^+} \dfrac{[x-2]}{\log(x-2)}$

When $x \to 2^+$, $x-2 \to 0^+$

or $\quad [x-2] = 0$

Also, $\log(x-2) \to \log 0^+ \to -\infty$

Thus, $L = \dfrac{\text{exact } 0}{-\infty} = 0$

3. When $x \to 0^+$ or 0^-, $\cos x \to 1^-$

or $\quad [\cos x] = 0$ for $x \to 0$

$\therefore \quad \lim\limits_{x\to 0} \dfrac{\sin[\cos x]}{1+[\cos x]} = \dfrac{\sin 0}{1+0} = 0$

4. L.H.L. of $f(x)$ at $x = 0$ is

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{-h-|-h|}{(-h)}$$

$$= \lim_{h\to 0} \frac{-h-h}{-h} = \lim_{h\to 0} \frac{-2h}{-h} = \lim_{h\to 0} 2 = 2$$

R.H.L. of $f(x)$ at $x = 0$ is

$$\lim_{h\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \frac{h-|h|}{h}$$

$$= \lim_{h\to 0} \frac{h-h}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0$$

Clearly, $\lim\limits_{x\to 0^-} f(x) \neq \lim\limits_{x\to 0^+} f(x)$

So, $\lim\limits_{x\to 0} f(x)$ does not exist.

5. Let $f(x) = \dfrac{e^{1/x}-1}{e^{1/x}+1}$

L.H.L. of $f(x)$ at $x = 0$ is

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} (0-h) = \lim_{h\to 0} \frac{e^{-1/h}-1}{e^{-1/h}+1}$$

$$= \lim_{h\to 0} \left(\frac{\frac{1}{e^{1/h}}-1}{\frac{1}{e^{1/h}}+1} \right) = -1$$

$$\left[\because h \to 0 \Rightarrow \frac{1}{h} \to \infty \Rightarrow e^{1/h} \to \infty \Rightarrow \frac{1}{e^{1/h}} \to 0 \right]$$

R.H.L. of $f(x)$ at $x = 0$ is

$$\lim_{x\to 0} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \frac{e^{1/h}-1}{e^{1/h}+1}$$

$$= \lim_{h\to 0} \left(\frac{1-\frac{1}{e^{1/h}}}{1+\frac{1}{e^{h}}} \right) \qquad \text{[Dividing Nr and Dr by } e^{1/h}\text{]}$$

$$= \frac{1-0}{1+0} = 1$$

Clearly, $\lim\limits_{x\to 0^-} f(x) \neq \lim\limits_{x\to 0^+} f(x)$

Hence, $\lim\limits_{x\to 0} f(x)$ does not exist.

6. $\lim\limits_{x\to 0^-} \dfrac{3x+|x|}{7x-5|x|} = \lim\limits_{x\to 0^-} \dfrac{3x-x}{7x+5x} = \dfrac{1}{6}$

and $\lim\limits_{x\to 0^+} \dfrac{3x+|x|}{7x-5|x|} = \lim\limits_{x\to 0} \dfrac{3x+x}{7x-5x} = 2$

Hence, the limit does not exist.

7. We have $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0} x = 0$

and $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0} x^2 = 0$.

Hence, $\lim\limits_{x\to 0} f(x)$ is equal to 0.

8. **a.** $\lim\limits_{x\to 1^+} f(x) = 3$ and $\lim\limits_{x\to 1^-} f(x) = 2$. Thus, $\lim\limits_{x\to 1} f(x)$ does not exist.

b. $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^-} f(x) = 3$. Thus, $\lim\limits_{x\to 2} f(x)$ exists.

c. $\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^-} f(x) = 3$. Thus, $\lim\limits_{x\to 3} f(x)$ exists.

d. $\lim\limits_{x\to 1.99^+} f(x) = \lim\limits_{x\to 1.99^-} f(x) = 3$. Thus, $\lim\limits_{x\to 1.99} f(x)$ exists.

9. $\lim\limits_{x\to 0^+} \dfrac{\tan(\text{sgn}(x))}{\text{sgn}(x)} = \dfrac{\tan 1}{1} = \tan 1$

$\lim\limits_{x\to 0^-} \dfrac{\tan(\text{sgn}(x))}{\text{sgn}(x)} = \dfrac{\tan(-1)}{(-1)} = \tan 1$

$\therefore \quad \lim\limits_{x\to 0} \dfrac{\tan(\text{sgn}(x))}{\text{sgn}(x)} = \tan 1$

10. $\lim\limits_{x\to 0^+} g\{f(x)\} = g(f(0^+)) = g((\sin 0^+)) = g(0^+) = (0)^2 + 1 = 1$

$\lim\limits_{x\to 0^-} g\{f(x)\} = g(f(0^-)) = g((\sin 0^-)) = g(0^-) = (0)^2 + 1 = 1$

Hence, $\lim\limits_{x\to 0} g\{f(x)\} = 1$

Exercise 2.2

1. We have $|f(x)| \le x^2$

$\therefore \quad \left| \dfrac{f(x)}{x} \right| \le |x|$

$\Rightarrow \quad \lim\limits_{x\to 0} \left| \dfrac{f(x)}{x} \right| \le \lim\limits_{x\to 0} |x|$

$\Rightarrow \quad \left| \lim\limits_{x\to 0} \dfrac{f(x)}{x} \right| \le 0$

$\Rightarrow \left|\lim_{x\to 0}\frac{f(x)}{x}\right| = 0$

$\Rightarrow \lim_{x\to 0}\frac{f(x)}{x} = 0$

$f(x) = \text{sgn}(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \\ 0, & x = 0 \end{cases}$

$\therefore \lim_{x\to 0^+} f(x) = 1$ and $\lim_{x\to 0^-} f(x) = -1$

Now $f(x)\times g(x) = \begin{cases} x^3, & x > 0 \\ -x^3, & x < 0 \\ 0, & x = 0 \end{cases}$

$\therefore \lim_{x\to 0^+} f(x)\cdot g(x) = \lim_{x\to 0^+} x^3 = 0$

and $\lim_{x\to 0^-} f(x)\cdot g(x) = \lim_{x\to 0^+}(-x^3) = 0$

Thus $\lim_{x\to 0} f(x)\cdot g(x)$ exists.

3. We have $f(x) = \begin{cases} \frac{\sin[x]}{[x]} & \text{if} \quad x \in (-\infty, 0)\cup[1, \infty) \\ 0 & \text{if} \quad x \in [0, 1) \end{cases}$

$\therefore \lim_{x\to 0-} f(x) = \lim_{h\to 0}\frac{\sin[-h]}{[-h]} = \lim_{h\to 0}\frac{\sin(-1)}{(-1)} = \sin 1$

and $\lim_{x\to 0+} f(x) = \lim_{h\to 0} 0 = 0$

Thus. $\lim_{x\to 0-} f(x) \neq \lim_{x\to 0+} f(x)$

$\therefore \lim_{x\to 0} f(x)$ does not exist.

4. $\lim_{x\to 0^-}(\sin x)^{\frac{1}{x}} = \left(\lim_{x\to 0^+}\sin x\right)^{\lim_{x\to 0^+}\frac{1}{x}} = 0^\infty = 0$

5. Let $\lim_{n\to\infty} b_n = b$

Now $b_{n+1} = \frac{1}{3}\left(2b_n + \frac{125}{b_n^2}\right)$

$\Rightarrow \lim_{n\to\infty} b_{n+1} = \frac{1}{3}\left(2\lim_{n\to\infty} b_n + \frac{125}{\lim_{n\to\infty} b_n^2}\right)$

$\Rightarrow b = \frac{1}{3}\left(2b + \frac{125}{b^2}\right)$ $\left(\because \lim_{n\to\infty} b_n = \lim_{n\to\infty} b_{n+1} = b\right)$

$\Rightarrow \frac{b}{3} = \frac{125}{3b^2}$

$\Rightarrow b^3 = 125$

$\Rightarrow b = 5$

6. $\frac{\cos(2x-4)-33}{2} < f(x) < \frac{x^2|4x-8|}{x-2}$

or $\lim_{x\to 2^-}\frac{\cos(2x-4)-33}{2} < \lim_{x\to 2^-} f(x) < \lim_{x\to 2^-}\frac{x^2|4x-8|}{x-2}$

or $-16 < \lim_{x\to 2^-} f(x) < \lim_{x\to 2^-}\frac{x^2(8-4x)}{x-2}$

or $-16 < \lim_{x\to 2^-} f(x) < -16$

or $\lim_{x\to 2^-} f(x) = -16$ (by sandwich theorem)

7. $\frac{x^2+x-2}{x+3} \le \frac{f(x)}{x^2} \le \frac{x^2+2x-1}{x+3}$

or $\lim_{x\to -1}\frac{x^2+x-2}{x+3} \le \lim_{x\to -1}\frac{f(x)}{x^2} \le \lim_{x\to -1}\frac{x^2+2x-1}{x+3}$

or $-1 \le \lim_{x\to -1}\frac{f(x)}{x^2} \le -1$

or $\lim_{x\to -1}\frac{f(x)}{x^2} = -1$ (Using Sandwich theorem)

or $\frac{\lim_{x\to -1} f(x)}{\lim_{x\to -1} x^2} = -1$

or $\lim_{x\to -1} f(x) = -\lim_{x\to -1} x^2 = -1$

Exercise 2.3

1. $\lim_{x\to 1}\frac{(2x-3)\left(\sqrt{x}-1\right)\times\left(\sqrt{x}+1\right)}{(x-1)(2x+3)\times\left(\sqrt{x}+1\right)} = \frac{-1}{5\times 2} = \frac{-1}{10}$

2. We have $L = \lim_{x\to 1}\frac{x^4-3x^3+2}{x^3-5x^2+3x+1}$

Clearly $x = 1$ is zero of Nr. and Dr..

So, factorizing, we get

$$L = \lim_{x\to 1}\frac{(x-1)(x^3-2x^2-2x-2)}{(x-1)(x^2-4x-1)}$$

$$= \lim_{x\to 1}\frac{x^3-2x^2-2x-2}{x^2-4x-1} = \frac{1-2-2-2}{1-4-1} = \frac{5}{4}$$

3. $\lim_{x\to\frac{\pi}{4}}\frac{1-\sin 2x}{1+\cos 4x}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to\frac{\pi}{4}}\frac{(\sin x-\cos x)^2}{2\cos^2 2x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to\frac{\pi}{4}}\frac{(\sin x-\cos x)^2}{2(\cos^2 x-\sin^2 x)^2} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to\frac{\pi}{4}}\frac{1}{2(\cos x+\sin x)^2} = \frac{1}{4}$$

4. $\lim_{x\to\pi/4}\frac{1-\cot^3 x}{2-\cot x-\cot^3 x}$

$$= \lim_{x\to\frac{\pi}{4}}\frac{(1-\cot x)}{(1-\cot x)}\frac{(1+\cot x+\cot^2 x)}{(\cot^2 x+\cot x+2)}$$

$$= \frac{1+1+1}{2+1+1} = \frac{3}{4}$$

5. $\lim_{x\to a}\frac{\sqrt{3x-a}-\sqrt{x+a}}{x-a}$

$$= \lim_{x\to a}\frac{\sqrt{3x-a}-\sqrt{x+a}}{(x-a)}\times\frac{\sqrt{3x-a}+\sqrt{x+a}}{\sqrt{3x-a}+\sqrt{x+a}}$$

$$= \lim_{x\to a}\frac{(3x-a)-(x+a)}{(x-a)\left(\sqrt{3x-a}+\sqrt{x+a}\right)}$$

$$= \frac{2}{2\sqrt{2a}} = \frac{1}{\sqrt{2a}}$$

6. We have $\lim_{x\to 0}\dfrac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$

$$= \lim_{x\to 0}\frac{2-(1+\cos x)}{\sin^2 x}\times\frac{1}{\sqrt{2}+\sqrt{1+\cos x}}$$

$$= \lim_{x\to 0}\frac{1-\cos x}{(1+\cos x)(1-\cos x)}\times\lim_{x\to 0}\frac{1}{\sqrt{2}+\sqrt{1+\cos x}}$$

$$= \lim_{x\to 0}\frac{1}{(1+\cos x)}\times\frac{1}{2\sqrt{2}}$$

$$= \frac{1}{4\sqrt{2}}$$

7. Rationalizing Nr., we get

$$= \lim_{x\to\sqrt{10}}\frac{7+2x-(\sqrt{5}+\sqrt{2})^2}{x^2-10}\cdot\frac{1}{\sqrt{7+2x}+(\sqrt{5}+\sqrt{2})}$$

$$= \frac{1}{\sqrt{7+2\sqrt{5}}+(\sqrt{5}+\sqrt{2})}\lim_{x\to\sqrt{10}}\frac{2(x-\sqrt{10})}{(x-\sqrt{10})(x+\sqrt{10})}$$

$$= \frac{1}{(\sqrt{5}+\sqrt{2})+(\sqrt{5}+\sqrt{2})}\cdot\frac{1}{\sqrt{10}}$$

$$= \frac{1}{2\sqrt{10}(\sqrt{5}+\sqrt{2})}$$

8. When n is even:

Given series

$$1^2-2^2+3^2-4^2+\cdots-n^2$$
$$= (1^2-2^2)+(3^2-4^2)+\cdots[(n-1)^2-n^2]$$
$$= -(1+2+3+4+\cdots+n)$$
$$= -\frac{n(n+1)}{2}$$

$\therefore$ Given $L = \lim_{n\to\infty}-\dfrac{n(n+1)}{2n^2} = -\dfrac{1}{2}$

When n is odd:

Given series

$$1^2-2^2+3^2-4^2+\cdots+n^2$$
$$= -1(1+2+3+\cdots+(n-1))+n^2$$
$$= -\frac{n(n-1)}{2}+n^2$$
$$= \frac{n(n+1)}{2}$$

$\therefore$ Given $L = \lim_{n\to\infty}\dfrac{n(n+1)}{2n^2} = \dfrac{1}{2}$

9. $\lim_{x\to\infty}\left[\sqrt{a^2x^2+ax+1}-\sqrt{a^2x^2+1}\right]$

$$= \lim_{x\to\infty}\frac{ax}{\sqrt{a^2x^2+ax+1}+\sqrt{a^2x^2+1}}$$ (Rationalizing)

$$= \lim_{x\to\infty}\frac{a}{\sqrt{a^2+\frac{a}{x}+\frac{1}{x^2}}+\sqrt{a^2+\frac{1}{x^2}}}$$

(Dividing numerator and denominator by x)

$$= \frac{a}{2a} = \frac{1}{2}$$

10. $\lim_{n\to\infty}\dfrac{1}{n^2}([1\cdot x]+[2\cdot x]+[3\cdot x]+\cdots+[n\cdot x])$

$$= \lim_{n\to\infty}\left\{\frac{\sum_{r=1}^{n}[rx]}{n^2}\right\}$$

$$= \lim_{n\to\infty}\left(\frac{\sum_{r=1}^{n}(rx-\{rx\})}{n^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{x.\frac{n(n+1)}{2}}{n^2}-\sum_{r=1}^{n}\frac{\{rx\}}{n^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{x.\left(1+\frac{1}{n}\right)}{2}-\sum_{r=1}^{n}\frac{\{rx\}}{n^2}\right)$$

$$= \frac{x}{2}-0 = \frac{x}{2}$$

11. $\lim_{x\to\infty}x^3\left\{\sqrt{x^2+\sqrt{1+x^4}}-x\sqrt{2}\right\}$

$$= \lim_{x\to\infty}x^3\left[\frac{x^2+\sqrt{1+x^4}-2x^2}{\sqrt{x^2+\sqrt{1+x^4}}+x\sqrt{2}}\right]$$

$$= \lim_{x\to\infty}\frac{x^3}{\left(\sqrt{x^2+\sqrt{1+x^4}}+x\sqrt{2}\right)}\frac{(1+x^4-x^4)}{\left(\sqrt{1+x^4}+x^2\right)}$$

$$= \lim_{x\to\infty}\frac{x^3}{\left(\sqrt{x^2+\sqrt{1+x^4}}+x\sqrt{2}\right)\left(\sqrt{1+x^4}+x^2\right)}$$

$$= \lim_{x\to\infty}\frac{x^3}{x^3\left(\sqrt{1+\sqrt{\frac{1}{x^4}+1}}+\sqrt{2}\right)\left(\sqrt{1+\frac{1}{x^4}}+1\right)}$$

$$= \frac{1}{2\sqrt{2}}\cdot\frac{1}{2} = \frac{1}{4\sqrt{2}}$$

12. $\lim_{x\to\infty}\dfrac{7x^2+1}{5x^2-1} = \lim_{x\to\infty}\dfrac{7+\frac{1}{x^2}}{5-\frac{1}{x^2}} = \dfrac{7}{5}$

$$\lim_{x\to\infty}\frac{x^5}{1-x^3} = \lim_{x\to\infty}\frac{x^2}{\frac{1}{x^3}-1} = -\infty$$

$$\therefore \lim_{x\to\infty}\left(\frac{7x^2+1}{5x^2-1}\right)^{\frac{x^5}{1-x^3}} = \lim_{x\to\infty}\left(\frac{7x^2+1}{5x^2-1}\right)^{\lim_{x\to\infty}\frac{x^5}{1-x^3}}$$
$$= \left(\frac{7}{5}\right)^{-\infty} = 0$$

$$L = \lim_{n\to\infty} \cos\left(\pi\sqrt{n^2+n}\right)$$

$$= \lim_{n\to\infty} (-1)^n \cos\left(n\pi - \pi\sqrt{n^2+n}\right)$$

$$= \lim_{n\to\infty} (-1)^n \cos\left(\pi\left(n-\sqrt{n^2+n}\right)\right)$$

$$= \lim_{n\to\infty} (-1)^n \cos\left(\frac{-n\pi}{n+\sqrt{n^2+n}}\right)$$

$$= (-1)^n \lim_{n\to\infty} \cos\left(\frac{n\pi}{n+\sqrt{n^2+n}}\right)$$

$$= (-1)^n \lim_{n\to\infty} \cos\left(\frac{\pi}{1+\sqrt{1+\frac{1}{n}}}\right)$$

$$= (-1)^n \cos\frac{\pi}{2}$$

$$= 0$$

14. $$\lim_{x\to 1} \frac{\left[\sum_{k=1}^{100} x^k\right] - 100}{(x-1)}$$

$$= \lim_{x\to 1} \frac{(x+x^2+x^3+\cdots+x^{100})-100}{(x-1)}$$

$$= \lim_{x\to 1} \frac{(x-1)+(x^2-1)+(x^3-1)+\cdots+(x^{100}-1)}{(x-1)}$$

$$= \lim_{x\to 1} \left\{\left(\frac{x-1}{x-1}\right)+\left(\frac{x^2-1}{x-1}\right)+\left(\frac{x^3-1}{x-1}\right)+\cdots+\left(\frac{x^{100}-1}{x-1}\right)\right\}$$

$$= \lim_{x\to 1}\left(\frac{x-1}{x-1}\right)+\lim_{x\to 1}\left(\frac{x^2-1}{x-1}\right)+\lim_{x\to 1}\left(\frac{x^3-1}{x-1}\right)+\cdots$$

$$\lim_{x\to 1}\left(\frac{x^{100}-1}{x-1}\right) = 1+2+3+\cdots+100$$

$$= \frac{100\times 101}{2} = 5050$$

15. $$\lim_{h\to 0}\left[\frac{2-\sqrt[3]{8+h}}{2h\sqrt[3]{8+h}}\right] = -\lim_{h\to 0}\left[\frac{\left(1+\frac{h}{8}\right)^{1/3}-1}{8.\frac{h}{8}\sqrt[3]{8+h}}\right]$$

$$= -\frac{1}{48}$$

Exercise 2.4

1. $$\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5} = \lim_{x\to 0} \frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots-x+\frac{x^3}{6}}{x^5}$$

$$= \lim_{x\to 0}\left(\frac{1}{5!}-\frac{x^2}{7!}+\cdots\right) = \frac{1}{120}$$

2. $$\lim_{x\to 0} \frac{e^x-1-x}{x^2} = \lim_{x\to 0} \frac{\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)-1-x}{x^2} = \frac{1}{2}$$

3. $$\lim_{x\to 0} \frac{e^x-e^{-x}-2x}{x-\sin x}$$

$$= \lim_{x\to 0} \frac{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)-\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots\right)-2x}{x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\ldots\right)}$$

$$= \lim_{x\to 0} \frac{2.\frac{x^3}{6}+2.\frac{x^5}{5!}+\ldots}{\frac{x^3}{6}-\frac{x^5}{5!}\ldots}$$

$$= \lim_{x\to 0} \frac{\frac{1}{3}+\frac{1}{60}x^2+\ldots}{\frac{1}{6}-\frac{1}{120}x^2+\ldots} = \frac{1/3}{1/6} = 2$$

4. $$\lim_{x\to 0} \frac{x^2}{e^{ax}-bx-1} = 2$$

$$\Rightarrow \lim_{x\to 0} \frac{x^2}{\left(1+\frac{ax}{1!}+\frac{a^2x^2}{2!}+\ldots\right)-bx-1} = 2$$

$$\Rightarrow \lim_{x\to 0} \frac{x}{(a-b)+\frac{a^2x}{2}+\ldots} = 2$$

Limit exists if $a-b=0$ or $a=b$

So, we have $\lim_{x\to 0} \frac{x}{\frac{a^2x}{2}} = 2$

$$\therefore \quad \frac{2}{a^2} = 2$$

$$\Rightarrow \quad a = \pm 1$$

If $a=1$, then $b=1$ and if $a=-1$, then $b=-1$

5. $$\lim_{x\to 0} \frac{x(1+a\cos x)-b\sin x}{x^3} = 1 \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$\Rightarrow \lim_{x\to 0} \frac{x+ax\cos x-b\sin x}{x^3} = 1$$

$$\Rightarrow \lim_{x\to 0} \frac{x+ax\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots\right)-b\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots\right)}{x^3} = 1$$

$$\Rightarrow \lim_{x\to 0} \frac{(1+a-b)x+\left(-\frac{a}{2}+\frac{b}{6}\right)x^3}{x^3} = 1$$

$$\Rightarrow \quad 1+a-b=0 \qquad (1)$$

and $$-\frac{a}{2}+\frac{b}{6} = 1 \qquad (2)$$

Solving equations (1) and (2), we get

$$a = -\frac{5}{2},\ b = -\frac{3}{2}.$$

Exercise 2.5

1. $\lim_{x\to 0}\frac{\sin x^\circ}{x}=\lim_{x\to 0}\frac{\sin\frac{\pi x}{180}}{x}=\frac{\pi}{180}$ $\left\{\because x^\circ=\frac{\pi x}{180}\text{ radian}\right\}$

2. $\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx}=\lim_{x\to 0}\left\{\frac{2\sin^2\frac{mx}{2}}{2\sin^2\frac{n}{2}x}\right\}$

$$=\lim_{x\to 0}\left[\left\{\frac{\sin\frac{mx}{2}}{\frac{mx}{2}}\right\}^2\frac{m^2x^2}{4}\times\frac{1}{\left\{\frac{\sin\frac{nx}{2}}{\frac{nx}{2}}\right\}^2}\times\frac{4}{n^2x^2}\right]$$

$$=\frac{m^2}{n^2}$$

3. We have

$$\lim_{x\to 0}\frac{\cot 2x-\operatorname{cosec}2x}{x}$$

$$=\lim_{x\to 0}\frac{\frac{\cos 2x}{\sin 2x}-\frac{1}{\sin 2x}}{x}=\lim_{x\to 0}\frac{\cos 2x-1}{x\sin 2x}$$

$$=\lim_{x\to 0}\frac{-(1-\cos 2x)}{x\sin 2x}=-\lim_{x\to 0}\frac{2\sin^2 x}{x(2\sin x\cos x)}$$

$$=-\lim_{x\to 0}\frac{\tan x}{x}=-1$$

4. $\lim_{x\to 0}\frac{\tan 2x-x}{3x-\sin x}=\lim_{x\to 0}\left\{\frac{\frac{2\tan 2x}{2x}-1}{3-\frac{\sin x}{x}}\right\}=\frac{1}{2}$

5. $L=\lim_{n\to\infty}n\cos\left(\frac{\pi}{4n}\right)\sin\left(\frac{\pi}{4n}\right)$

$$=\lim_{n\to\infty}\frac{\cos\left(\frac{\pi}{4n}\right)\sin\left(\frac{\pi}{4n}\right)}{\left(\frac{\pi}{4n}\right)\frac{4}{\pi}}$$

$$=\cos(0)\times 1\times\frac{\pi}{4}=\frac{\pi}{4}$$

6. $\lim_{x\to 0}\frac{\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)}{\sin^{-1}x}=\lim_{x\to 0}\frac{2\tan^{-1}x}{\sin^{-1}x}=2$

7. $\lim_{h\to 0}\frac{2\left[\sqrt{3}\sin\left(\frac{\pi}{6}+h\right)-\cos\left(\frac{\pi}{6}+h\right)\right]}{\sqrt{3}h\left(\sqrt{3}\cos h-\sin h\right)}$

$$=\lim_{h\to 0}\frac{\frac{4}{\sqrt{3}}\left[\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{6}+h\right)-\frac{1}{2}\cos\left(\frac{\pi}{6}+h\right)\right]}{h\left(\sqrt{3}\cos h-\sin h\right)}$$

$$=\lim_{h\to 0}\frac{4}{\sqrt{3}}\times\frac{\sin h}{h}\frac{1}{\left(\sqrt{3}\cos h-\sin h\right)}=\frac{4}{3}$$

8. $\lim_{x\to 0}\frac{8\left(1-\cos\frac{x^2}{4}\right)\left(1-\cos\frac{x^2}{2}\right)}{x^8}$ $\left(\text{form }\frac{0}{0}\right)$

$$=\lim_{x\to 0}\frac{8\times 2\sin^2\frac{x^2}{8}\times 2\sin^2\frac{x^2}{4}}{x^8}$$

$$=\lim_{x\to 0}\frac{32}{64\times 16}\cdot\frac{\sin^2\frac{x^2}{8}}{\frac{x^4}{64}}\cdot\frac{\sin^2\frac{x^2}{4}}{\frac{x^4}{16}}$$

$$=\frac{1}{32}$$

9. $\lim_{\substack{x\to 0\\y\to 0}}\frac{y^2+\sin x}{x^2+\sin y^2}=\lim_{x\to 0}\frac{x+\sin x}{x^2+\sin x}=\lim_{x\to 0}\frac{1+\frac{\sin x}{x}}{x+\frac{\sin x}{x}}=2$

10. $\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2}=\lim_{x\to 1}\frac{(1-x)}{\tan\left(\frac{\pi}{2}-\frac{\pi x}{2}\right)}$

$$=\frac{2}{\pi}\lim_{x\to 1}\frac{\frac{\pi}{2}(1-x)}{\tan\left(\frac{\pi}{2}(1-x)\right)}$$

$$=\frac{2}{\pi}$$

11. $\lim_{x\to 0}\frac{x\tan 2x-2x\tan x}{4\sin^4 x}=\lim_{x\to 0}\frac{x}{4\sin^4 x}\left[\frac{2\tan x}{1-\tan^2 x}-2\tan x\right]$

$$=\lim_{x\to 0}\frac{x\tan^3 x}{2\sin^4 x(1-\tan^2 x)}$$

$$=\frac{1}{2}\lim_{x\to 0}\frac{x}{\sin x}\cdot\frac{1}{\cos^3 x}\cdot\frac{1}{1-\tan^2 x}$$

$$=\frac{1}{2}.1.\frac{1}{1^3}.\frac{1}{1-0}=\frac{1}{2}$$

12. We have $\lim_{x\to 2}\frac{\tan(x-2)\cdot(x^2+(k-2)x-2k)}{(x-2)^2}=5$

$$\Rightarrow\quad\lim_{x\to 2}\frac{\tan(x-2)}{x-2}\cdot\frac{(x^2+(k-2)x-2k)}{x-2}=5$$

$$\Rightarrow\quad\lim_{x\to 2}\frac{x^2+(k-2)x-2k}{x-2}=5$$

$$\Rightarrow\quad x^2+(k-2)x-2k=(x-2)(x+k)$$

For the value of limit to be 5, $k=3$

Exercise 2.6

1. $\lim_{x\to\infty}x\left(a^{1/x}-1\right)=\lim_{x\to\infty}\left[\frac{a^{1/x}-1}{1/x}\right]=\log_e a$

2. $\lim_{x\to 0}\frac{x\left(2^x-1\right)}{1-\cos x}=\lim_{x\to 0}\frac{2^x-1}{x}\times\frac{x^2}{1-\cos x}$

$$=\log 2\lim_{x\to 0}\frac{x^2}{2\sin^2\frac{x}{2}}$$

$$=2\log 2=\log 4$$

3. $\lim_{x\to 2}\dfrac{\sin(e^{x-2}-1)}{\log(x-1)} = \lim_{t\to 0}\dfrac{\sin(e^{t}-1)}{\log(1+t)}$ [Putting $x=2+t$]

$= \lim_{t\to 0}\dfrac{\sin(e^{t}-1)}{e^{t}-1}\times\dfrac{e^{t}-1}{t}\times\dfrac{t}{\log(1+t)}$

$= 1\times 1\times 1 = 1$

4. $\lim_{x\to 0}\dfrac{e^{x^2}-\cos x}{x^2} = \lim_{x\to 0}\left[\dfrac{e^{x^2}-1}{x^2}+\dfrac{1-\cos x}{x^2}\right]$

$= 1+\lim_{x\to 0}\dfrac{2\sin^2\frac{x}{2}}{x^2} = 1+\dfrac{1}{2} = \dfrac{3}{2}$

5. $\lim_{x\to 0}\dfrac{e^{x}+e^{-x}-2}{x^2}$ $\left(\text{form }\dfrac{0}{0}\right)$

$= \lim_{x\to 0}\dfrac{1}{e^{x}}\left(\dfrac{e^{x}-1}{x}\right)^2$

$= 1$

6. $\lim_{x\to a}\dfrac{\log(x-a)}{\log(e^{x}-a^{a})}$

$= \lim_{x\to a}\dfrac{\frac{1}{x-a}}{\frac{e^{x}}{e^{x}-e^{a}}}$ (Applying L′ Hospital's rule)

$= \lim_{x\to a}\dfrac{e^{x}-e^{a}}{e^{x}(x-a)}$

$= \lim_{x\to a}\dfrac{e^{a}(e^{x-a}-1)}{e^{x}(x-a)}$

$= 1$

7. $\lim_{x\to 0} a^{\sin x}\times\dfrac{(a^{\tan x-\sin x}-1)}{(\tan x-\sin x)} = a^{0}\ln a = \ln a$

8. $\lim_{x\to 0}\dfrac{(1-3^{x}-4^{x}+12^{x})}{\sqrt{(2\cos x+7)}-3}$

$= \lim_{x\to 0}\dfrac{(3^{x}-1)(4^{x}-1)}{\sqrt{(2\cos x+7)}-3}$

$= \lim_{x\to 0}\dfrac{(3^{x}-1)(4^{x}-1)(\sqrt{(2\cos x+7)}+3)}{(2\cos x+7-9)}$

$= \lim_{x\to 0}\dfrac{\frac{(3^{x}-1)}{x}\times\frac{(4^{x}-1)}{x}(\sqrt{(2\cos x+7)}+3)}{\frac{-2(1-\cos x)}{x^2}}$

$= \dfrac{(\ln 3)(\ln 4)6}{-2\times\frac{1}{2}} = -6\ln 3\times\ln 4$

$= -12\ln 2\times\ln 3$

9. $L = \lim_{x\to 0}\dfrac{(729)^{x}-(243)^{x}-(81)^{x}+9^{x}+3^{x}-1}{x^3}$

$= \lim_{x\to 0}\dfrac{(3^{x}-1)}{x}\times\dfrac{(9^{x}-1)}{x}\times\dfrac{(27^{x}-1)}{x}$

$= (\ln 3)(\ln 9)(\ln 27)$

$= 6(\ln 3)^3$

Exercise 2.7

1. $\left[\lim_{x\to\infty}\left(1+\dfrac{1}{x/2}\right)^{x/2}\right]^2 = e^2$

2. $\lim_{x\to 1}(\log_3 3x)^{\log_x 3} = \lim_{x\to 1}(\log_3 3+\log_3 x)^{\frac{1}{\log_3 x}}$

$= \lim_{x\to 1}(1+\log_3 x)^{\frac{1}{\log_3 x}}$ (1^{∞} form)

$= e$

3. Let $A = \lim_{x\to\infty}\left(\dfrac{x+2}{x+1}\right)^{x+3}$

$= \lim_{x\to\infty}\left(1+\dfrac{1}{x+1}\right)^{x+3}$

$= \lim_{x\to\infty}\left[\left(1+\dfrac{1}{x+1}\right)^{x+1}\right]^{\frac{(x+3)}{(x+1)}}$

$= \left[\lim_{x\to\infty}\left(1+\dfrac{1}{x+1}\right)^{x+1}\right]^{\lim_{x\to\infty}\frac{1+\frac{3}{x}}{1+\frac{1}{x}}}$

$= e^{1}$

4. $\lim_{x\to\infty}\left(1+\dfrac{1}{a+bx}\right)^{c+dx} = \lim_{x\to\infty}\left\{\left(1+\dfrac{1}{a+bx}\right)^{a+bx}\right\}^{\frac{c+dx}{a+bx}} = e^{d/b}$

$\left\{\because \lim_{x\to\infty}\left(1+\dfrac{1}{a+bx}\right)^{a+bx} = e \text{ and } \lim_{x\to\infty}\dfrac{c+dx}{a+bx} = \dfrac{d}{b}\right\}$

5. Given limit takes 1^{∞} form: Therefore,

$L = \lim_{x\to 7/2}(2x^2-9x+8)^{\cot(2x-7)}$

$= \lim_{x\to 7/2}((2x-7)(x-1)+1)^{\cot(2x-7)}$

$= e^{\lim_{x\to 7/2}((2x-7)(x-1))\cot(2x-7)}$

$= e^{\lim_{x\to 7/2}\frac{(2x-7)(x-1)}{\tan(2x-7)}}$

$= e^{5/2}$

6. $ax^2+bx+c = a(x-x_1)(x-x_2)$

$\Rightarrow \lim_{x\to x_1}(1+\sin(ax^2+bx+c))^{\frac{1}{x-x_1}}$ (1^{∞} form)

$= e^{\lim_{x\to x_1}\frac{\sin(a(x-x_1)(x-x_2))}{(x-x_1)}}$

$= e^{\lim_{x\to x_1}\frac{\sin(a(x-x_1)(x-x_2))}{a(x-x_1)(x-x_2)}\cdot a(x-x_2)}$

$= e^{a(x_1-x_2)}$

7. Given limit takes 1^{∞} form. Therefore,

$L = \lim_{x\to 0}\left\{\sin^2\left(\dfrac{\pi}{2-px}\right)\right\}^{\sec^2\left(\frac{\pi}{2-qx}\right)}$

$= \exp\left\{\lim_{x\to 0}\left[\sin^2\left(\dfrac{\pi}{2-px}\right)-1\right]\sec^2\left(\dfrac{\pi}{2-qx}\right)\right\}$

$$= \exp\left\{-\lim_{x\to 0}\frac{\cos^2\left(\frac{\pi}{2-px}\right)}{\cos^2\left(\frac{\pi}{2-qx}\right)}\right\}$$

$$= \exp\left\{-\lim_{x\to 0}\frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{2-px}\right)}{\sin^2\left(\frac{\pi}{2}-\frac{\pi}{2-qx}\right)}\right\}$$

Now, $\lim_{x\to 0}\dfrac{\sin^2\left(\frac{\pi}{2}-\frac{\pi}{2-px}\right)}{\sin^2\left(\frac{\pi}{2}-\frac{\pi}{2-qx}\right)}$

$$= \lim_{x\to 0}\frac{\dfrac{\sin^2\left(\frac{\pi}{2}-\frac{\pi}{2-px}\right)}{\left(\frac{\pi}{2}-\frac{\pi}{2-px}\right)^2}}{\dfrac{\sin^2\left(\frac{\pi}{2}-\frac{\pi}{2-qx}\right)}{\left(\frac{\pi}{2}-\frac{\pi}{2-qx}\right)^2}} \times \frac{\left(\frac{\pi}{2}-\frac{\pi}{2-px}\right)^2}{\left(\frac{\pi}{2}-\frac{\pi}{2-qx}\right)^2}$$

$$= \lim_{x\to 0}\frac{\left(\frac{\pi}{2}-\frac{\pi}{2-px}\right)^2}{\left(\frac{\pi}{2}-\frac{\pi}{2-qx}\right)^2}$$

$$= \lim_{x\to 0}\frac{\left(\frac{-\pi px}{2(2-px)}\right)^2}{\left(\frac{-\pi qx}{2(2-qx)}\right)^2}$$

$$= p^2/q^2$$

$\therefore \quad L = e^{-p^2/q^2}$

Exercise 2.8

1. $\lim_{x\to 1}\dfrac{\cos\frac{\pi}{2}x}{1-\sqrt{x}}$ $\left(\text{form } \frac{0}{0}\right)$

$$= \lim_{x\to 1}\frac{-\frac{\pi}{2}\sin\left(\frac{\pi}{2}x\right)}{-\frac{1}{2\sqrt{x}}}$$

$$= \frac{\frac{\pi}{2}\times 1}{\frac{1}{2\sqrt{1}}} = \frac{\pi}{2}\times 2 = \pi$$

2. $\lim_{x\to 1}\dfrac{1+\log x - x}{1-2x+x^2}$ $\left(\text{form } \frac{0}{0}\right)$

$$= \lim_{x\to 1}\frac{0+\frac{1}{x}-1}{0-2+2x}$$ (Using L'Hospital rule)

$$= \lim_{x\to 1}\frac{1-x}{-2x(1-x)} = \lim_{x\to 1}\frac{1}{-2x} = \frac{-1}{2}$$

3. $\lim_{x\to \pi/2}\tan x \log\sin x = \lim_{x\to\pi/2}\dfrac{\log\sin x}{\cot x}$

$$= \lim_{x\to\pi/2}\frac{\frac{1}{\sin x}\cos x}{-\operatorname{cosec}^2 x} = 0$$ (Applying L' Hospital's rule)

4. $\lim_{x\to 0}\dfrac{\log\cos x}{x} = \lim_{x\to 0}\dfrac{-\tan x}{1} = 0$

5. $\lim_{x\to 0}\dfrac{2^x-1}{(1+x)^{1/2}-1}$

$$= \lim_{x\to 0}\frac{2^x\log 2}{\frac{1}{2}(1+x)^{-1/2}}$$ (Applying L' Hospital's rule)

$$= 2\log 2$$

$$= \log 4$$

6. $\lim_{x\to\pi/4}(2-\tan x)^{1/\ln\tan x}$

$$= e^{\lim_{x\to\pi/4}(2-\tan x-1)\times\frac{1}{\ln\tan x}}$$ (1^∞ form)

$$= e^{\lim_{x\to\pi/4}\left(\frac{1-\tan x}{\ln\tan x}\right)}$$

$$= e^{\lim_{x\to\pi/4}\frac{-\sec^2 x}{\frac{1}{\tan x}\sec^2 x}}$$

$$= e^{-\lim_{x\to\pi/4}\tan x}$$

$$= e^{-1}$$

7. Let $y = x^x$

$\therefore \quad \log y = x\log x$

$\therefore \quad \lim_{y\to 0}\log y = \lim_{x\to 0} x\log x$

$$= \lim_{x\to 0}\frac{\log x}{1/x}$$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

$$= \lim_{x\to 0}\frac{1/x}{-1/x^2}$$

$$= \lim_{x\to 0}(-x)$$

$$= 0$$

$\therefore \quad y = 1$

$$\lim_{x\to 0^+} x^{x^x} = (0^+)^1 = 0$$

8. Since $\lim_{x\to a}\dfrac{a^x - x^a}{x^x - a^a} = -1$,

$$\lim_{x\to a}\frac{a^x\log a - ax^{a-1}}{x^x(1+\log x)} = -1$$

or $\dfrac{a^a[\log a - 1]}{a^a[1+\log a]} = -1$

or $\log a - 1 = -1 - \log a$

or $2\log a = 0$

or $\log a = 0$

or $a = 1$

le Correct Answer Type

(1) $\lim\limits_{x\to 0+}\left[\dfrac{\sin(\text{sgn}\,x)}{\text{sgn}(x)}\right] = \lim\limits_{x\to 0^+}\left[\dfrac{\sin 1}{1}\right] = 0$

$= \lim\limits_{x\to 0^-}\left[\dfrac{\sin(\text{sgn}\,x)}{\text{sgn}(x)}\right]$

$= \lim\limits_{x\to 0^-}\left[\dfrac{\sin(-1)}{-1}\right]$

$= \lim\limits_{x\to 0^-}[\sin 1]$

Hence, the given limit is 0.

(2) $\dfrac{[x]^2}{x^2} = \begin{cases} 0, \text{ if } 0<x<1 \\ \dfrac{1}{x^2}, \text{if } -1<x<0 \end{cases} \Rightarrow l$ does not exist

$\dfrac{[x^2]}{x^2} = \begin{cases} 0, \text{ if } 0<x<1 \\ 0, \text{ if } -1<x<0 \end{cases} \Rightarrow m$ exists and is equal to 0

3. (2) $\text{cosec}\,\dfrac{\pi x}{2} \to 1^+$ when $x \to 1$

or $\left[\text{cosec}\,\dfrac{\pi x}{2}\right] = 1$

$\therefore$ limit = 1

4. (3) $\lim\limits_{x\to 0}\dfrac{a^{\sqrt{x}} - a^{1/\sqrt{x}}}{a^{\sqrt{x}} + a^{1/\sqrt{x}}}$, $a > 1$

Put $x = t^2$. Then

$\lim\limits_{t\to 0}\dfrac{a^t - a^{1/t}}{a^t + a^{1/t}}$

or $\lim\limits_{t\to 0}\dfrac{a^{t-1/t} - 1}{a^{t-1/t} + 1} = \dfrac{a^{-\infty} - 1}{a^{-\infty} + 1} = \dfrac{0-1}{0+1} = -1$

5. (3) If $f(x) = \sin\left(\dfrac{1}{x}\right)$ and $g(x) = \dfrac{1}{x}$, then both $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to 0} g(x)$ do not exist, but $\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)} = 0$ exists.

6. (1) L.H.L. $= \lim\limits_{x\to -1^-}\dfrac{1}{\sqrt{|x|-\{-x\}}} = \lim\limits_{x\to -1^-}\dfrac{1}{\sqrt{-x-(x+2)}}$

$= \lim\limits_{x\to -1^-}\dfrac{1}{\sqrt{-2x-2}} = \infty$

R.H.L. $= \lim\limits_{x\to -1^+}\dfrac{1}{\sqrt{|x|-\{-x\}}} = \lim\limits_{x\to -1^-}\dfrac{1}{\sqrt{-x-(x+1)}}$

$= \lim\limits_{x\to -1^-}\dfrac{1}{\sqrt{-2x-1}} = 1$

Hence, the limit does not exist.

7. (2) $x_{n+1} = \sqrt{2+x_n}$

or $\lim\limits_{n\to\infty} x_{n+1} = \sqrt{2 + \lim\limits_{n\to\infty} x_n}$

or $t = \sqrt{2+t}$ $\left(\because \lim\limits_{x\to\infty} x_{n+1} = \lim\limits_{x\to\infty} x_n = t\right)$

or $t^2 - t - 2 = 0$

or $(t-2)(t+1) = 0$

or $t = 2$ $(\because x_n > 0\ \forall\ n,\ t > 0)$

8. (3) $\lim\limits_{x\to 0^-}\dfrac{\sum\limits_{r=1}^{2n+1}[x^r] + (n+1)}{1 + [x] + |x| + 2x} = \lim\limits_{x\to 0^-}\dfrac{\sum\limits_{r=1}^{2n+1}[x^r] + (n+1)}{x}$

$= \lim\limits_{x\to 0^-}\dfrac{[x] + [x^2] + [x^3] + \ldots + [x^{2n+1}] + (n+1)}{x}$

$= \lim\limits_{x\to 0^-}\dfrac{\underbrace{(-1) + (-1) + \cdots + (-1)}_{(n+1)\text{times}} + (n+1)}{x} = \lim\limits_{x\to 0^-}\dfrac{0}{x} = 0$

9. (2) $\lim\limits_{x\to\infty}\dfrac{\sin^4 x - \sin^2 x + 1}{\cos^4 x - \cos^2 x + 1} = \lim\limits_{x\to\infty}\dfrac{(1-\cos^2 x)^2 - (1-\cos^2 x) + 1}{\cos^4 x - \cos^2 x + 1}$

$= \lim\limits_{x\to\infty}\dfrac{\cos^4 x - \cos^2 x + 1}{\cos^4 x - \cos^2 x + 1}$

$= 1$

10. (3) We have

$f(x) + g(x) + h(x) = \dfrac{x^2 - 4x + 17 - 4x - 2}{x^2 + x - 12}$

$= \dfrac{x^2 - 8x + 15}{x^2 + x - 12} = \dfrac{(x-3)(x-5)}{(x-3)(x+4)}$

$\therefore \quad \lim\limits_{x\to 3}[f(x) + g(x) + h(x)] = \lim\limits_{x\to 3}\dfrac{(x-3)(x-5)}{(x-3)(x+4)} = -\dfrac{2}{7}$

11. (4) We have

$\lim\limits_{x\to\pi}\dfrac{1+\cos^3 x}{\sin^2 x} = \lim\limits_{x\to\pi}\dfrac{(1+\cos x)(1-\cos x+\cos^2 x)}{(1-\cos x)(1+\cos x)}$

$= \lim\limits_{x\to\pi}\dfrac{1-\cos x+\cos^2 x}{1-\cos x} = \dfrac{1+1+1}{1+1} = \dfrac{3}{2}$

12. (1) $\lim\limits_{x\to 2}\dfrac{\sqrt{1+\sqrt{2+x}} - \sqrt{3}}{x-2}$

$= \lim\limits_{x\to 2}\dfrac{1+\sqrt{2+x}-3}{\left(\sqrt{1+\sqrt{2+x}} + \sqrt{3}\right)(x-2)}$ (Rationalizing)

$= \lim\limits_{x\to 2}\dfrac{\sqrt{2+x}-2}{\left(\sqrt{1+\sqrt{2+x}} + \sqrt{3}\right)(x-2)}$

$= \lim\limits_{x\to 2}\dfrac{(x-2)}{(\sqrt{1+\sqrt{2+x}} + \sqrt{3})(\sqrt{2+x}+2)(x-2)}$

(Rationalizing)

$= \dfrac{1}{(2\sqrt{3})\,4} = \dfrac{1}{8\sqrt{3}}$

13. (2) $\lim\limits_{x\to 2}\dfrac{2^x + 2^{3-x} - 6}{\sqrt{2^{-x}} - 2^{1-x}}$

$= \lim\limits_{x\to 2}\dfrac{(2^x)^2 - 6\times 2^x + 2^3}{\sqrt{2^x} - 2}$ [Multiplying N^r and D^r by 2^x]

$= \lim\limits_{x\to 2}\dfrac{(2^x - 4)(2^x - 2)(\sqrt{2^x}+2)}{(\sqrt{2^x} - 2)(\sqrt{2^x}+2)}$

$= \lim\limits_{x\to 2}\dfrac{(2^x - 4)(2^x - 2)(\sqrt{2^x}+2)}{(2^x - 4)}$

$= \lim\limits_{x\to 2}(2^x - 2)(\sqrt{2^x}+2) = (2^2 - 2)(2+2) = 8$

14. (1) $\lim\limits_{x\to 2}\left[\left(\frac{x(x-2)(x+2)}{(x-2)(x^2+2x+4)}\right)^{-1}-\left(\frac{\sqrt{x}(\sqrt{x}+\sqrt{2})}{(\sqrt{x}-\sqrt{2})(\sqrt{x}+\sqrt{2})}-\frac{\sqrt{2}}{\sqrt{x}-\sqrt{2}}\right)^{-1}\right]$

$= \lim\limits_{x\to 2}\left[\frac{x^2+2x+4}{x(x+2)}-\left(\frac{\sqrt{x}-\sqrt{2}}{\sqrt{x}-\sqrt{2}}\right)^{-1}\right]$

$= \lim\limits_{x\to 2}\left[\frac{x^2+2x+4}{x(x+2)}-1\right]=\frac{12}{8}-1=\frac{1}{2}$

15. (3) $\lim\limits_{x\to -2^-}\frac{ae^{1/|x+2|}-1}{2-e^{1/|x+2|}} = \lim\limits_{x\to -2^-}\frac{a-e^{-1/|x+2|}}{2e^{-1/|x+2|}-1} = -a$

$$\lim_{x\to -2^-}\sin\left(\frac{x^4-16}{x^5+32}\right)=\lim_{x\to -2^-}\sin\left(\frac{\frac{x^4-(-2)^4}{x-(-2)}}{\frac{x^5-(-2)^5}{x-(-2)}}\right)$$

$$= \sin\left(-\frac{2}{5}\right)$$

or $\quad a=\sin\frac{2}{5}$

16. (2) $\lim\limits_{x\to 1}\frac{(1-x)(1-x^2)\cdots(1-x^{2n})}{\{(1-x)(1-x^2)\cdots(1-x^n)\}^2}$

$$= \lim_{x\to 1}\frac{\left(\frac{1-x}{1-x}\right)\left(\frac{1-x^2}{1-x}\right)\cdots\left(\frac{1-x^{2n}}{1-x}\right)}{\left(\left(\frac{1-x}{1-x}\right)\left(\frac{1-x^2}{1-x}\right)\cdots\left(\frac{1-x^n}{1-x}\right)\right)^2}$$

$$= \frac{1\times 2\times 3\cdots(2n)}{(1\times 2\times 3\cdots n)^2} = \frac{(2n)!}{n!n!} = {}^{2n}C_n$$

17. (1) Let $\sin^{-1}x=\theta$. Then, $x=\sin\theta$.

Now, $x\to\frac{1}{\sqrt{2}}$ or $\sin\theta\to\frac{1}{\sqrt{2}}$ or $\theta\to\frac{\pi}{4}$. Therefore,

$$\lim_{x\to\frac{1}{\sqrt{2}}}\frac{x-\cos\left(\sin^{-1}x\right)}{1-\tan\left(\sin^{-1}x\right)} = \lim_{\theta\to\frac{\pi}{4}}\frac{\sin\theta-\cos\theta}{1-\tan\theta}$$

$$= \lim_{\theta\to\frac{\pi}{4}}\frac{(\sin\theta-\cos\theta)}{(\cos\theta-\sin\theta)}\cos\theta$$

$$= \lim_{\theta\to\frac{\pi}{4}}-\cos\theta=-\frac{1}{\sqrt{2}}$$

18. (1) i. $\lim\limits_{x\to\infty}\sec^{-1}\left(\frac{x}{\sin x}\right) = \sec^{-1}\left(\frac{\infty}{\sin\infty}\right)$

$$= \sec^{-1}\left(\frac{\infty}{\text{any value between } -1 \text{ to } 1}\right)$$

$$= \sec^{-1}(\pm\infty)=\frac{\pi}{2}$$

ii. $\lim\limits_{x\to\infty}\sec^{-1}\left(\frac{\sin x}{x}\right) = \sec^{-1}\left(\frac{\sin\infty}{\infty}\right)$

$$= \sec^{-1}\left(\frac{\text{any value between } -1 \text{ to } 1}{\infty}\right)$$

$= \sec^{-1}0 =$ not defined

Hence, (i) exists but (ii) does not exist.

19. (4) $\lim\limits_{x\to\infty}\left(\frac{x^3}{3x^2-4}-\frac{x^2}{3x+2}\right) = \lim\limits_{x\to\infty}\frac{x^3(3x+2)-x^2(3x^2-4)}{(3x^2-4)(3x+2)}$

$$= \lim_{x\to\infty}\frac{2x^3+4x^2}{9x^3+6x^2-12x-8}$$

$$= \lim_{x\to\infty}\frac{2+\frac{4}{x}}{9+\frac{6}{x}-\frac{12}{x^2}-\frac{8}{x^3}}$$

$= 2/9$

20. (3) $\lim\limits_{n\to\infty}\frac{n(2n+1)^2}{(n+2)\left(n^2+3n-1\right)} = \lim\limits_{n\to\infty}\frac{\left(2+\frac{1}{n}\right)^2}{\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}-\frac{1}{n^2}\right)}$

$$= \frac{(2+0)^2}{(1+0)(1+0+0)} = 4$$

21. (3) $\lim\limits_{x\to\infty}\frac{(2x+1)^{40}(4x-1)^5}{(2x+3)^{45}} = \lim\limits_{x\to\infty}\frac{\left(2+\frac{1}{x}\right)^{40}\left(4-\frac{1}{x}\right)^5}{\left(2+\frac{3}{x}\right)^{45}}$

(Dividing numerator and denominator by x^{45})

$$= \frac{2^{40}4^5}{2^{45}} = 2^5 = 32$$

22. (2) $\lim\limits_{x\to\infty}\left[\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right] = \lim\limits_{x\to\infty}\frac{x+\sqrt{x+\sqrt{x}}-x}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}$

(Rationalizing)

$$\lim_{x\to\infty}\frac{\sqrt{1+x^{-1/2}}}{\sqrt{1+\sqrt{x^{-1}+x^{-3/2}}}+1}=\frac{1}{2}$$

23. (4) The given limit is

$$\lim_{x\to\infty}\frac{\frac{2}{x}+2+\frac{\sin 2x}{x}}{\left(2+\frac{\sin 2x}{x}\right)e^{\sin x}}$$

$$= \frac{0+2+0}{(2+0)\times(\text{a value between } \frac{1}{e} \text{ and } e)}$$

$\left[\because \lim\limits_{x\to\infty}\sin x\in(-1,1)\right]$

Hence, limit does not exist.

24. (4) $\lim\limits_{x\to\infty}\frac{(x+1)^{10}+(x+2)^{10}+\cdots+(x+100)^{10}}{x^{10}+10^{10}}$

$$= \lim_{x\to\infty}\frac{x^{10}\left[\left(1+\frac{1}{x}\right)^{10}+\left(1+\frac{2}{x}\right)^{10}+\cdots+\left(1+\frac{100}{x}\right)^{10}\right]}{x^{10}\left[1+\frac{10^{10}}{x^{10}}\right]}$$

$= 100$

25. (3) Since the highest degree of x is 1/2, divide numerator and denominator by $\sqrt{x}$. Then we have limit $\frac{2}{\sqrt{2}}$ or $\sqrt{2}$.

26. (1) $\lim_{n\to\infty} \dfrac{n\cdot 3^n}{n(x-2)^n + n\cdot 3^{n+1} - 3^n} = \dfrac{1}{3}$

or $\lim_{n\to\infty} \dfrac{1}{\dfrac{(x-2)^n}{3^n} + 3 - \dfrac{1}{n}} = \dfrac{1}{3}$ (Dividing N^r and D^r by $n\times 3^n$)

For $\lim_{n\to\infty}$ to be equal to 1/3,

$$\lim_{n\to\infty}\left(\frac{x-2}{3}\right)^n \to 0$$

$\therefore \quad 2 \le x < 5$

27. (3) $\lim_{n\to\infty} n^2\left(x^{1/n} - x^{\frac{1}{n+1}}\right) = \lim_{n\to\infty} n^2 \cdot x^{\frac{1}{n+1}}\left(x^{\frac{1}{n}-\frac{1}{n+1}} - 1\right)$

$$= \lim_{n\to\infty} x^{\frac{1}{n+1}}\left(x^{\frac{1}{n(n+1)}} - 1\right) n^2$$

$$= \lim_{n\to\infty} x^{\frac{1}{n+1}} \cdot \frac{x^{\frac{1}{n(n+1)}} - 1}{\frac{1}{n(n+1)}} \cdot \frac{n^2}{n(n+1)}$$

$$= 1 \cdot \log_e x \cdot 1 = \log_e x$$

28. (1) Given $g(x) = \lim_{n\to\infty} \dfrac{1}{\left(\frac{3}{\pi}\tan^{-1} 2x\right)^{2n} + 5} = 0$

or $\lim_{n\to\infty}\left[\left(\frac{3}{\pi}\tan^{-1} 2x\right)^2\right]^n \to \infty$

or $\left(\frac{3}{\pi}\tan^{-1} 2x\right)^2 > 1$

or $|\tan^{-1} 2x| > \dfrac{\pi}{3}$

i.e., $\tan^{-1} 2x < -\dfrac{\pi}{3}$ or $\tan^{-1} 2x > \dfrac{\pi}{3}$

i.e., $2x < -\sqrt{3}$ or $2x > \sqrt{3}$, i.e., $|2x| > \sqrt{3}$

29. (2) $L = \lim_{x\to\infty} \dfrac{\ln\left(x^2 + e^x\right)}{\ln\left(x^4 + e^{2x}\right)} = \lim_{x\to\infty} \dfrac{\ln e^x\left(1 + \frac{x^2}{e^x}\right)}{\ln e^{2x}\left(1 + \frac{x^4}{e^{2x}}\right)}$

$$= \lim_{x\to\infty} \frac{x + \ln\left(1 + \frac{x^2}{e^x}\right)}{2x + \ln\left(1 + \frac{x^4}{e^{2x}}\right)}$$

$$= \lim_{x\to\infty} \frac{1 + \frac{1}{x}\ln\left(1 + \frac{x^2}{e^x}\right)}{2 + \frac{1}{x}\ln\left(1 + \frac{x^4}{e^{2x}}\right)}$$

Note that as $x \to \infty$, $\dfrac{x^2}{e^x} \to 0$ and as $x \to \infty$, $\dfrac{x^2}{e^{2x}} \to 0$ (Using L'Hospital's rule)

Hence $L = \dfrac{1}{2}$.

30. (3) $\lim_{n\to\infty}\left[\dfrac{2}{2-\frac{1}{n^2}}\cdot\dfrac{1}{n}\cos\left(\dfrac{1+1/n}{2-1/n}\right) - \dfrac{1}{\left(\frac{1}{n}-2\right)}\cdot\dfrac{1}{\left(1+\frac{1}{n^2}\right)}\cdot\dfrac{1}{n}\right]$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\frac{2}{2-\frac{1}{n^2}}\cdot\cos\left(\frac{1+\frac{1}{n}}{2-\frac{1}{n}}\right) - \frac{1}{\left(\frac{1}{n}-2\right)}\cdot\frac{1}{\left(1+\frac{1}{n^2}\right)}\right]$$

$$= 0\times\left[\frac{2}{2}\times\cos\frac{1}{2} + \frac{1}{2}\times\frac{1}{1}\right] = 0.$$

31. (3) Given $f(x) = x^2 - \pi^2$

$$\lim_{x\to-\pi}\frac{x^2-\pi^2}{\sin(\sin x)} = \lim_{h\to 0}\frac{(-\pi+h)^2-\pi^2}{\sin(\sin(-\pi+h))}$$

$$= \lim_{h\to 0}\frac{-2h\pi+h^2}{-\sin(\sin h)}$$

$$= \lim_{h\to 0}\frac{h-2\pi}{\frac{-\sin(\sin h)}{\sin h}\times\frac{\sin h}{h}} = 2\pi$$

32. (3) $\lim_{x\to 1}\dfrac{x\sin(x-[x])}{x-1}$

Now, L.H.L. $= \lim_{h\to 0}\dfrac{(1-h)\sin(1-h-[1-h])}{(1-h)-1}$

$$= \lim_{h\to 0}\frac{(1-h)\sin(1-h)}{-h} = -\infty$$

R.H.L. $= \lim_{h\to 0}\dfrac{(1+h)\sin(1+h-[1+h])}{(1+h)-1} = \lim_{h\to 0}\dfrac{(1+h)\sin h}{h} = 1$

Hence, the limit does not exist.

33. (1) $\lim_{x\to-\infty}\dfrac{x^2\tan\frac{1}{x}}{\sqrt{8x^2+7x+1}} = \lim_{x\to-\infty}\dfrac{x^2\tan\frac{1}{x}}{-x\sqrt{8+\frac{7}{x}+\frac{1}{x^2}}}$

$$= -\lim_{x\to-\infty}\frac{\tan\frac{1}{x}}{\frac{1}{x}\sqrt{8+\frac{7}{x}+\frac{1}{x^2}}} = -\frac{1}{2\sqrt{2}}$$

34. (3) $\lim_{x\to 0}\dfrac{x^a\sin^b x}{\sin x^c} = \lim_{x\to 0} x^a\left(\dfrac{\sin x}{x}\right)^b\left(\dfrac{x^c}{\sin x^c}\right)x^{b-c} = \lim_{x\to 0} x^{a+b-c}$

This limit will have non-zero value if $a + b = c$.

35. (1) $\dfrac{x^4(\cot^4 x - \cot^2 x + 1)}{(\tan^4 x - \tan^2 x + 1)}$

$$\frac{x^4(1-\tan^2 x+\tan^4 x)}{\tan^4 x(\tan^4 x-\tan^2 x+1)} = \frac{x^4}{\tan^4 x},\ x \ne 0$$

$$\therefore \quad \lim_{x\to 0}\frac{x^4(\cot^4 x-\cot^2 x+1)}{(\tan^4 x-\tan^2 x+1)} = \lim_{x\to 0}\frac{x^4}{\tan^4 x} = 1$$

36. (2) $\lim_{x\to 1}\dfrac{1-x^2}{\sin 2\pi x} = -\lim_{x\to 1}\dfrac{2\pi(1-x)(1+x)}{2\pi\sin(2\pi-2\pi x)}$

$$= -\lim_{x\to 1}\frac{(2\pi-2\pi x)}{\sin(2\pi-2\pi x)}\frac{1+x}{2\pi} = \frac{-1}{\pi}$$

37. (4) We know that $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=\begin{cases}2\tan^{-1}x, & x\geq 0\\ -2\tan^{-1}x, & x\leq 0\end{cases}$

or $\lim\limits_{x\to 0^+}\frac{1}{x}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=\lim\limits_{x\to 0^+}\frac{2\tan^{-1}x}{x}=2$

and $\lim\limits_{x\to 0^-}\frac{1}{x}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=\lim\limits_{x\to 0^-}\left[-\frac{2\tan^{-1}x}{x}\right]=-2$

38. (1) $\lim\limits_{y\to 0}\left\{\frac{x\{\sec(x+y)-\sec x\}}{y}+\sec(x+y)\right\}$

$=\lim\limits_{y\to 0}\left[\frac{x}{y}\left\{\frac{\cos x-\cos(x+y)}{\cos(x+y)\cos x}\right\}\right]+\lim\limits_{y\to 0}\sec(x+y)$

$=\lim\limits_{y\to 0}\left[\frac{x\,2\sin\left(x+\frac{y}{2}\right)\sin\left(\frac{y}{2}\right)}{y\cos(x+y)\cos x}\right]+\sec x$

$=\lim\limits_{y\to 0}\left[\frac{x\sin\left(x+\frac{y}{2}\right)}{\cos(x+y)\cos x}\times\frac{\sin\left(\frac{y}{2}\right)}{\frac{y}{2}}\right]+\sec x$

$=x\tan x\sec x+\sec x$

$=\sec x\,(x\tan x+1)$

39. (1) $\lim\limits_{x\to 1}\frac{1+\sin\pi\left(\frac{3x}{1+x^2}\right)}{1+\cos\pi x}=\lim\limits_{x\to 1}\frac{1-\cos\left(\frac{3\pi}{2}-\frac{3\pi x}{1+x^2}\right)}{1-\cos(\pi-\pi x)}$

$=\lim\limits_{x\to 1}\frac{2\sin^2\left(\frac{3\pi}{4}-\frac{3\pi x}{2(1+x^2)}\right)}{2\sin^2\left(\frac{\pi}{2}-\frac{\pi x}{2}\right)}$

$=\lim\limits_{x\to 1}\left(\frac{\frac{3\pi}{4}-\frac{3\pi x}{2(1+x^2)}}{\frac{\pi}{2}-\frac{\pi x}{2}}\right)^2$

$=\lim\limits_{x\to 1}9\left(\frac{\frac{1}{2}-\frac{x}{1+x^2}}{1-x}\right)^2$

$=\lim\limits_{x\to 1}9\left(\frac{x-1}{2(1+x^2)}\right)^2=0$

40. (2) $\lim\limits_{n\to\infty}\cos^{2n}x=\begin{cases}1, & x=r\pi, r\in I\\ 0, & x\neq r\pi, r\in I\end{cases}$

Here, for $x=10$, $\lim\limits_{n\to\infty}\cos^{2n}(x-10)=1$ and in all other cases, it is zero. Therefore,

$$\lim_{n\to\infty}\sum_{x=1}^{\infty}\cos^{2n}(x-10)=1$$

41. (2) $\lim\limits_{x\to -1}\left(\frac{x^4+x^2+x+1}{x^2-x+1}\right)^{\lim\limits_{x\to -1}\frac{1-\cos(x+1)}{(x+1)^2}}$

$=\left(\frac{2}{3}\right)^{\lim\limits_{x\to -1}\frac{\sin^2\left(\frac{x+1}{2}\right)}{2\left(\frac{x+1}{2}\right)^2}}$

$=\left(\frac{2}{3}\right)^{\frac{1}{2}}$

42. (2) Given limit is

$\lim\limits_{x\to\infty}(x+1)[\tan^{-1}(x+5)-(x+1)]+4\tan^{-1}(x+5)$

$=\lim\limits_{x\to\infty}\left[(x+1)\tan^{-1}\frac{4}{1+(x+1)(x+5)}+4\tan^{-1}(x+5)\right]$

$=\lim\limits_{x\to\infty}\left[(x+1)\frac{\tan^{-1}\frac{4}{x^2+6x+6}}{\left(\frac{4}{x^2+6x+6}\right)}\times\frac{4}{x^2+6x+6}+4\tan^{-1}(x+5)\right]$

$=0+4\times\frac{\pi}{2}=2\pi$

43. (2) We know that $\lim\limits_{x\to 0}\frac{\sin x}{x}\to 1^-$ and $\lim\limits_{x\to 0}\frac{x}{\sin x}\to 1^-$.

So, $\lim\limits_{x\to 0}\left[100\frac{x}{\sin x}\right]+\lim\limits_{x\to 0}\left[99\frac{\sin x}{x}\right]=100+98=198.$

44. (3) $\lim\limits_{x\to a}\sqrt{a^2-x^2}\cot\frac{\pi}{2}\sqrt{\frac{a-x}{a+x}}$

$=\lim\limits_{x\to a}\frac{\sqrt{a^2-x^2}}{\tan\frac{\pi}{2}\sqrt{\frac{a-x}{a+x}}}$

$=\frac{2}{\pi}\lim\limits_{x\to a}\frac{\frac{\pi}{2}\sqrt{\frac{a-x}{a+x}}}{\tan\frac{\pi}{2}\sqrt{\frac{a-x}{a+x}}}(a+x)=\frac{4a}{\pi}$

45. (2) $\min(y^2-4y+11)=\min[(y-2)^2+7]=7$

or $L=\lim\limits_{x\to 0}\left[\min(y^2-4y+11)\frac{\sin x}{x}\right]$

$=\lim\limits_{x\to 0}\left[\frac{7\sin x}{x}\right]$

= [a value slightly lesser than 7]

$(|\sin x|<|x|$, when $x\to 0)$

$=6$

46. (4) $\lim\limits_{x\to 0}\frac{1-\cos x\sqrt{\cos 2x}}{x^2}$

$=\lim\limits_{x\to 0}\frac{(1-\cos x\sqrt{\cos 2x})(1+\cos x\sqrt{\cos 2x})}{x^2(1+\cos x\sqrt{\cos 2x})}$

$=\lim\limits_{x\to 0}\frac{1-\cos^2 x(\cos 2x)}{2x^2}$

$$= \lim_{x\to 0} \frac{1-\cos^2 x(2\cos^2 x-1)}{2x^2}$$

$$= \lim_{x\to 0} \frac{(1-\cos^2 x)\,(1+2\cos^2 x)}{2x^2}$$

$$= \lim_{x\to 0} \frac{\sin^2 x\cdot(1+2\cos^2 x)}{2x^2}$$

$$= \frac{3}{2}$$

$$\lim_{x\to\infty}\frac{1}{x+1}\tan\left(\frac{\pi x+1}{2x+2}\right) = \lim_{x\to\infty}\frac{1}{x+1}\cot\left(\frac{\pi}{2}-\frac{\pi x+1}{2x+2}\right)$$

$$= \lim_{x\to\infty}\frac{1}{x+1}\cot\left(\frac{2\pi-2}{4(x+1)}\right)$$

$$= \frac{2}{\pi-1}\cdot\lim_{x\to\infty}\frac{\frac{\pi-1}{2(x+1)}}{\tan\frac{(\pi-1)}{2(x+1)}}$$

$$= \frac{2}{\pi-1}$$

) We have

$$\lim_{x\to 1}\frac{1-\sqrt{x}}{\left(\cos^{-1}x\right)^2} = \lim_{x\to 1}\frac{\left(1-\sqrt{x}\right)\left(1+\sqrt{x}\right)}{\left(\cos^{-1}x\right)^2\left(1+\sqrt{x}\right)}$$

$$= \lim_{x\to 1}\frac{1-x}{\left(\cos^{-1}x\right)^2\left(1+\sqrt{x}\right)}$$

$$= \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2\left(1+\sqrt{\cos\theta}\right)}, \text{ where } x=\cos\theta$$

$$[\because x\to 1 \text{ or } \cos\theta\to 1 \text{ or } \theta\to 0]$$

$$= \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}\cdot\frac{1}{\left(1+\sqrt{\cos\theta}\right)}$$

$$= \lim_{\theta\to 0}\frac{2\sin^2\frac{\theta}{2}}{4\frac{\theta^2}{4}}\left(\frac{1}{1+\sqrt{\cos\theta}}\right)$$

$$= \frac{1}{2}\lim_{\theta\to 0}\left(\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right)^2\frac{1}{\left(1+\sqrt{\cos\theta}\right)}$$

$$= \frac{1}{2}(1)^2\frac{1}{(1+1)} = \frac{1}{4}$$

(2) $$L = \lim_{x\to\pi/2}\frac{\sin(x\cos x)}{\sin\left(\frac{\pi}{2}-x\sin x\right)}$$

$$= \lim_{x\to\frac{\pi}{2}}\frac{\sin(x\cos x)}{(x\cos x)}\cdot\frac{\left(\frac{\pi}{2}-x\sin x\right)}{\sin\left(\frac{\pi}{2}-x\sin x\right)}\cdot\frac{x\cos x}{\left(\frac{\pi}{2}-x\sin x\right)}$$

$$= 1\times 1\lim_{x\to\pi/2}\frac{x\cos x}{\left(\frac{\pi}{2}-x\sin x\right)}$$

Put $x = \pi/2 + h$. Then

$$L = \lim_{h\to 0}\frac{\left(\frac{\pi}{2}+h\right)\cos\left(\frac{\pi}{2}+h\right)}{\frac{\pi}{2}-\left(\frac{\pi}{2}+h\right)\sin\left(\frac{\pi}{2}+h\right)}$$

$$= \lim_{h\to 0}\frac{-\left(\frac{\pi}{2}+h\right)\sin h}{\frac{\pi}{2}(1-\cos h)-h\cos h}$$

$$= \lim_{h\to 0}\frac{-\left(\frac{\pi}{2}+h\right)\left(\frac{\sin h}{h}\right)}{\frac{\pi}{2}\frac{(1-\cos h)}{h}-\cos h} \text{ (Divide N}^r \text{ and D}^r \text{ by } h)$$

$$= \frac{-\left(\frac{\pi}{2}+0\right)1}{0-1} = \frac{\pi}{2}$$

50. (1) $$\lim_{x\to 0^+}\left[\left(1-e^x\right)\frac{\sin x}{|x|}\right] = \lim_{x\to 0^+}\left[(0^-)\frac{\sin x}{x}\right] = [0^-] = -1.$$

$$\lim_{x\to 0^-}\left[\left(1-e^x\right)\frac{\sin x}{|x|}\right] = \lim_{x\to 0^-}\left[(0^+)\frac{\sin x}{-x}\right] = [0^-] = -1$$

Hence, $$\lim_{x\to 0}\left[\left(1-e^x\right)\frac{\sin x}{|x|}\right] = -1.$$

51. (4) $$\lim_{x\to 0}\frac{x\left(e^x-1\right)}{1-\cos x} = \lim_{x\to 0}\frac{2x\left(e^x-1\right)}{4\sin^2\frac{x}{2}}$$

$$= 2\lim_{x\to 0}\left[\frac{(x/2)^2}{\sin^2\frac{x}{2}}\right]\left(\frac{e^x-1}{x}\right) = 2$$

52. (2) $$f(x) = \lim_{n\to\infty} n(x^{1/n}-1)$$

$$= \lim_{n\to\infty}\frac{x^{1/n}-1}{1/n}$$

$$= \lim_{m\to 0}\frac{x^m-1}{m} \qquad \left(\text{where } \frac{1}{n} \text{ is replaced by } m\right)$$

$$= \ln x$$

or $$f(xy) = \ln(xy) = \ln x + \ln y = f(x) + f(y)$$

53. (1) $$(1+x)^{2/x} = (1+x)^{2/x} - [(1+x)^{2/x}]$$

Now, $$\lim_{x\to 0}(1+x)^{2/x} = e^2$$

or $$\lim_{x\to 0}\{(1+x)^{2/x}\} = e^2 - [e^2] = e^2 - 7$$

54. (2) $$L = \lim_{x\to\infty}\frac{(2^{x^n})^{\frac{1}{e^x}}-(3^{x^n})^{\frac{1}{e^x}}}{x^n} = \lim_{x\to\infty}\frac{(3)^{\frac{x^n}{e^x}}\left(\left(\frac{2}{3}\right)^{\frac{x^n}{e^x}}-1\right)}{x^n}$$

Now, $$\lim_{x\to\infty}\frac{x^n}{e^x} = \lim_{x\to\infty}\frac{n!}{e^x} = 0$$

(Differentiating numerator and denominator n times for L′ Hospital's rule)

Hence, $L = \lim_{x\to\infty}(3)^{\frac{x^n}{e^x}} \lim_{x\to\infty} \frac{\left(\left(\frac{2}{3}\right)^{\frac{x^n}{e^x}} - 1\right)}{\frac{x^n}{e^x}} \lim_{x\to\infty} \frac{1}{e^x}$

$= 1 \times \log(2/3) \times 0 = 0$

55. (2) $\lim_{x\to 0} \frac{\sin(x^2)}{\ln(\cos(2x^2 - x))}$

$= \lim_{x\to 0} \frac{\sin(x^2)}{\log\left(1 - 2\sin^2\left(\frac{2x^2 - x}{2}\right)\right)}$

$= \lim_{x\to 0} \frac{\sin(x^2)x^2}{x^2 \log\left(1 - 2\sin^2\left(\frac{2x^2 - x}{2}\right)\right)} \left[-2\sin^2\left(\frac{2x^2 - x}{2}\right)\right]$

$= \lim_{x\to 0} - \frac{x^2}{\frac{2\sin^2\left(\frac{2x^2 - x}{2}\right)}{\left(\frac{2x^2 - x}{2}\right)^2} \left(\frac{2x^2 - x}{2}\right)^2}$.

$= \lim_{x\to 0} - \frac{2x^2}{(2x^2 - x)^2} = \lim_{x\to 0} - \frac{2}{(2x-1)^2} = -2$

56. (4) $\lim_{x\to\infty} \frac{e^{1/x^2} - 1}{2\tan^{-1}(x^2) - \pi} = \lim_{t\to 0^+} \frac{e^{t^2} - 1}{2\cot^{-1} t^2 - \pi}$

$= \lim_{t\to 0^+} \frac{e^{t^2} - 1}{-2\tan t^2}$

$= \lim_{t\to 0^+} -\frac{1}{2} \frac{e^{t^2} - 1}{t^2 \frac{\tan t^2}{t^2}} = -\frac{1}{2}$

57. (3) $\lim_{x\to 0} \frac{(2^m + x)^{1/m} - (2^n + x)^{1/n}}{x}$

$= \lim_{x\to 0} \frac{(2^m + x)^{1/m} - 2}{x} - \lim_{x\to 0} \frac{(2^n + x)^{1/n} - 2}{x}$

$= \lim_{a\to 2} \frac{a - 2}{a^m - 2^m} - \lim_{b\to 2} \frac{b - 2}{b^n - 2^n}$

[Putting $2^m + x = a^m$ and $2^n + x = b^n$]

$= \frac{1}{m2^{m-1}} - \frac{1}{n2^{n-1}}$

58. (3) $\lim_{n\to\infty} \left[\frac{1}{n} + \frac{e^{1/n}}{n} + \frac{e^{2/n}}{n} + \cdots + \frac{e^{(n-1)/n}}{n}\right]$

$= \lim_{n\to\infty} \left[\frac{1 + e^{1/n} + (e^{1/n})^2 + \cdots + (e^{1/n})^{n-1}}{n}\right]$

$= \lim_{n\to\infty} \frac{1.\left[(e^{1/n})^n - 1\right]}{n(e^{1/n} - 1)} = (e-1) \lim_{n\to\infty} \frac{1}{\left(\frac{e^{1/n} - 1}{1/n}\right)}$

$= (e-1) \times 1 = (e-1)$

59. (3) $L = \lim_{x\to 1} \frac{nx^n(x-1) - (x^n - 1)}{(e^x - e)\sin \pi x}$

Put $x = 1 + h$ so that as $x \to 1$, $h \to 0$. Therefore

$L = -\lim_{h\to 0} \frac{h \cdot n(1+h)^n - \left((1+h)^n - 1\right)}{e(e^h - 1)\sin \pi h}$

$= -\lim_{h\to 0} \frac{n \cdot h(1 + {}^nC_1 h + {}^nC_2 h^2 + {}^nC_3 h^3 + \cdots) - (1 + {}^nC_1 h + {}^nC_2 h^2 + {}^nC_3 h^3 + \cdots - 1)}{\pi e(h^2)\left(\frac{e^h - 1}{h}\right)\left(\frac{\sin \pi h}{\pi h}\right)}$

$= -\frac{n^2 - {}^nC_2}{\pi e} = -\left[\frac{2n^2 - n(n-1)}{2\pi e}\right]$

$= -\frac{n^2 + n}{2(\pi e)} = -\frac{n(n+1)}{2(\pi e)}$

If $n = 100$, then $L = -\left(\frac{5050}{\pi e}\right)$.

60. (2) $\lim_{x\to 0} \frac{\log(1 + x + x^2) + \log(1 - x + x^2)}{\sec x - \cos x}$

$= \lim_{x\to 0} \frac{\log\left[(1+x^2)^2 - x^2\right]}{(1 - \cos^2 x)/\cos x}$

$= \lim_{x\to 0} \frac{\log(1 + x^2 + x^4)}{\sin x \tan x}$

$= \lim_{x\to 0} \frac{\log(1 + x^2(1+x^2))}{x^2(1+x^2)} \cdot x^2(1+x^2) \cdot \frac{1}{\frac{\sin x}{x} \cdot \frac{\tan x}{x} \cdot x^2}$

$= 1$ $\left(\text{as } \lim_{x\to 0} \frac{\log(1+x)}{x} = 1\right)$

61. (3) $\lim_{x\to\infty} \left(\sqrt[3]{x^3 + 2x^2} - \sqrt{x^2 + x}\right)$

$= \lim_{x\to\infty} x\left[\left(1 + \frac{2}{x}\right)^{\frac{1}{3}} - \left(1 + \frac{1}{x}\right)^{\frac{1}{2}}\right]$

$= \lim_{x\to\infty} x\left[\left(1 + \frac{1}{3} \cdot \frac{2}{x} + \ldots\right) - \left(1 + \frac{1}{2} \cdot \frac{1}{x} + \ldots\right)\right]$

$= \lim_{x\to\infty} x\left(\frac{2}{3x} - \frac{1}{2x}\right)$

$= \frac{1}{6}$

62. (2) $\lim_{x\to 0} \frac{1 + \sin x - \cos x + \log(1 - x)}{x^3}$

$= \lim_{x\to 0} \frac{1 + \left(x - \frac{x^3}{3!} + \cdots\right) - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots\right)}{x^3}$

$= -\frac{1}{3!} - \frac{1}{3} = -\frac{1}{2}$

2) $\cos(\tan x) - \cos x = 2\sin\left(\frac{x+\tan x}{2}\right)\sin\left(\frac{x-\tan x}{2}\right)$

or $\lim_{x\to 0}\frac{\cos(\tan x)-\cos x}{x^4}$

$$=\lim_{x\to 0}\frac{2\sin\left(\frac{x+\tan x}{2}\right)\sin\left(\frac{x-\tan x}{2}\right)}{x^4}$$

$$=\lim_{x\to 0}\frac{2\sin\left(\frac{x+\tan x}{2}\right)\sin\left(\frac{x-\tan x}{2}\right)}{x^4\left(\frac{x+\tan x}{2}\right)\left(\frac{x-\tan x}{2}\right)}\left(\frac{x^2-\tan^2 x}{4}\right)$$

$$=\frac{1}{2}\lim_{x\to 0}\frac{x^2-\tan^2 x}{x^4}$$

$$=\frac{1}{2}\lim_{x\to 0}\frac{x^2-\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+\cdots\right)^2}{x^4}$$

$$=\frac{1}{2}\lim_{x\to 0}\frac{1}{x^2}\left(1-\left(1+\frac{x^2}{3}+\frac{2}{15}x^4+\cdots\right)^2\right)=-\frac{1}{3}$$

(1) $\lim_{x\to 0}\left(\frac{\sin 3x}{x^3}+\frac{a}{x^2}+b\right)=\lim_{x\to 0}\frac{\sin 3x+ax+bx^3}{x^3}$

$$=\lim_{x\to 0}\frac{3\frac{\sin 3x}{3x}+a+bx^2}{x^2}$$

For existence,

$(3+a)=0$

or $a=-3$

$\therefore\quad L=\lim_{x\to 0}\frac{\sin 3x-3x+bx^3}{x^3}$

$=27\lim_{t\to 0}\frac{\sin t-t}{t^3}+b=0\ (3x=t)$

$=-\frac{27}{6}+b=0$

or $b=\frac{9}{2}$

6. (2) $\lim_{x\to 0}\frac{x^n\sin^n x}{x^n-\sin^n x}=\lim_{x\to 0}\frac{x^n\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)^n}{x^n-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)^n}$

$$=\lim_{x\to 0}\frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)^n}{1-\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots\right)^n}$$

$$=\lim_{x\to 0}\frac{x^n\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots\right)^n}{1-\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots\right)^n}$$

For $n=2$,

$$\lim_{x\to 0}\frac{x^2\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots\right)^2}{1-\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots\right)^2}$$

$$=\lim_{x\to 0}\frac{x^2\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots\right)^2}{\left(2-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots\right)\left(\frac{x^2}{3!}-\frac{x^4}{5!}+\cdots\right)}$$

$$=\frac{1(1-0+\cdots)^2}{(2-0+0)\left(\frac{1}{3!}-0+\cdots\right)}$$

$=3$

66. (3) $\lim_{x\to 0}\left(\frac{1+\tan x}{1+\sin x}\right)^{\operatorname{cosec} x}=\lim_{x\to 0}\frac{(1+\tan x)^{\frac{1}{\sin x}}}{(1+\sin x)^{\frac{1}{\sin x}}}$

$$=\frac{\lim_{x\to 0}\left((1+\tan x)^{\frac{1}{\tan x}}\right)^{\frac{1}{\cos x}}}{(1+\sin x)^{\frac{1}{\sin x}}}$$

$$=\frac{e^{\frac{1}{\cos 0}}}{e}=1$$

67. (3) $\lim_{x\to 1}(2-x)^{\tan\frac{\pi x}{2}}=\lim_{x\to 1}\{1+(1-x)\}^{\tan\frac{\pi x}{2}}$

$$=e^{\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2}}$$

$$=e^{\lim_{x\to 1}(1-x)\cot\left(\frac{\pi}{2}-\frac{\pi x}{2}\right)}$$

$$=e^{\lim_{x\to 1}\frac{(1-x)}{\tan\left(\frac{\pi}{2}-\frac{\pi x}{2}\right)}}$$

$$=e^{\frac{2}{\pi}\lim_{x\to 1}\frac{\frac{\pi}{2}(1-x)}{\tan\left(\frac{\pi}{2}(1-x)\right)}}$$

$$=e^{2/\pi}$$

68. (1) $\lim_{m\to\infty}\left(\cos\frac{x}{m}\right)^m=\lim_{m\to\infty}\left[1-\left(1-\cos\frac{x}{m}\right)\right]^m$

$$=\lim_{m\to\infty}\left[1-2\sin^2\frac{x}{2m}\right]^m$$

$$=e^{\lim_{m\to\infty}\left(-2\sin^2\frac{x}{2m}\right)m}=1$$

69. (2) $\lim_{n\to\infty}\left(\frac{n^2-n+1}{n^2-n-1}\right)^{n(n-1)}=\lim_{n\to\infty}\left(\frac{n(n-1)+1}{n(n-1)-1}\right)^{n(n-1)}$

$$=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n(n-1)}\right)^{n(n-1)}}{\left(1-\frac{1}{n(n-1)}\right)^{n(n-1)}}=\frac{e}{e^{-1}}=e^2$$

70. (3) 1^∞ form

$$L = e^{\lim_{n\to\infty} n\left(\left(\frac{n}{n+1}\right)^\alpha + \sin\frac{1}{n} - 1\right)} = e^{\lim_{n\to\infty} n\sin\frac{1}{n} + \lim_{n\to\infty} n\left(\left(\frac{n}{n+1}\right)^\alpha - 1\right)}$$

Consider $\lim_{n\to\infty} n\left(\left(\frac{n}{n+1}\right)^\alpha - 1\right) = \lim_{n\to\infty} n\left(\left(\frac{1}{1+1/n}\right)^\alpha - 1\right)$

Put $n = \frac{1}{y}$. Then

$$\lim_{y\to 0}\frac{1}{y}\left(\left(\frac{1}{1+y}\right)^\alpha - 1\right) = \lim_{y\to 0}\frac{1-(1+y)^\alpha}{y} = -\alpha$$

$\therefore \quad L = e^{1-\alpha}$

71. (3) $\lim_{x\to\infty}\left[\left(\frac{e}{1-e}\right)\left(\frac{1}{e} - \frac{x}{1+x}\right)\right]^x = e^{\lim_{x\to\infty}\left[\left(\frac{e}{1-e}\right)\left(\frac{1}{e}-\frac{x}{1+x}\right)-1\right]x}$

$$= e^{\lim_{x\to\infty}\left[\frac{1}{1-e} - \frac{e}{1-e}\frac{x}{1+x} - 1\right]x}$$

$$= e^{\lim_{x\to\infty}\left[\frac{e}{1-e} - \frac{e}{1-e}\frac{x}{1+x}\right]x}$$

$$= e^{\lim_{x\to\infty}\frac{e}{1-e}\left[1-\frac{x}{1+x}\right]x}$$

$$= e^{\frac{e}{1-e}\lim_{x\to\infty}\frac{x}{1+x}}$$

$$= e^{\frac{e}{1-e}\lim_{x\to\infty}\frac{1}{1+\frac{1}{x}}}$$

$$= e^{\frac{e}{1-e}}$$

72. (2) $\lim_{x\to 0}\left(\frac{1^x + 2^x + \cdots + n^x}{n}\right)^{1/x} = e^{\lim_{x\to 0}\left(\frac{1^x-1}{n} + \frac{2^x-1}{n} + \cdots + \frac{n^x-1}{n}\right)\frac{1}{x}}$

$$= e^{\lim_{x\to 0}\frac{1}{n}\left\{\frac{1^x-1}{x} + \frac{2^x-1}{x} + \cdots + \frac{n^x-1}{x}\right\}}$$

$$= e^{\frac{1}{n}[\log 1 + \log 2 + \cdots + \log n]}$$

$$= e^{\frac{1}{n}(\log n!)}$$

$$= e^{\log(n!)^{\frac{1}{n}}} = (n!)^{\frac{1}{n}}$$

73. (3) $\lim_{x\to 1}\frac{p - q + qx^p - px^q}{1 - x^p - x^q + x^{p+q}} \quad \left(\frac{0}{0}\right)$

$$= \lim_{x\to 1}\frac{pqx^{p-1} - pqx^{q-1}}{-px^{p-1} - qx^{q-1} + (p+q)x^{p+q-1}} \quad \left(\frac{0}{0}\right)$$

(L' Hospital's rule)

$$= \lim_{x\to 1}\frac{pq(p-1)x^{p-2} - pq(q-1)x^{q-2}}{-p(p-1)x^{p-2} - q(q-1)x^{q-2} + (p+q)(p+q-1)x^{p+q-2}}$$

(L' Hospital's rule)

$$= \frac{p-q}{2}$$

74. (1) $\lim_{x\to\infty}\frac{(\log x)^3 + x\cdot 3(\log x)^2\times\frac{1}{x}}{1+2x}$ (Using L' Hospital's rule)

$$= \lim_{x\to\infty}\frac{3(\log x)^2\times\frac{1}{x} + 6(\log x)\times\frac{1}{x}}{2}$$

$$= \lim_{x\to\infty}\frac{3(\log x)^2 + 6\log x}{2x}$$

$$= \lim_{x\to\infty}\frac{6\log x\times\frac{1}{x} + \frac{6}{x}}{2}$$ (Using L' Hospital's rule)

$$= \lim_{x\to\infty}\frac{6\log x + 6}{2x}$$

$$= \lim_{x\to\infty}\frac{6\left(\frac{1}{x}\right) + 0}{2}$$ (Using L' Hospital's rule)

$$= \frac{\frac{6}{\infty}}{2} = 0$$

75. (2) $\lim_{x\to\infty}\frac{\cot^{-1}(x^{-a}\log_a x)}{\sec^{-1}(a^x\log_x a)}$ $(a > 1)$

$$= \lim_{x\to\infty}\frac{\cot^{-1}\left(\frac{\log_a x}{x^a}\right)}{\sec^{-1}\left(\frac{a^x}{\log_a x}\right)}$$

as $x\to\infty$, $\left(\frac{\log_a x}{x^a}\right)\to 0$ and $\left(\frac{a^x}{\log_a x}\right)\to\infty$

(using L'Hospital rule)

$\therefore \quad I = \frac{\pi/2}{\pi/2} = 1$

76. (3) $L = \lim_{n\to\infty}\frac{e^n}{\left(1+\frac{1}{n}\right)^{n^2}}$

$$= \lim_{n\to\infty}\frac{e^n}{e^{n^2\log_e\left(1+\frac{1}{n}\right)}}$$

$$= \lim_{n\to\infty} e^{n - n^2\log_e\left(1+\frac{1}{n}\right)}$$

Put $n = \frac{1}{y}$

$\therefore \quad L = \lim_{y\to 0} e^{\frac{y - \log_e(1+y)}{y^2}}$

$$= \lim_{y\to 0} e^{\frac{y - \left(y - \frac{y^2}{2} + \cdots\right)}{y^2}}$$

$$= \lim_{y\to 0} e^{1/2}$$

$$= e^{1/2}$$

iple Correct Answers Type

1. **(2), (3)**

$$\text{R.H.L.} = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} a(1+h) = a$$

$$\text{L.H.L.} = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} a\left\{1+\frac{2}{a}(1-h)\right\} = 1+\frac{2}{a}$$

$\lim_{x\to 1} f(x)$ exists $\Rightarrow$ R.H.L. = L.H.L. Therefore,

$$a = 1+\frac{2}{a} \text{ or } a = 2, -1$$

2. **(1), (4)**

$$f(1+0) = \lim_{h\to 0}\left(|1+h-1| - [1+h]\right) = \lim_{h\to 0}(h-1) = -1$$

$$f(1-0) = \lim_{h\to 0}\left(|1-h-1| - [1-h]\right) = \lim_{h\to 0}(h-0) = 0$$

3. **(1), (3)**

$$\text{Limit} = \lim_{n\to\infty} \frac{an(1+n)-\left(1+n^2\right)}{1+n} = \lim_{n\to\infty} \frac{(a-1)n^2+an-1}{n+1}$$

$= \infty$ if $a-1 \neq 0$

If $a-1=0$, limit $= \lim_{n\to\infty} \frac{an-1}{n+1} = a = b$

$\therefore \quad a = b = 1$

4. **(1), (2), (3)**

$$L = \lim_{x\to 0} \frac{\sin x^n}{(\sin x)^m} = \lim_{x\to 0} \frac{\frac{\sin x^n}{x^n}x^n}{\frac{(\sin x)^m}{x^m}x^m} = \lim_{x\to 0} x^{n-m}$$

If $n = m$, then

$L =$ (a very small value near to zero)$^{\text{exactly zero}} = 1$

If $n > m$, then

$L =$ (a very small value near to zero)$^{\text{positive integer}} = 0$

If $n < m$, then

$L =$ (a very small value near to zero)$^{\text{negative integer}} = \infty$

5. **(1), (2)**

For $n>0$, $\lim_{x\to 0} x^n \sin(1/x^2) = 0 \times$ (any value between -1 to 1) $= 0$

For $n<0$, $\lim_{x\to 0} x^n \sin(1/x^2) = \infty \times$ (any value between -1 to 1) $= \pm\infty$

6. **(1), (2), (3)**

$$L = \lim_{x\to 0} \frac{1}{x^3}\left(\frac{1}{\sqrt{1+x}} - \frac{1+ax}{1+bx}\right)$$

$$= \lim_{x\to 0} \frac{1+bx-(1+ax)\sqrt{1+x}}{x^3} \lim_{x\to 0} \frac{1}{\sqrt{1+x}\ (1+bx)}$$

$$= \lim_{x\to 0} \frac{1+bx-(1+ax)(1+x)^{1/2}}{x^3}$$

$$= \lim_{x\to 0} \frac{1+bx-(1+ax)\left(1+\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3+\ldots\right)}{x^3}$$

$$= \lim_{x\to 0} \frac{1+bx-(1+ax)\left(1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\ldots..\right)}{x^3}$$

$$= \lim_{x\to 0} \frac{\left(b-\frac{1}{2}-a\right)x+\left(-\frac{a}{2}+\frac{1}{8}\right)x^2+\left(\frac{a}{8}-\frac{1}{16}\right)x^3}{x^3}$$

Limit exists, if $b-\frac{1}{2}-a=0$ and $-\frac{1}{8}+\frac{a}{2}=0$

$\therefore \quad a = \frac{1}{4}$ and $b = \frac{3}{4}$

$\therefore \quad L = -\frac{1}{32}$

7. **(1), (2), (3)**

$$\lim_{x\to\infty} \frac{\log_e x}{\{x\}} = \frac{\text{positive infinity}}{\text{a value between 0 and 1}} = \infty$$

$$\lim_{x\to 2^+} \frac{x}{x^2-x-2} = \lim_{x\to 2^+} \frac{x}{(x-2)(x+1)} = \lim_{h\to 0} \frac{2+h}{h(3+h)} = \infty$$

$$\lim_{h\to -1} \frac{x}{x^2-x-2} = -\lim_{h\to -1^-} \frac{x}{(x-2)(x+1)}$$

$$= \lim_{h\to 0} \frac{-1-h}{(-3-h)(-h)} = -\lim_{h\to 0} \frac{1+h}{(3+h)(h)} = -\infty$$

8. **(1), (2), (3)**

(1) $$\lim_{x\to 0^+} \frac{[x+|x|]}{x} = \lim_{x\to 0^+} \frac{[x+x]}{x} = \lim_{x\to 0^+} \frac{[2x]}{x} = 0$$

$$\lim_{x\to 0^-} \frac{[x+|x|]}{x} = \lim_{x\to 0^-} \frac{[x-x]}{x} = \lim_{x\to 0^-} \frac{0}{x} = 0$$

(2) $$\lim_{x\to 0^+} \frac{xe^{\frac{1}{x}}}{1+e^{\frac{1}{x}}} = \lim_{x\to 0^+} \frac{x}{1+e^{-\frac{1}{x}}} = 0$$

$$\lim_{x\to 0^-} \frac{xe^{\frac{1}{x}}}{1+e^{\frac{1}{x}}} = 0$$

(3) $$\lim_{x\to 3^+} (x-3)^{\frac{1}{5}} \operatorname{sgn}(x-3) = \lim_{x\to 3^+} (x-3)^{\frac{1}{5}} \times 1 = 0$$

$$\lim_{x\to 3-} (x-3)^{\frac{1}{5}} \operatorname{sgn}(x-3) = \lim_{x\to 3^-} (x-3)^{\frac{1}{5}} \times (-1) = 0$$

(4) $$\lim_{x\to 0^+} \frac{\tan^{-1}|x|}{x} = \lim_{x\to 0^+} \frac{\tan^{-1}x}{x} = 1$$

$$\lim_{x\to 0^-} \frac{\tan^{-1}|x|}{x} = \lim_{x\to 0^+} \frac{\tan^{-1}(-x)}{x} = -1$$

9. **(2), (3)**

Since the greatest integer function is discontinuous at integral values of x, for a given limit to exist both left- and right-hand limits must be equal.

$$\text{L.H.L.} = \lim_{x\to 1^-} (2-x+a[x-1]+b[1+x])$$

$$= 2-1+a(-1)+b(1) = 1-a+b$$

$$\text{R.H.L.} = \lim_{x\to 1^+} (2-x+a[x-1]+b[1+x])$$

$$= 2-1+a(0)+b(2) = 1+2b$$

On comparing, we have $-a = b$.

$$L = \lim_{x\to a}\frac{|2\sin x-1|}{2\sin x-1}$$

For $a = \pi/6$,

$$\text{L.H.L.} = \lim_{x\to \frac{\pi^-}{6}}\frac{1-2\sin x}{2\sin x-1} = -1$$

$$\text{R.H.L.} = \lim_{x\to \frac{\pi^+}{6}}\frac{2\sin x-1}{2\sin x-1} = 1$$

Hence, the limit does not exist.

For $a = \pi$, $\lim_{x\to\pi}\frac{1-2\sin x}{2\sin x-1} = -1$ (as in neighborhood of π, $\sin x$ is less than ½).

For $a = \frac{\pi}{2}$, $\lim_{x\to\pi/2}\frac{2\sin x-1}{2\sin x-1} = 1$ (as in neighborhood of $\pi/2$, $\sin x$ approaches 1).

11. (2), (3), (4)

$$f(x) = \lim_{x\to\infty}\frac{x}{x^{2n}+1} = \begin{cases} x, & x^2<1 \\ 0, & x^2>1 \\ 1/2, & x=1 \\ -1/2, & x=-1 \end{cases}$$

$\therefore \quad f(1^+) = f(-1^-) = 0$

$f(1^-) = 1,\ f(-1^+) = -1$

$f(1) = 1/2$

12. (1), (3)

$$\lim_{n\to\infty}\frac{-3+\frac{(-1)^n}{n}}{4+\frac{(-1)^n}{n}} = \frac{-3}{4}$$

13. (1), (2), (3), (4)

We have $\lim_{x\to0^+} f(x) = \lim_{x\to0^+}\frac{\tan^2\{x\}}{\left(x^2-[x]^2\right)} = \lim_{x\to0^+}\frac{\tan^2 x}{x^2} = 1$ (1)

$(\because \quad x\to0^+,\ [x]=0 \Rightarrow \{x\}=x)$

Also, $\lim_{x\to0^-} f(x) = \lim_{x\to0^-}\sqrt{\{x\}\cot\{x\}} = \sqrt{\cot 1}$ (2)

$(\because \quad x\to0^-,\ [x]=-1 \Rightarrow \{x\}=x+1 \Rightarrow \{x\}\to1)$

Also, $\cot^{-1}\left(\lim_{x\to0^-} f(x)\right)^2 = \cot^{-1}(\cot 1) = 1.$

Also, $\tan^{-1}\left(\lim_{x\to0^+} f(x)\right) = \tan^{-1}1 = \frac{\pi}{4}$

14. (1), (2), (3), (4)

$$f(x) = \frac{3x^2+ax+a+1}{(x+2)(x-1)}$$

As $x\to1$, $D^r\to0$. Hence as $x\to1$, $N^r\to0$. Therefore,

$3+2a+1=0$ or $a=-2$

As $x\to-2$, $D^r\to0$. Hence as $x\to-2$, $N^r\to0$. Therefore,

$12-2a+a+1=0$ or $a=13$

Now, $\lim_{x\to1} f(x) = \lim_{x\to1}\frac{3x^2-2x-1}{(x+2)(x-1)} = \lim_{x\to1}\frac{(3x+1)(x-1)}{(x+2)(x-1)} = \frac{4}{3}$

Now, $\lim_{x\to-2}\frac{3x^2+13x+14}{(x+2)(x-1)} = \lim_{x\to-2}\frac{(3x+7)(x+2)}{(x+2)(x-1)} = -\frac{1}{3}$

15. (2), (3)

Case I: $x\neq m\pi$ (m is an integer)

$$\lim_{x\to\infty}\frac{1}{1+n\sin^2 nx} = \frac{1}{\infty} = 0$$

Case II: $x = m\pi$ (m is an integer)

$$\lim_{n\to\infty}\frac{1}{1+n\sin^2 nx} = \frac{1}{1} = 1$$

16. (1), (2), (3)

$$\lim_{x\to5^-}\frac{x^2-9x+20}{x-[x]} = \lim_{x\to5^-}\frac{(x-5)(x-4)}{x-4} = \lim_{x\to5^-}(x-5) = 0$$

$$\lim_{x\to5^+}\frac{x^2-9x+20}{x-[x]} = \lim_{x\to5^+}\frac{(x-5)(x-4)}{x-5} = \lim_{x\to5^+}(x-4) = 1$$

Hence, limit does not exist.

17. (1), (3)

Since $x^2>0$ and limit equals 2, $f(x)$ must be a positive quantity. Also, since $\lim_{x\to0}\frac{f(x)}{x^2} = 2.$, denominator → zero and limit is finite. Therefore, $f(x)$ must be approaching zero or $\lim_{x\to0}[f(x)] = 0^+$.

Hence, $\lim_{x\to0}[f(x)] = 0^+$.

$$\lim_{x\to0^+}\left[\frac{f(x)}{x}\right] = \lim_{x\to0^+}\left[x\frac{f(x)}{x^2}\right] = 0$$

and $\lim_{x\to0^-}\left[\frac{f(x)}{x}\right] = \lim_{x\to0^-}\left[x\frac{f(x)}{x^2}\right] = -1$

Hence, $\lim_{x\to0}\left[\frac{f(x)}{x}\right]$ does not exist.

18. (1), (2), (3)

$$f(\alpha) = \lim_{x\to2}\left(\sin^x\alpha+\cos^x\alpha\right)^{\frac{1}{(x-2)}} \qquad (1^\infty \text{ form})$$

$$= \begin{cases} e^{\lim\limits_{x\to2}\frac{\sin^x\alpha+\cos^x\alpha-1}{x-2}}, & \alpha\in\left(0,\frac{\pi}{2}\right) \\ 1, & \alpha = 0,\frac{\pi}{2} \end{cases}$$

Now $e^{\lim\limits_{x\to2}\frac{\sin^x\alpha+\cos^x\alpha-1}{x-2}} = e^{\lim\limits_{x\to2}\frac{\sin^x\alpha+\cos^x\alpha-\sin^2\alpha-\cos^2\alpha}{x-2}}$

$$= e^{\lim\limits_{x\to2}\frac{\sin^2\alpha(\sin^{x-2}\alpha-1)+\cos^2\alpha(\cos^{x-2}\alpha-1)}{x-2}}$$

$$= e^{\sin^2\alpha\log_e\sin\alpha+\cos^2\alpha\log_e\cos\alpha}$$

$$= e^{\log_e(\sin\alpha)^{\sin^2\alpha}+\log_e(\cos\alpha)^{\cos^2\alpha}}$$

$$= e^{\log_e(\sin\alpha)^{\sin^2\alpha}(\cos\alpha)^{\cos^2\alpha}}$$

$$= (\sin\alpha)^{\sin^2\alpha}(\cos\alpha)^{\cos^2\alpha}$$

$$\therefore \quad f(x) = \begin{cases} (\cos\alpha)^{\cos^2\alpha}\cdot(\sin\alpha)^{\sin^2\alpha}, & \alpha\in\left(0,\frac{\pi}{2}\right) \\ 1, & \alpha = 0,\frac{\pi}{2} \end{cases}$$

mprehension Type

), 2. (2), 3. (4)

We have $f(x) = \dfrac{\sin^{-1}(1-\{x\})\cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}(1-\{x\})}$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0} \frac{\sin^{-1}(1-\{0+h\})\cos^{-1}(1-\{0+h\})}{\sqrt{2\{0+h\}}(1-\{0+h\})}$$

$$= \lim_{h\to 0} \frac{\sin^{-1}(1-h)\cos^{-1}(1-h)}{\sqrt{2h}(1-h)}$$

$$= \lim_{h\to 0} \frac{\sin^{-1}(1-h)}{(1-h)} \lim_{h\to 0} \frac{\cos^{-1}(1-h)}{\sqrt{2h}}$$

In second limit, put $1 - h = \cos\theta$. Then

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} \frac{\sin^{-1}(1-h)}{(1-h)} \lim_{\theta\to 0} \frac{\cos^{-1}(\cos\theta)}{\sqrt{2(1-\cos\theta)}}$$

$$= \lim_{h\to 0} \frac{\sin^{-1}(1-h)}{(1-h)} \lim_{\theta\to 0} \frac{\theta}{2\sin(\theta/2)} \quad (\because \theta > 0)$$

$$= \sin^{-1} 1 \times 1 = \pi/2$$

and $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$

$$= \lim_{h\to 0} \frac{\sin^{-1}(1-\{0-h\})\cos^{-1}(1-\{0-h\})}{\sqrt{2\{0-h\}}(1-\{0-h\})}$$

$$= \lim_{h\to 0} \frac{\sin^{-1}(1+h-1)\cos^{-1}(1+h-1)}{\sqrt{2(-h+1)}(1+h-1)}$$

$$= \lim_{h\to 0} \frac{\sin^{-1} h}{h} \lim_{h\to 0} \frac{\cos^{-1} h}{\sqrt{2(1-h)}}$$

$$= 1\,\frac{\pi/2}{\sqrt{2}} = \frac{\pi}{2\sqrt{2}}$$

4. (3), 5. (4), 6. (3)

We have $A_i = \dfrac{x-a_i}{|x-a_i|}$, $i = 1, 2, \ldots, n$,

and $a_1 < a_2 < \cdots < a_{n-1} < a_n$.

Let x be in the left neighborhood of a_m.

Then $x - a_i < 0$ for $i = m, m+1, \ldots, n$

and $x - a_i > 0$ for $i = 1, 2, \ldots, m-1$

and $A_i = \dfrac{x-a_i}{-(x-a_i)} = -1$ for $i = m, m+1, \ldots, n$

$$A_i = \frac{x-a_i}{x-a_i} = 1 \text{ for } i = 1, 2, \ldots, m-1$$

Similarly, if x is in the right neighborhood of a_m, then

$x - a_i < 0$ for $i = m+1, \ldots, n$, and $x - a_i > 0$ for $i = 1, 2, \ldots, m$.

Therefore,

$$A_i = \frac{x-a_i}{-(x-a_i)} = -1 \text{ for } i = m+1, \ldots, n$$

and $A_i = \dfrac{x-a_i}{x-a_i} = 1$ for i = 1, 2, ..., m

Now, $\lim_{x\to a_m^-} (A_1 A_2 \cdots A_n) = (-1)^{n-m+1}$

and $\lim_{x\to a_m^+} (A_1 A_2 \cdots A_n) = (-1)^{n-m}$

Hence, $\lim_{x\to a_m} (A_1 A_2 \cdots A_n)$ does not exist.

7. (2), 8. (4), 9.(3)

$$L = \lim_{x\to 0} \frac{\sin x + ae^x + be^{-x} + c\log_e(1+x)}{x^3}$$

$$\left(x - \frac{x^3}{3!}\right) + a\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right) + b\left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!}\right) + c\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)$$

$$= \lim_{x\to 0} \frac{}{x^3}$$

$$(a+b) + (1+a-b+c)x + \left(\frac{a}{2} + \frac{b}{2} - \frac{c}{2}\right)x^2 + \left(-\frac{1}{3!} + \frac{a}{3!} - \frac{b}{3!} + \frac{c}{3}\right)x^3$$

$$= \lim_{x\to 0} \frac{}{x^3}$$

$\Rightarrow \quad a + b = 0,\ 1 + a - b + c = 0,\ \dfrac{a}{2} + \dfrac{b}{2} - \dfrac{c}{2} = 0$

and $L = -\dfrac{1}{3!} + \dfrac{a}{3!} - \dfrac{b}{3!} + \dfrac{c}{3}$

Solving first three equations we get $c = 0$, $a = -1/2$, $b = 1/2$.

$\therefore \quad L = -1/3$

Equation $ax^2 + bx + c = 0$ reduces to $x^2 - x = 0 \Rightarrow x = 0, 1$

$||x + c| - 2a| < 4b$ reduces to $||x| + 1| < 2$

$\Rightarrow \quad -2 < |x| + 1 < 2$

$\Rightarrow \quad 0 \le |x| < 1$

$\Rightarrow \quad x \in [-1, 1]$

10. (3) $\lim_{x\to 0^+} F(x) = \lim_{x\to 0^+} (p_1 a_1^x + p_2 a_2^x + \cdots + p_n a_n^x)^{1/x}$ (1^∞ form)

$$= e^{\lim_{x\to 0}\left(\frac{p_1 a_1^x + p_2 a_2^x + \cdots + p_n a_n^x - 1}{x}\right)}$$

$$= e^{\lim_{x\to 0}(p_1 a_1^x \ln a_1 + p_2 a_2^x \ln a_2 + \cdots + p_n a_n^x \ln a_n)}$$

$$= e^{(p_1 \ln a_1 + p_2 \ln a_2 + \cdots + p_n \ln a_n)}$$

$$= e^{\left(\ln a_1^{p_1} + \ln a_2^{p_2} + \cdots + \ln a_n^{p_n}\right)}$$

$$= e^{\left(\ln a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n}\right)} = a_1^{p_1} \cdot a_2^{p_2}\, a_3^{p_3} \cdots a_n^{p_n}$$

11. (3) $\lim_{x\to\infty} F(x) = L = \lim_{x\to\infty} \left(p_1 a_1^x + p_2 a_2^x + \cdots + p_n a_n^x\right)^{1/x}$ (∞^0 form)

$$\therefore \quad \ln L = \lim_{x\to\infty} \frac{\log_e\left(p_1 a_1^x + p_2 a_2^x + \cdots + p_n a_n^x\right)}{x}$$

Using L'Hospital's rule

$$\ln L = \lim_{x\to\infty} \frac{p_1 a_1^x \ln a_1 + p_2 a_2^x \ln a_2 + \cdots + p_n a_n^x \ln a_n}{p_1 a_1^x + p_2 a_2^x + \cdots + p_n a_n^x} \quad (1)$$

Dividing by a_1^x and taking limit, we get

$$\left(\because \lim_{x\to\infty}\left(\frac{a_2}{a_1}\right)^x = 0,\ \lim_{x\to\infty}\left(\frac{a_3}{a_1}\right)^x = 0 \text{ etc.}\right)$$

$$\ln L = \frac{p_1 \ln a_1}{p_1} = \ln a_1$$

or $\quad L = a_1$

12. (4) Let $\lim_{x\to-\infty} F(x) = L$

$$\therefore \quad \ln L = \lim_{x\to\infty} \frac{p_1 a_1^x \ln a_1 + p_2 a_2^x \ln a_2 + \cdots + p_n a_n^x \ln a_n}{p_1 a_1^x + p_2 a_2^x + \cdots + p_n a_n^x}$$

Dividing by $(a_n)^x$ and taking $\lim_{x\to\infty}\left(\frac{a_1}{a_n}\right)^x, \left(\frac{a_2}{a_n}\right)^x$, etc. vanish.

Therefore,

$$\ln L = \frac{p_n \ln a_n}{p_n}$$

or $L = a_n$

13. (3), 14. (2)

$$L = \lim_{x\to\infty}\left(x+1-\sqrt{ax^2+x+3}\right)$$

$$= \lim_{x\to\infty}\left(\frac{(x+1)^2-(ax^2+x+3)}{x+1+\sqrt{ax^2+x+3}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{(1-a)x^2+x-2}{x+1+\sqrt{ax^2+x+3}}\right)$$

L exists finitely if $1-a=0$ or $a=1$

$$\therefore \quad L = \lim_{x\to\infty}\left(\frac{x-2}{x+1+\sqrt{x^2+x+3}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{1-\frac{2}{x}}{1+\frac{1}{x}+\sqrt{1+\frac{1}{x}+\frac{3}{x^2}}}\right)$$

$$= \frac{1}{2}$$

15. (1), 16. (4)

We have $\lim_{x\to\infty} x^n f(x) = p$

$$\Rightarrow \quad \lim_{x\to\infty}\frac{x^{n+1}f(x)}{x} = p$$

Using L'Hospital rule, we get

$$\lim_{x\to\infty}\frac{(n+1)x^n f(x)+x^{n+1}f'(x)}{1} = p$$

$$\Rightarrow \quad (n+1)p + \lim_{x\to\infty} x^{n+1}f'(x) = p$$

$$\Rightarrow \quad \lim_{x\to\infty} x^{n+1}f'(x) = -np$$

Further $\lim_{x\to\infty}\frac{x^{n+2}f'(x)}{x} = -np$

Using L'Hospital rule, we get

$$\lim_{x\to\infty}\frac{(n+2)x^{n+1}f'(x)+x^{n+2}f''(x)}{1} = -np$$

$$\Rightarrow \quad -np(n+2) + \lim_{x\to\infty} x^{n+2}f''(x) = -np$$

$$\Rightarrow \quad \lim_{x\to\infty} x^{n+2}f''(x) = np(1+n)$$

17. (4), 18. (1)

Since $\lim_{x\to\infty}\frac{x^2 f(x)}{2x^5+3} = 6$, which is finite and non-zero, f(x) must be polynomial of degree '3'.

Also $f(1)=3, f(3)=7$ and $f(5)=11$

$$\therefore \quad f(x) = \lambda(x-1)(x-3)(x-5)+(2x+1)$$

$$\Rightarrow \quad \lim_{x\to\infty}\frac{x^2(\lambda(x-1)(x-3)(x-5)+2x+1)}{2x^5+3} = 6$$

$$\Rightarrow \quad \frac{\lambda}{2} = 6$$

$$\Rightarrow \quad \lambda = 12$$

$$\therefore \quad f(x) = 12(x-1)(x-3)(x-5)+(2x+1)$$

So, $f(0) = 12(-1)(-3)(-5)+1 = -179$

$$\lim_{x\to 1}\frac{x-1}{\sin(f(x)-2x-1)}$$

$$= \lim_{x\to 1}\frac{x-1}{\sin[12(x-1)(x-3)(x-5)]}$$

$$= \lim_{x\to 1}\frac{12(x-1)(x-3)(x-5)}{\sin[12(x-1)(x-3)(x-5)]}\cdot\frac{1}{12(x-3)(x-5)}$$

$$= \frac{1}{96}$$

19. (3), 20. (1)

$$\lim_{x\to 0}\frac{f(x)}{\sin^2 x} = 8$$

$$\Rightarrow \quad \lim_{x\to 0}\frac{\frac{f(x)}{x^2}}{\frac{\sin^2 x}{x^2}} = 8$$

$$\Rightarrow \quad \lim_{x\to 0}\frac{f(x)}{x^2} = 8$$

Also, $\lim_{x\to 0}\frac{g(x)}{2\cos x - xe^x + x^3 + x - 2} = \lambda$

$$\Rightarrow \quad \lim_{x\to 0}\frac{\frac{g(x)}{x^2}}{\frac{-2(1-\cos x)}{x^2}-\frac{e^x-1}{x}+x} = \lambda$$

$$\Rightarrow \quad \lim_{x\to 0}\frac{\frac{g(x)}{x^2}}{\frac{-4\sin^2\frac{x}{2}}{x^2}-1+0} = \lambda$$

$$\Rightarrow \quad \lim_{x\to 0}\frac{g(x)}{x^2} = -2\lambda$$

Now, $\lim_{x\to 0}\left(1+2f(x)\right)^{\frac{1}{g(x)}}$ (one power infinity form)

$$= e^{\lim_{x\to 0}\frac{2f(x)}{g(x)}} = e^{\lim_{x\to 0}\frac{2(f(x)/x^2)}{(g(x)/x^2)}}$$

$$= e^{\lim_{x\to 0}\frac{16}{-2\lambda}}$$

$$= e^{-\frac{8}{\lambda}}$$

$$= e^{-1} \text{ (given)}$$

$$\therefore \quad \lambda = 8$$

$$\lim_{x\to 0}\left(1+f(x)\right)^{\frac{1}{2g(x)}} = e^{\lim_{x\to 0}\frac{f(x)}{2g(x)}}$$

$$= e^{\lim_{x\to 0}\frac{(f(x)/x^2)}{2(g(x)/x^2)}}$$

$$= e^{\lim_{x\to 0}\frac{8}{2\times(-16)}}$$

$$= e^{-\frac{1}{4}}$$

ix Match Type

1. **a → s; b → r; c → p; d → q.**

a. Let $x+1=h$. Then,

$$\lim_{x\to -1}\frac{\sqrt[3]{(7-x)}-2}{(x+1)}=\lim_{h\to 0}\frac{(8-h)^{1/3}-2}{h}$$

$$=\lim_{h\to 0}\frac{2\left(1-\frac{h}{8}\right)^{1/3}-2}{h}$$

$$=2\lim_{h\to 0}\frac{\left(1-\frac{1}{3}\frac{h}{8}\right)-1}{h}$$

$$=-\frac{1}{12}$$

b. We have

$$\lim_{x\to \pi/4}\frac{\tan^3 x-\tan x}{\cos\left(x+\pi/4\right)}$$

$$=\lim_{x\to \pi/4}\frac{\tan x\left(\tan x-1\right)\left(\tan x+1\right)}{\cos\left(x+\pi/4\right)}$$

$$=\lim_{x\to \pi/4}\frac{\tan x\left(\sin x-\cos x\right)\left(\tan x+1\right)}{\cos x\cos\left(x+\pi/4\right)}$$

$$=-\lim_{x\to \pi/4}\frac{\tan x\left(\cos x-\sin x\right)\left(\tan x+1\right)}{\cos x\cos\left(x+\pi/4\right)}$$

$$=-\sqrt{2}\lim_{x\to \pi/2}\frac{\tan x\left(\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x\right)\left(\tan x+1\right)}{\cos x\cos\left(x+\pi/4\right)}$$

$$=-\sqrt{2}\lim_{x\to \pi/4}\frac{\tan x\left(\tan x+1\right)}{\cos x}$$

$$=-\sqrt{2}\times 2\times\sqrt{2}=-4$$

c. $\lim_{x\to 1}\frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}=\lim_{x\to 1}\frac{(2x-3)(\sqrt{x}-1)}{(2x+3)(x-1)}$

$$=\lim_{x\to 1}\frac{(2x-3)(\sqrt{x}-1)}{(2x+3)(\sqrt{x}-1)(\sqrt{x}+1)}$$

$$=\lim_{x\to 1}\frac{(2x-3)}{(2x+3)(\sqrt{x}+1)}$$

$$=\frac{2-3}{(2+3)(\sqrt{1}+1)}=-1/10$$

d. $\lim_{x\to\infty}\frac{\log x^n-[x]}{[x]}=\lim_{x\to\infty}\frac{\log x^n}{[x]}-\lim_{x\to\infty}\frac{[x]}{[x]}$

$$=0-1=-1$$

2. **a → q; b → r; c → q; d → p.**

We know that $\lim_{x\to 0}\frac{\sin x}{x}=1$ (but a value is smaller than 1)

or $\left[\lim_{x\to 0}100\frac{\sin x}{x}\right]=99$

and $\left[\lim_{x\to 0}100\frac{x}{\sin x}\right]=100$

Also, $\lim_{x\to 0}\frac{\sin^{-1}x}{x}=1$ (but a value is more than 1)

or $\left[\lim_{x\to 0}100\frac{\sin^{-1}x}{x}\right]=100$

and $\left[\lim_{x\to 0}100\frac{x}{\sin^{-1}x}\right]=99$

$\lim_{x\to 0}\frac{\tan x}{x}=1$ (but a value is bigger than 1)

or $\left[\lim_{x\to 0}100\frac{\tan x}{x}\right]=100$

and $\left[\lim_{x\to 0}100\frac{\tan^{-1}x}{x}\right]=99$

Hence,

a. $\lim_{x\to 0}\left(\left[100\frac{\sin x}{x}\right]+\left[100\frac{\tan x}{x}\right]\right)=199$

b. $\lim_{x\to 0}\left(\left[100\frac{x}{\sin x}\right]+\left[100\frac{\tan x}{x}\right]\right)=200$

c. $\lim_{x\to 0}\left(\left[100\frac{\sin^{-1}x}{x}\right]+\left[100\frac{\tan^{-1}x}{x}\right]\right)=199$

d. $\lim_{x\to 0}\left(\left[100\frac{x}{\sin^{-1}x}\right]+\left[100\frac{\tan^{-1}x}{x}\right]\right)=198$

3. **a → q; b → p, q, r; c → r, s; d → r, s.**

a. Here, $a>0$. Therfore,

$$\lim_{x\to\infty}\frac{(\sqrt{(x^2-x+1)}-ax-b)(\sqrt{x^2-x+1)}+ax+b)}{(\sqrt{(x^2-x+1)}+ax+b)}$$

$$=0$$

or $\lim_{x\to\infty}\frac{(x^2-x+1)-(ax+b)^2}{\sqrt{(x^2-x+1)}+ax+b}=0$

or $\lim_{x\to\infty}\frac{(1-a^2)x^2-(1+2ab)x+(1-b^2)}{\sqrt{(x^2-x+1)}+ax+b}=0$

or $\lim_{x\to\infty}\frac{(1-a^2)x-(1+2ab)+\frac{(1-b^2)}{x}}{\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}+a+\frac{b}{x}}=0$

This is possible only when

$1-a^2=0$ and $1+2ab=0$

$\therefore\quad a=\pm 1$

or $\quad a=1\quad(\because\ a>0)$

$\therefore\quad b=-1/2$ (1)

$\therefore\quad (a,2b)\equiv(1,-1)$

b. Divide numerator and denominator by $e^{1/x}$. Then,

$$\lim_{x\to\infty}\frac{(1+a^3)e^{-\frac{1}{x}}+8}{e^{-\frac{1}{x}}+(1-b^3)}=2$$

or $\frac{0+8}{0+1-b^3}=2$

or $1-b^3=4$

$\therefore\quad b^3=-3$ or $b=-3^{1/3}$

Then, $a\in R$. Therefore,

$(a,b^3)\equiv(a,-3)$

c. $\lim_{x\to\infty}(\sqrt{(x^4-x^2+1)}-ax^2-b)=0$

Put $x=\frac{1}{t}$. Then

$$\lim_{t\to 0}\left(\sqrt{\left(\frac{1}{t^4}-\frac{1}{t^2}+1\right)}-\frac{a}{t^2}-b\right)=0$$

or $$\lim_{t\to 0}\frac{\sqrt{(1-t^2+t^4)}-a-bt^2}{t^2}=0 \qquad (1)$$

Since R.H.S. is finite, numerator must be equal to 0 at $t\to 0$.

Therefore, $1-a=0$ or $a=1$.

From equation (1),

$$\lim_{t\to 0}\frac{\sqrt{(1-t^2+t^4)}-1-bt^2}{t^2}=0$$

$$\lim_{t\to 0}(-1+t^2)\left(\frac{(1-t^2+t^4)^{1/2}-(1)^{1/2}}{(1-t^2+t^4)-1}\right)=b$$

or $$(-1)\left(\frac{1}{2}\right)=b \text{ or } a=1, b=-\frac{1}{2} \text{ or } (a,-4b)\equiv(1,2)$$

d. $\lim_{x\to -a}\frac{x^7-(-a)^7}{x-(-a)}=7$ or $7a^6=7$ or $a^6=1$ or $a=-1$

4. (4) $\lim_{x\to\infty}\left(\frac{x^3+x^2+x+\sin x}{x^2+2\cos x}-a\sin x-bx+c\right)=4$

$$\Rightarrow \lim_{x\to\infty}\left(\frac{x+1+\frac{1}{x}+\frac{\sin x}{x^2}}{1+\frac{2\cos x}{x^2}}-a\sin x-bx+c\right)=4$$

$$\Rightarrow \lim_{x\to\infty}\left(x+1-a\sin x-bx+c\right)=4$$

$$\Rightarrow \lim_{x\to\infty}\left((1-b)x-a\sin x+1+c\right)=4$$

$\Rightarrow$ $1-b=0$, $a=0$ and $1+c=4$

$\Rightarrow$ $b=1$, $a=0$ and $c=3$

5. (2) a. $\lim_{x\to 0}x(-1)^{[1/x]}$

L.H.L.$=\lim_{h\to 0}(0-h)(-1)^{\left[\frac{1}{0-h}\right]}=0$

R.H.L.$=\lim_{h\to 0}(0+h)(-1)^{\left[\frac{1}{0+h}\right]}=0$

b. $\lim_{x\to 2}(-1)^{[x]}$

L.H.L.$=\lim_{h\to 0}(-1)^{[2-h]}=(-1)^1=-1$

R.H.L.$=\lim_{h\to 0}(-1)^{[2+h]}=(-1)^2=1$

So limit does not exist.

c. $\lim_{x\to\frac{3}{2}}(x-[x])$

L.H.L.$=\lim_{h\to 0}\frac{3}{2}-h-\left[\frac{3}{2}-h\right]=\lim_{h\to 0}\frac{3}{2}-h-1=\frac{1}{2}$

R.H.L. $\lim_{h\to 0}\frac{3}{2}+h-\left[\frac{3}{2}+h\right]=\lim_{h\to 0}\frac{3}{2}+h-1=\frac{1}{2}$

L.H.L. = R.H.L. $=\frac{1}{2}$

d. $\lim_{x\to 0}[x]\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)$

L.H.L. $=\lim_{h\to 0}[0-h]\left(\frac{e^{\frac{1}{0-h}}-1}{e^{\frac{1}{0-h}}+1}\right)$

$$=\lim_{h\to 0}[-h]\left(\frac{e^{-\frac{1}{h}}-1}{e^{-\frac{1}{h}}+1}\right)=(-1)\times(-1)=1$$

R.H.L.$=\lim_{h\to 0}[0+h]\left(\frac{e^{\frac{1}{h}}-1}{e^{\frac{1}{h}}+1}\right)=0$

Limit does not exist

Numerical Value Type

1. (2) We have

$$L=\lim_{n\to\infty}\prod_{n=2}^{n}\frac{n^2-1}{n^2}$$

$$=\lim_{n\to\infty}\prod_{n=2}^{n}\frac{n-1}{n}\cdot\prod_{n=2}^{n}\frac{n+1}{n}$$

$$=\lim_{n\to\infty}\left(\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{n-1}{n}\right)\left(\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdots\frac{n+1}{n}\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\cdot\frac{n+1}{2}=\frac{1}{2}$$

2. (2) $\lim_{x\to\infty}\frac{2x-3}{x}<\lim_{x\to\infty}f(x)<\lim_{x\to\infty}\frac{2x^2+5x}{x^2}$

or $$\lim_{x\to\infty}\frac{2-\frac{3}{x}}{1}<\lim_{x\to\infty}f(x)<\lim_{x\to\infty}\frac{2+\frac{5}{x^2}}{1}$$

or $$\lim_{x\to\infty}f(x)=2$$

3. (0) $\lim_{x\to 0^+}f(g(h(x)))=f(g(0^+))=f(1^+)=0$

$\lim_{x\to 0^-}f(g(h(x)))=f(g(0^+))=f(1^+)=0$

Hence, $\lim_{x\to 0}f(g(h(x)))=0$

4. (1) $\lim_{x\to\infty}\left(f(x)+\frac{3f(x)-1}{f^2(x)}\right)=3$

or $$\left(\lim_{x\to\infty}f(x)+\frac{3\lim_{x\to\infty}f(x)-1}{\left(\lim_{x\to\infty}f(x)\right)^2}\right)=3$$

or $$\left(y+\frac{3y-1}{y^2}\right)=3$$

or $y^3-3y^2+3y-1=0$

or $(y-1)^3=0$

or $y=1$

5. (3) $\lim_{x\to 2}\dfrac{(10-x)^{1/3}-2}{x-2}=\lim_{h\to 0}\dfrac{(8-h)^{1/3}-2}{h}$ (Put $x=2+h$)

$$=\lim_{h\to 0}\frac{2\left(1-\frac{h}{8}\right)^{1/3}-2}{h}=2\lim_{h\to 0}\frac{\left(1-\frac{h}{8}\right)^{1/3}-1}{h}$$

$$=2\lim_{h\to 0}\frac{1-\frac{1}{3}\frac{h}{8}-1}{h}=-\frac{1}{12}$$

6. (1) $\lim_{x\to 0}\dfrac{p\sin 2x+(1-\cos 2x)}{x+\tan x}=1$

$$\Rightarrow\quad \lim_{x\to 0}\frac{2p\left(\frac{\sin 2x}{2x}\right)+\left(\frac{1-\cos 2x}{x}\right)}{1+\frac{\tan x}{x}}=1$$

$$\Rightarrow\quad \frac{2p+0}{1+1}=1$$

$$\Rightarrow\quad p=1$$

7. (25) We have $\lim_{x\to 1}\dfrac{100}{(1-x^{50})(1+x^{50})}-\dfrac{50}{1-x^{50}}$

$$=\lim_{x\to 1}\frac{100-50(1+x^{50})}{(1-x^{50})(1+x^{50})}$$

$$=\lim_{x\to 1}\frac{50(1-x^{50})}{(1-x^{50})(1+x^{50})}$$

$$=\lim_{x\to 1}\frac{50}{1+x^{100}}$$

$$=25$$

8. (12) We have, $\lim_{x\to 2}\dfrac{\sqrt[3]{60+x^2}-\sqrt[3]{64}}{\sin(x-2)}$

$$=\lim_{x\to 2}\frac{\left[(60+x^2)^{\frac{1}{3}}-64^{\frac{1}{3}}\right]\left[(60+x^2)^{\frac{2}{3}}+(60+x^2)^{\frac{1}{3}}64^{\frac{1}{3}}+64^{\frac{2}{3}}\right]}{(x-2)\frac{\sin(x-2)}{(x-2)}\left[(60+x^2)^{\frac{2}{3}}+(60+x^2)^{\frac{1}{3}}64^{\frac{1}{3}}+64^{\frac{2}{3}}\right]}$$

$$=\lim_{x\to 2}\frac{60+x^2-64}{(x-2)[16+4\times 4+16]}$$

$$=\lim_{x\to 2}\frac{(x-2)(x+2)}{48(x-2)}$$

$$=\lim_{x\to 2}\frac{x+2}{48}=\frac{4}{48}=\frac{1}{12}$$

9. (4) We have,

$$y=\lim_{x\to\infty}\left(\frac{20^x-1}{19(5^x)}\right)^{\frac{1}{x}}\qquad(\infty^0\text{ form})$$

$$\therefore\quad \log_e y=\log_e\lim_{x\to\infty}\left(\frac{20^x-1}{19(5^x)}\right)^{\frac{1}{x}}$$

$$=\lim_{x\to\infty}\log_e\left(\frac{20^x-1}{19(5^x)}\right)^{\frac{1}{x}}$$

$$=\lim_{x\to\infty}\frac{1}{x}\log_e\left(\frac{20^x-1}{19(5^x)}\right)$$

$$=\lim_{x\to\infty}\frac{\log_e(20^x-1)-\log_e 19-\log_e 5^x}{x}$$

$$=\lim_{x\to\infty}\frac{\frac{20^x\times\log_e 20}{20^x-1}-0-\frac{5^x\log_e 5}{5^x}}{1}$$

$$=\lim_{x\to\infty}\left(\frac{\log_e 20}{1-\frac{1}{20^x}}-\log_e 5\right)$$

$$=\log_e 20-\log_e 5=\log_e 4$$

$$\Rightarrow\quad y=4$$

10. **(0)** $\lim_{n\to\infty}\left[\sqrt[3]{(n+1)^2}-\sqrt[3]{(n-1)^2}\right]$

$$=\lim_{n\to\infty}n^{2/3}\left[\left(1+\frac{1}{n}\right)^{2/3}-\left(1-\frac{1}{n}\right)^{2/3}\right]$$

$$=\lim_{n\to\infty}n^{2/3}\left[\left(1+\frac{2}{3}\cdot\frac{1}{n}+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}\frac{1}{n^2}\cdots\right)\right.$$

$$\left.-\left(1-\frac{2}{3}\cdot\frac{1}{n}+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}\frac{1}{n^2}\cdots\right)\right]$$

$$=\lim_{n\to\infty}n^{2/3}\left[\frac{4}{3}\cdot\frac{1}{n}+\frac{8}{81}\cdot\frac{1}{n^3}+\cdots\right]$$

$$=\lim_{n\to\infty}\left[\frac{4}{3}\cdot\frac{1}{n^{1/3}}+\frac{8}{81}\cdot\frac{1}{n^{7/3}}+\cdots\right]=0$$

11. **(6)** Clearly, n is even. Then,

$$\lim_{n\to\infty}(2^{1+3+5+\cdots+n/2\text{ terms}}\cdot 3^{2+4+6+\cdots+n/2\text{ terms}})^{\frac{1}{(n^2+1)}}$$

$$=\lim_{n\to\infty}\left(2^{\frac{n^2}{4}}\cdot 3^{\frac{n(n+2)}{4}}\right)^{\frac{1}{(n^2+1)}}$$

$$=\lim_{n\to\infty}2^{\frac{n^2}{4(n^2+1)}}\cdot 3^{\frac{n(n+2)}{4(n^2+1)}}$$

$$=2^{\lim_{n\to\infty}\frac{1}{4\left(1+\frac{1}{n^2}\right)}}\cdot 3^{\lim_{n\to\infty}\frac{\left(1+\frac{2}{n}\right)}{4\left(1+\frac{1}{n^2}\right)}}$$

$$=2^{\frac{1}{4}}3^{\frac{1}{4}}=(6)^{\frac{1}{4}}$$

12. **(0)** Let $L=\lim_{x\to\infty}\dfrac{\log_e(\log_e x)}{e^{\sqrt{x}}}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

$$=\lim_{x\to\infty}\frac{\frac{1}{x\log_e x}}{e^{\sqrt{x}}\frac{1}{2\sqrt{x}}}=\lim_{x\to\infty}\frac{2\sqrt{x}}{e^{\sqrt{x}}x\log_e x}$$

$$=\lim_{x\to\infty}\frac{2}{e^{\sqrt{x}}\sqrt{x}\log_e x}$$

$$=0$$

13. (6) $L = -\lim_{x\to 0} \dfrac{\dfrac{d}{dx}\prod_{r=2}^{n}(\cos rx)^{1/r}}{2x}$ (Using L'Hospital's rule)

Let $y = \prod_{r=2}^{n}(\cos rx)^{1/r}$

or $\ln y = \sum_{r=2}^{n}\left(\frac{1}{r}\ln(\cos rx)\right)$

or $\frac{1}{y}\frac{dy}{dx} = -\sum_{r=2}^{n}\tan(rx)$

or $\frac{d}{dx} = -y\sum_{r=2}^{n}\tan(rx)$

or $L = \lim_{x\to 0}\dfrac{y\cdot\sum_{r=2}^{n}\tan(rx)}{2x}$

$= \frac{1}{2}[2+3+4+\cdots+n]$

$= \frac{1}{2}\left[\frac{n(n+1)}{2}-1\right] = \frac{n^2+n-2}{4}$

or $\frac{n^2+n-2}{4} = 10$

or $n^2+n-42=0$

or $(n+7)(n-6)=0$

or $n=6$

14. (0.5) Let $x = 1/y$

$\therefore \lim_{x\to\infty}\left(x - x^2\log_e\left(1+\frac{1}{x}\right)\right) = \lim_{y\to 0}\left(\frac{1}{y} - \frac{\log_e(1+y)}{y^2}\right)$

$= \lim_{y\to 0}\left(\frac{y-\log_e(1+y)}{y^2}\right)$

$= \lim_{y\to 0}\left(\frac{y-\left(y-\frac{y^2}{2}\right)}{y^2}\right) = 1/2$

15. (3) $S_n = \frac{n(n+1)}{2}$ and $S_n - 1 = \frac{(n+2)(n-1)}{2}$

$\therefore \frac{S_n}{S_n-1} = \frac{n(n+1)}{2}\cdot\frac{2}{(n+2)(n-1)} = \left(\frac{n}{n-1}\right)\left(\frac{n+1}{n+2}\right)$

$\therefore P_n = \left(\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdots\frac{n}{n-1}\right)\left(\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}\cdots\frac{n+1}{n+2}\right)$

$= \left(\frac{n}{1}\right)\left(\frac{3}{n+2}\right)$

$\therefore \lim_{n\to\infty} P_n = 3$

16. (8) Since RHS is finite quantity, at $x \to 1$, numerator must be 0. Therefore,

$0 + b + 4 = 0$

$b = -4$

Then $\lim_{x\to 1}\dfrac{a\sin(x-1)-4\cos(x-1)+4}{(x^2-1)} = -2$

Put $x = 1+h$. Then $\lim_{h\to 0}\dfrac{a\sin h + 4(1-\cos h)}{h(2+h)} = -2$

or $\lim_{h\to 0}\dfrac{a\left(\frac{\sin h}{h}\right)+4\left(\frac{1-\cos h}{h}\right)}{2+h} = -2$

or $\frac{a(1)+0}{2} = -2$

or $a = -4$

or $|a+b| = 8$

17. (6) Put $x = 1+h$. Then,

$f(a) = \lim_{h\to 0}\dfrac{(1+h)^a - a(1+h)+a-1}{h^2}$

$= \lim_{h\to 0}\dfrac{\left(1+ah+\frac{a(a-1)}{2!}h^2+\cdots\right)-a-ah+a-1}{h^2}$

$\therefore f(a) = \frac{a(a-1)}{2}$

$\therefore f(4) = 6$

18. (5) We have $\lim_{x\to 1}\sin^{-1}\left(\frac{k}{\log_e x} - \frac{k}{x-1}\right)$

Now, $\lim_{x\to 1}\left(\frac{k}{\log_e x} - \frac{k}{x-1}\right) = k\lim_{h\to 0}\frac{h-\log_e(1+h)}{h^2}$

$= k\lim_{h\to 0}\dfrac{h-\left(h-\frac{h^2}{2}+\ldots\right)}{h^2} = \frac{k}{2}$

Now $\sin^{-1}\frac{k}{2}$ is defined if $-1 \le \frac{k}{2} \le 1$

$\Rightarrow -2 \le k \le 2$

19. (3) $\lim_{x\to 1}(1+ax+bx^2)^{\frac{c}{x-1}} = e^3$

or $e^{\lim_{x\to 1}(1+ax+bx^2-1)^{\frac{c}{x-1}}} = e^3$

or $e^{\lim_{x\to 1}\frac{c(ax+bx^2)}{x-1}} = e^3$

or $\lim_{x\to 1}\frac{c(ax+bx^2)}{x-1} = 3$

or $\lim_{h\to 0}\frac{c(a(1+h)+b(1+h)^2)}{1+h-1} = 3$

or $\lim_{h\to 0}\frac{(ca+b)+(ac+2b)h+bh^2}{h} = 3$

or $ca + b = 0$ and $ac + 2b = 3$

or $b = 3$ and $ac = -3$

Also, the form must be 1^∞ for which $a + b = 0$, i.e., $a = -3$ and $c = 1$.

20. (7) We have

$L = \lim_{x\to 0}\dfrac{2f(x) - 3af(2x) + bf(8x)}{\sin^2 x}$

For the limit to exist, we have

$2f(0) - 3af(0) + bf(0) = 0$

or $3a - b = 2$ $[\because f(0) \neq 0, \text{given}]$ (1)

or $L = \lim_{x\to 0} \dfrac{2f'(x) - 6a\, f'(2x) + 8b\, f'(8x)}{2x}$

For the limit to exist, we have

$2f'(0) - 6af'(0) + 8bf'(0) = 0$

$\Rightarrow \quad 3a - 4b = 1 \qquad [\because f'(0) \neq 0, \text{given}]$ (2)

Solving equations (1) and (2), we have $a = 7/9$ and $b = 1/3$.

(4) $\lim_{x\to 0} \dfrac{e^{-x^2/2} - \cos x}{x^3 \sin x}$

$$= \lim_{x\to 0} \frac{\left(1 - \frac{(x^2/2)}{1!} + \frac{(x^2/2)^2}{2!}\right) - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right)}{x^3\left(x - \frac{x^3}{3!}\right)}$$

$$= \lim_{x\to 0} \frac{\left(\frac{x^4}{8}\right) - \left(\frac{x^4}{24}\right)}{x^4\left(1 - \frac{x^2}{3!}\right)} = \frac{1}{12}$$

(3) $L = \lim_{x\to 0} \dfrac{(\cos x - 1)(\cos x - e^x)}{x^n}$

$$= -\lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)(\cos x - e^x)}{(1+\cos x)x^n}$$

$$= \lim_{x\to 0} \frac{\left(\frac{\sin x}{x}\right)^2 \left(\frac{1-\cos x}{x} + \frac{e^x - 1}{x}\right)}{x^{n-3}} \frac{1}{1+\cos x}$$

If L is finite nonzero, then $n = 3$ (as for $n = 1, 2$, $L = 0$, and for $n = 4$, $L = \infty$.)

(2) $\lim_{x\to 0}\left[1 + x + \dfrac{f(x)}{x}\right]^{1/x} = e^3$

or $\lim_{x\to 0} e^{\lim_{x\to 0}\left[1+x+\frac{f(x)}{x}-1\right]\frac{1}{x}} = e^3$

or $\lim_{x\to 0} e^{\lim_{x\to 0}\left[1+\frac{f(x)}{x^2}\right]} = e^3$ or $\lim_{x\to 0} \dfrac{f(x)}{x^2} = 2$

Now, $\lim_{x\to 0}\left[1 + \dfrac{f(x)}{x}\right]^{1/x} = e^{\lim_{x\to 0}\left[1+\frac{f(x)}{x}-1\right]\frac{1}{x}} = e^{\lim_{x\to 0}\frac{f(x)}{x^2}} = e^2$

Archives

JEE MAIN

Single Correct Answer Type

1. (2) $f(x)$ is a positive increasing function. Therefore,

$$0 < f(x) < f(2x) < f(3x)$$

$$\Rightarrow \quad 0 < 1 < \frac{f(2x)}{f(x)} < \frac{f(3x)}{f(x)}$$

$$\Rightarrow \quad \lim_{x\to\infty} 1 < \lim_{x\to\infty} \frac{f(2x)}{f(x)} < \lim_{x\to\infty} \frac{f(3x)}{f(x)} \qquad \left(\because \lim_{x\to\infty} \frac{f(3x)}{f(x)} = 1\right)$$

By Sandwich theorem, we get

$$\lim_{x\to\infty} \frac{f(2x)}{f(x)} = 1$$

2. (2) $\lim_{x\to 2}\left(\dfrac{\sqrt{1-\cos 2(x-2)}}{x-2}\right) = \lim_{x\to 2} \dfrac{\sqrt{2}\,|\sin(x-2)|}{x-2}$

Now $\lim_{x\to 2^+} \dfrac{\sqrt{2}\,|\sin(x-2)|}{x-2} = \lim_{x\to 2^+} \dfrac{\sqrt{2}\sin(x-2)}{x-2} = \sqrt{2}$

and $\lim_{x\to 2^-} \dfrac{\sqrt{2}\,|\sin(x-2)|}{x-2} = \lim_{x\to 2^-} \dfrac{-\sqrt{2}\sin(x-2)}{(x-2)} = -\sqrt{2}$

Hence, limit does not exist.

3. (4) $\lim_{x\to 0} \dfrac{(1-\cos 2x)(3+\cos x)}{x\tan 4x} = \lim_{x\to 0} \dfrac{(2\sin^2 x)(3+\cos x)}{x\left(\frac{\tan 4x}{4x}\right)\times 4x}$

$$= \lim_{x\to 0} \frac{2\sin^2 x(3+\cos x)}{4x^2}$$

$$= \frac{2}{4}(3+1) = 2$$

4. (4) $\lim_{x\to 0} \dfrac{\sin(\pi\cos^2 x)}{x^2} = \lim_{x\to 0} \dfrac{\sin(\pi - \pi\sin^2 x)}{x^2}$

$$= \lim_{x\to 0} \frac{\sin(\pi\sin^2 x)}{\pi\sin^2 x} \times \frac{\pi\sin^2 x}{x^2} = \pi$$

5. (3) $\lim_{x\to 0} \dfrac{(1-\cos 2x)(3+\cos x)}{x\tan 4x} = \lim_{x\to 0} \dfrac{(2\sin^2 x)(3+\cos x)}{x\left(\frac{\tan 4x}{4x}\right)\times 4x}$

$$= \lim_{x\to 0} \frac{2\sin^2 x(3+\cos x)}{4x^2}$$

$$= \frac{2}{4}(3+1) = 2$$

6. (2) $p = \lim_{x\to 0^+} (1+\tan^2\sqrt{x})^{\frac{1}{2x}}$ (1^∞ form)

$$= e^{\lim_{x\to 0^+}(1+\tan^2\sqrt{x}-1)\frac{1}{2x}} = e^{\lim_{x\to 0^+}\frac{(\tan\sqrt{x})^2}{2(\sqrt{x})^2}} = e^{\frac{1}{2}}$$

$$\therefore \quad \log_e p = \log e^{\frac{1}{2}} = \frac{1}{2}$$

7. (3) $\lim_{x\to\frac{\pi}{2}} \dfrac{\cot x - \cos x}{(\pi - 2x)^3} = \lim_{x\to\frac{\pi}{2}} \dfrac{\cot x(1-\sin x)}{8\left(\frac{\pi}{2} - x\right)^3}$

$$= \frac{1}{8}\lim_{x\to\frac{\pi}{2}} \frac{\tan\left(\frac{\pi}{2}-x\right)}{\left(\frac{\pi}{2}-x\right)} \cdot \frac{(1-\sin^2 x)}{\left(\frac{\pi}{2}-x\right)^2} \cdot \frac{1}{1+\sin x}$$

$$= \frac{1}{8}\lim_{x\to\frac{\pi}{2}} \frac{\tan\left(\frac{\pi}{2}-x\right)}{\left(\frac{\pi}{2}-x\right)} \cdot \frac{\cos^2 x}{\left(\frac{\pi}{2}-x\right)^2} \cdot \frac{1}{1+\sin x}$$

$$= \frac{1}{8}\lim_{x\to\frac{\pi}{2}} \frac{\tan\left(\frac{\pi}{2}-x\right)}{\left(\frac{\pi}{2}-x\right)} \cdot \frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\left(\frac{\pi}{2}-x\right)^2} \cdot \frac{1}{1+\sin x}$$

$$= \frac{1}{8}\cdot 1\cdot 1\cdot \frac{1}{2} = \frac{1}{16}$$

8. (4) $\lim_{x\to 0^+} x\left(\left[\frac{1}{x}\right]+\left[\frac{2}{x}\right]+\ldots+\left[\frac{15}{x}\right]\right)$

$= \lim_{x\to 0^+}\left(\left(\frac{1}{x}+\frac{2}{x}+\frac{3}{x}+\ldots+\frac{15}{x}\right)-\left(\left\{\frac{1}{x}\right\}+\left\{\frac{2}{x}\right\}+\ldots+\left\{\frac{15}{x}\right\}\right)\right)$

(Where $\{k\}$ represents fractional part of k)

$= \lim_{x\to 0^+}\left(x\frac{120}{x}-x\left(\left\{\frac{1}{x}\right\}+\left\{\frac{2}{x}\right\}+\ldots+\left\{\frac{15}{x}\right\}\right)\right)$

$= 120 - 0$ $\quad(\because\ 0<\{k\}<1)$

$= 120$

JEE ADVANCED

Single Correct Answer Type

1. (4) $e^{\ln(1+b^2)} = 2b\sin^2\theta$

or $\sin^2\theta = \frac{1+b^2}{2b} = 1$ as $\frac{1+b^2}{2b} \geq 1$

$\theta = \pm\pi/2$

2. (2) Given $\lim_{x\to\infty}\left(\frac{x^2+x+1}{x+1}-ax-b\right)=4$

or $\lim_{x\to\infty}\frac{x^2+x+1-ax^2-ax-bx-b}{(x+1)}=4$

or $\lim_{x\to\infty}\frac{(1-a)x^2+(1-a-b)x+(1-b)}{(x+1)}=4$

or $1-a=0$ and $1-a-b=4$

or $b=-4,\ a=1$

Multiple Correct Answers Type

1. (1), (3)

$$L = \lim_{x\to 0}\frac{a-\sqrt{a^2-x^2}-\frac{x^2}{4}}{x^4}$$

$$= \lim_{x\to 0}\left(\frac{1}{x^2\left(a+\sqrt{a^2-x^2}\right)}-\frac{1}{4x^2}\right)$$

$$= \lim_{x\to 0}\frac{(4-a)-\sqrt{a^2-x^2}}{4x^2\left(a+\sqrt{a^2-x^2}\right)}$$

Numerator $\to 0$ if $a=2$ and then $L=\frac{1}{64}$.

2. (3), (4)

$$f(1^+) = \lim_{h\to 0}\frac{1-(1+h)(1+h)}{h}\cos\frac{1}{h}$$

$$= \lim_{h\to 0}\frac{-h^2-2h}{h}\cos\frac{1}{h} = \lim_{h\to 0}(-h-2)\cos\frac{1}{h}$$

Thus, $\lim_{x\to 1^+} f(x)$ does not exist.

$$f(1^-) = \lim_{h\to 0}\frac{1-(1-h)(1+h)}{h}\cos\frac{1}{h}$$

$$= \lim_{h\to 0}\frac{1-(1-h^2)}{h}\cos\frac{1}{h}$$

$$= \lim_{h\to 0}\frac{h^2}{h}\cos\frac{1}{h} = \lim_{h\to 0} h\cos\frac{1}{h} = 0$$

Numerical Value Type

1. (2) $\lim_{x\to 1}\left\{\frac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{\frac{1-x}{1-\sqrt{x}}} = \frac{1}{4}$

$$\Rightarrow \lim_{x\to 1}\left\{\frac{\frac{\sin(x-1)}{(x-1)}-a}{\frac{\sin(x-1)}{(x-1)}+1}\right\}^{1+\sqrt{x}} = \frac{1}{4}$$

$$\Rightarrow \left(\frac{1-a}{2}\right)^2 = \frac{1}{4}$$

$$\Rightarrow a=0,\ a=2$$

2. (2) $m \geq 2$ and $n \geq 2$

$$\lim_{\alpha\to 0}\frac{e^{\cos(\alpha^n)}-e}{\alpha^n}$$

$$= e\times\lim_{\alpha\to 0}\frac{e^{\cos(\alpha^n)-1}-1}{\cos(\alpha^n)-1}\times\left(\frac{\cos(\alpha^n)-1}{(\alpha^n)^2}\right)\frac{\alpha^{2n}}{\alpha^n}$$

$$= e\times\lim_{\alpha\to 0}\left(\frac{e^{\cos(\alpha^n)-1}-1}{\cos(\alpha^n)-1}\right)\times\lim_{\alpha\to 0}\left(\frac{\cos(\alpha^n)-1}{\alpha^{2n}}\right)\times\lim_{\alpha\to 0}\alpha^{2n-m}$$

$$= e\times 1\times\lim_{\alpha\to 0}\frac{-2\sin^2\frac{\alpha^n}{2}}{\alpha^{2n}}\times\lim_{\alpha\to 0}\alpha^{2n-m}$$

$$= e\times 1\times\left(-\frac{1}{2}\right)\times\lim_{\alpha\to 0}\alpha^{2n-m}$$

Now, $\lim_{\alpha\to 0}\alpha^{2n-m}$ must be equal to 1.

i.e., $2n-m=0$

or $\frac{m}{n}=2$

3. (7) $\lim_{x\to 0}\frac{x^2\left\{\beta x-\frac{(\beta x)^3}{3!}+\ldots\right\}}{\alpha x-\left(x-\frac{x^3}{3!}+\ldots\right)}=1$

$$\Rightarrow \lim_{x\to 0}\frac{x^3\left(\beta-\frac{\beta^3x^2}{3!}+\ldots\right)}{(\alpha-1)x+\frac{x^3}{3!}-\frac{x^5}{5!}+\ldots}=1$$

$$\Rightarrow \lim_{x\to 0}\frac{x^2\left(\beta-\frac{\beta^3x^2}{3!}+\ldots\right)}{(\alpha-1)+\frac{x^2}{3!}-\frac{x^4}{5!}+\ldots}=1$$

$\alpha-1=0$ or $\alpha=1$

$$\Rightarrow \lim_{x\to 0}\frac{\beta-\frac{\beta^3}{3}x^2\ldots}{\frac{1}{3!}-\frac{x^2}{5!}+\ldots}=1$$

$$\therefore\quad \beta=\frac{1}{3!}=\frac{1}{6}$$

$$\therefore\quad 6(\alpha+\beta)=6\left(1+\frac{1}{6}\right)=7$$

Exercise 3.1

1. (a) Let $f(x) = \sqrt{\sin x}$

$\therefore \quad f(x+h) = \sqrt{\sin(x+h)}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{\sin(x+h)}-\sqrt{\sin x}}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h\left(\sqrt{\sin(x+h)}+\sqrt{\sin x}\right)}$$

$$= \lim_{h\to 0}\frac{2\sin\left(\frac{h}{2}\right)\cos\left(\frac{2x+h}{2}\right)}{h\left(\sqrt{\sin(x+h)}+\sqrt{\sin x}\right)}$$

$$= \lim_{h\to 0}\frac{\left(\sin\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\lim_{h\to 0}\frac{\cos\left(x+\frac{h}{2}\right)}{\left(\sqrt{\sin(x+h)}+\sqrt{\sin x}\right)}$$

$$= \frac{\cos x}{\sqrt{\sin x}+\sqrt{\sin x}} = \frac{\cos x}{2\sqrt{\sin x}}$$

(b) Let $f(x) = \cos^2 x$.

$\therefore \quad f(x+h) = \cos^2(x+h)$

$$\therefore \quad \frac{d}{dx}\left(f(x)\right) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\cos^2(x+h)-\cos^2 x}{h}$$

$$= \lim_{h\to 0}\frac{\sin^2 x-\sin^2(x+h)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x+x+h)\sin\left(x-(x+h)\right)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(2x+h)\sin(-h)}{h}$$

$$= -\lim_{h\to 0}\frac{\sin h}{h}\cdot\lim_{h\to 0}\sin(2x+h)$$

$$= -\sin 2x$$

(c) $$\frac{d(\sin^{-1}x)}{dx} = \lim_{h\to 0}\frac{\sin^{-1}(x+h)-\sin^{-1}x}{h}$$

$$= \lim_{h\to 0}\frac{\sin^{-1}\left[(x+h)\sqrt{1-x^2}-x\sqrt{1-(x+h)^2}\right]}{h}$$

$$= \lim_{h\to 0}\frac{\sin^{-1}\left[(x+h)\sqrt{1-x^2}-x\sqrt{1-(x+h)^2}\right]}{\left[(x+h)\sqrt{1-x^2}-x\sqrt{1-(x+h)^2}\right]}$$

$$\times\frac{(x+h)\sqrt{1-x^2}-x\sqrt{1-(x+h)^2}}{h}$$

$$= 1\times\lim_{h\to 0}\frac{(x+h)^2(1-x^2)-x^2(1-(x+h)^2)}{h\left[(x+h)\sqrt{1-x^2}+x\sqrt{1-(x+h)^2}\right]}$$

$$= \lim_{h\to 0}\frac{(x+h)^2-x^2}{h\left[(x+0)\sqrt{1-x^2}+x\sqrt{1-(x+0)^2}\right]}$$

$$= \lim_{h\to 0}\frac{h(2x+h)}{h\left[2x\sqrt{1-x^2}\right]}$$

$$= \lim_{h\to 0}\frac{2x+h}{2x\sqrt{1-x^2}}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

2. Let $f(x) = \sqrt{4-x}$. Then $f(x+h) = \sqrt{4-(x+h)}$. Therefore,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{4-(x+h)}-\sqrt{4-x}}{h}$$

$$= \lim_{h\to 0}\frac{\left\{\sqrt{4-(x+h)}-\sqrt{4-x}\right\}\left\{\sqrt{4-(x+h)}+\sqrt{4-x}\right\}}{h\left\{\sqrt{4-(x+h)}+\sqrt{4-x}\right\}}$$

$$= \lim_{h\to 0}\frac{4-(x+h)-(4-x)}{h\left\{\sqrt{4-(x+h)}+\sqrt{4-x}\right\}}$$

$$= \lim_{h\to 0}\frac{-h}{h\left\{\sqrt{4-x-h}+\sqrt{4-x}\right\}} = \frac{-1}{2\sqrt{4-x}}$$

3. $$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$$

$$= \lim_{x\to 0}\frac{(2^x+2^{-x})\frac{\sin x}{x}\sqrt{\tan^{-1}(x^2-x+1)}}{(7x^2+3x+1)^3}$$

$$= \frac{2\times 1\sqrt{\tan^{-1}1}}{(0+0+1)^3}$$

$$= \sqrt{\pi}$$

4. We have $|f(x)-f(y)| \le |x-y|^3,\ x \ne y$

$$\therefore \quad \left|\frac{f(x)-f(y)}{x-y}\right| \le |x-y|^2$$

$$\Rightarrow \quad \lim_{y\to x}\left|\frac{f(x)-f(y)}{x-y}\right| \le \lim_{y\to x}|x-y|^2$$

$$\Rightarrow \quad \left|\lim_{y\to x}\frac{f(x)-f(y)}{x-y}\right| \le \left|\lim_{y\to x}(x-y)^2\right|$$

$$\Rightarrow \quad |f'(x)| \le 0$$

$$\Rightarrow \quad |f'(x)| = 0 \qquad (\because |f'(x)| \ge 0)$$

$$\therefore \quad f'(x) = 0$$

$$\Rightarrow \quad f(x) = c \text{ (constant)}$$

Exercise 3.2

1. $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Let $x = \tan\theta$. Therefore,

$$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$
$$= \sin^{-1}(\sin 2\theta)$$
$$= 2\theta$$
$$= 2\tan^{-1}x$$
$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(2\tan^{-1}x)$$
$$= 2\frac{d}{dx}(\tan^{-1}x)$$
$$= \frac{2}{1+x^2}$$

2. $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

Put $x = \tan\theta$. Then,

$$y = \tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$$
$$= \tan^{-1}(\tan 3\theta)$$
$$= 3\theta$$
$$= 3\tan^{-1}x$$
$$\therefore \quad \frac{dy}{dx} = \frac{3}{1+x^2}$$

3. $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$

Let $x = \cos\theta$. Therefore,

$$y = \sec^{-1}\left(\frac{1}{2\cos^2\theta-1}\right)$$
$$= \sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$$
$$= \sec^{-1}(\sec 2\theta)$$
$$= 2\theta$$
$$= 2\cos^{-1}x$$
$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2}{\sqrt{1-x^2}}$$

4. $y = \tan^{-1}\frac{4x}{1+5x^2} + \tan^{-1}\frac{2+3x}{3-2x}$

$$= \tan^{-1}\frac{5x-x}{1+5x\cdot x} + \tan^{-1}\frac{\frac{2}{3}+x}{1-\frac{2}{3}x}$$

$$= \tan^{-1}5x - \tan^{-1}x + \tan^{-1}\frac{2}{3} + \tan^{-1}x$$

$$= \tan^{-1}5x + \tan^{-1}\frac{2}{3}$$

$$\therefore \quad \frac{dy}{dx} = \frac{5}{1+25x^2}$$

5. Let $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$. Putting $x = \tan\theta$, we get

$$y = \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$
$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$
$$= \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$
$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{1}{2}\theta = \frac{1}{2}\tan^{-1}x$$
$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{1+x^2}\right)$$

6. $y = \tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right)$

Put $x = \sin\theta$. Then,

$$y = \tan^{-1}\left(\frac{\sin\theta}{1+\sqrt{1-\sin^2\theta}}\right) = \tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right)$$

$$= \tan^{-1}\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan^{-1}\tan\frac{\theta}{2} = \frac{\theta}{2}$$

So, $y = \frac{\sin^{-1}x}{2}$ or $\frac{dy}{dx} = \frac{1}{2\sqrt{1-x^2}}$

7. $\sin^{-1}\sqrt{1-x} = \sin^{-1}\sqrt{1-(\sqrt{x})^2} = \cos^{-1}\sqrt{x}$

$$\therefore \quad y = 2\cos^{-1}\sqrt{x}$$

or $$\frac{dy}{dx} = 2\times\frac{-1}{\sqrt{1-x}}\times\frac{1}{2\sqrt{x}} = \frac{-1}{\sqrt{x-x^2}}$$

8. $\frac{dy}{dx} = \frac{1}{2\sqrt{\sin\sqrt{x}}}\times\cos\sqrt{x}\times\frac{1}{2\sqrt{x}}$

9. Let $y = e^{\sin x^2}$.

Putting $x^2 = v$ and $u = \sin x^2 = \sin v$, we get

$y = e^u$, $u = \sin v$, and $v = x^2$

$$\therefore \quad \frac{dy}{du} = e^u, \ \frac{du}{dv} = \cos v, \text{ and } \frac{dv}{dx} = 2x$$

Now, $$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dv}\times\frac{dv}{dx}$$
$$= e^u\cos v\,2x = e^{\sin v}\cos v\,2x$$
$$= e^{\sin x^2}\cos x^2\,2x$$

10. $\frac{d}{dx}\left[\log\sqrt{\sin\sqrt{e^x}}\right] = \frac{d}{dx}\left[\frac{1}{2}\log\left(\sin\sqrt{e^x}\right)\right]$

$$= \frac{1}{2}\cot\sqrt{e^x}\,\frac{1}{2\sqrt{e^x}}e^x = \frac{1}{4}e^{x/2}\cot\left(e^{x/2}\right)$$

11. Let $y = a^{(\sin^{-1}x)^2}$.

Using chain rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left\{a^{(\sin^{-1}x)^2}\right\}$$

$$= a^{(\sin^{-1}x)^2} \log a \frac{d}{dx}\{(\sin^{-1}x)^2\}$$

$$= a^{(\sin^{-1}x)^2} (\log a)\, 2\,(\sin^{-1}x)^1 \frac{d}{dx}(\sin^{-1}x)$$

$$= a^{(\sin^{-1}x)^2} (\log a)\, 2 \sin^{-1} x \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{2\log a \sin^{-1} x}{\sqrt{1-x^2}} a^{(\sin^{-1}x)^2}$$

12. Let $y = \left[\log\left\{e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right\}\right] = \log e^x + \log\left(\frac{x-2}{x+2}\right)^{3/4}$

$$= x + \frac{3}{4}[\log(x-2) - \log(x+2)]$$

$$\therefore \quad \frac{dy}{dx} = 1 + \frac{3}{4}\left[\frac{1}{x-2} - \frac{1}{x+2}\right] = 1 + \frac{3}{(x^2-4)} = \frac{x^2-1}{x^2-4}$$

13. $y = \sin^{-1}\left[\sqrt{x-ax} - \sqrt{a-ax}\right]$

$$= \sin^{-1}\left[\sqrt{x}\sqrt{1-a} - \sqrt{a}\sqrt{1-x}\right]$$

$$= \sin^{-1}\left[\sqrt{x}\sqrt{1-(\sqrt{a})^2} - \sqrt{a}\sqrt{1-(\sqrt{x})^2}\right]$$

$$= \sin^{-1}\sqrt{x} - \sin^{-1}\sqrt{a}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}}$$

14. $y = x^3 e^x \sin x$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(x^3 e^x \sin x)$$

$$= \left\{\frac{d}{dx}(x^3)\right\} e^x \sin x + x^3 \left\{\frac{d}{dx}(e^x)\right\} \sin x + x^3 e^x \left\{\frac{d}{dx}(\sin x)\right\}$$

$$= 3x^2 e^x \sin x + x^3 e^x \sin x + x^3 e^x \cos x$$

$$= x^2 e^x (3 \sin x + x \sin x + x \cos x)$$

15. $y = \log\sqrt{\frac{1+\sin x}{1-\sin x}} = \frac{1}{2}\left[\log_e(1+\sin x) - \log_e(1-\sin x)\right]$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}\left\{\frac{1}{1+\sin x}\frac{d}{dx}(1+\sin x) - \frac{1}{1-\sin x}\frac{d}{dx}(1-\sin x)\right\}$$

$$= \frac{1}{2}\left\{\frac{\cos x}{1+\sin x} + \frac{\cos x}{1-\sin x}\right\}$$

$$= \frac{1}{2}\cos x\left(\frac{2}{1-\sin^2 x}\right) = \frac{\cos x}{\cos^2 x} = \sec x$$

or $$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{3}} = \sec\frac{\pi}{3} = 2$$

16. Using quotient rule, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x+\sin x}{x+\cos x}\right)$$

$$= \frac{(x+\cos x).\frac{d}{dx}(x+\sin x) - (x+\sin x).\frac{d}{dx}(x+\cos x)}{(x+\cos x)^2}$$

$$= \frac{(x+\cos x)\cdot(1+\cos x) - (x+\sin x)\cdot(1-\sin x)}{(x+\cos x)^2}$$

$$= \frac{x+\cos x + x\cos x + \cos^2 x - x - \sin x + x\sin x + \sin^2 x}{(x+\cos x)^2}$$

$$= \frac{\cos x - \sin x + x\cos x + x\sin x + \cos^2 x + \sin^2 x}{(x+\cos x)^2}$$

$$= \frac{\cos x - \sin x + x(\cos x + \sin x) + 1}{(x+\cos x)^2}$$

17. $\frac{dy}{dx} = 1(1+x^2)(1+x^4)\cdots(1+x^{2^n}) + 2x(1+x)(1+x^4)\cdots(1+x^{2^n})$

$$+ 4x^3(1+x)(1+x^2)(1+x^8)\cdots(1+x^{2^n})$$

$$\vdots$$

$$+ 2^n x^{2^n-1}(1+x)(1+x^2)\cdots(1+x^{2^{n-1}})$$

or $$\left.\frac{dy}{dx}\right|_{x=0} = 1$$

18. We have $x\sqrt{1+y} + y\sqrt{1+x} = 0$

or $x\sqrt{1+y} = -y\sqrt{1+x}$

or $x^2(1+y) = y^2(1+x)$ [On squaring both sides]

or $x^2 - y^2 = y^2 x - x^2 y$

or $(x+y)(x-y) = -xy(x-y)$

or $x + y = -xy$ [$\because$ $x - y \neq 0$ as $x = y$ does not satisfy the given equation]

or $y = -\frac{x}{1+x}$

or $$\frac{dy}{dx} = -\left\{\frac{(1+x)\times 1 - x(0+1)}{(1+x^2)}\right\} = -\frac{1}{(1+x^2)}$$

19. Since g is the inverse function of f, we have $f(g(x)) = x$

$\Rightarrow \quad \frac{d}{dx}(f(g(x))) = 1$

$\Rightarrow \quad f'(g(x)).\, g'(x) = 1$

$\Rightarrow \quad \sin(g(x))\, g'(x) = 1$

$\Rightarrow \quad g'(x) = \frac{1}{\sin\{g(x)\}}$

Exercise 3.3

1. Given $x^3 + y^3 - 3axy = 0$. From

$$\frac{dy}{dx} = -\frac{\text{differentiating } f \text{ w.r.t. } x \text{ keeping } y \text{ as constant}}{\text{differentiating } f \text{ w.r.t. } y \text{ keeping } x \text{ as constant}}$$

$$= -\frac{3x^2 - 3ay}{3y^2 - 3ax} = \frac{ay - x^2}{y^2 - ax}$$

2. We have

$$y = b\tan^{-1}\left(\frac{x}{a} + \tan^{-1}\frac{y}{x}\right)$$

or $\tan\frac{y}{b}=\frac{x}{a}+\tan^{-1}\frac{y}{x}$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{b}\sec^2\left(\frac{y}{b}\right)\frac{dy}{dx}=\frac{1}{a}+\frac{1}{1+\left(\frac{y}{x}\right)^2}\times\frac{x\frac{dy}{dx}-y}{x^2}$$

or $$\frac{1}{b}\sec^2\left(\frac{y}{b}\right)\frac{dy}{dx}=\frac{1}{a}+\frac{x\frac{dy}{dx}-y}{x^2+y^2}$$

or $$\frac{dy}{dx}\left\{\frac{1}{b}\sec^2\left(\frac{y}{b}\right)-\frac{x}{x^2+y^2}\right\}=\frac{1}{a}-\frac{y}{x^2+y^2}$$

or $$\frac{dy}{dx}=\frac{\frac{1}{a}-\frac{y}{x^2+y^2}}{\frac{1}{b}\sec^2\left(\frac{y}{b}\right)-\frac{x}{x^2+y^2}}$$

3. We have,

$$\log_c(\log_e x-\log_e y)=e^{x^2y}(1-\log_e x) \qquad \ldots(1)$$

For $x=e$, $\log_c(1-\log_e y)=0$.

$\therefore \quad y=1$

Differentiating (1), w.r.t. x, we get

$$\frac{1}{\log_e x-\log_e y}\cdot\left(\frac{1}{x}-\frac{1}{y}y'\right)$$

$$=e^{x^2y}\cdot(2xy+x^2y')(1-\log_e x)-\frac{1}{x}e^{x^2y}$$

Putting $x=e$ and $y=1$, we get

$$\frac{1}{1-0}\cdot\left(\frac{1}{e}-y'\right)=0-\frac{1}{e}\cdot e^{e^2}$$

$$\therefore \quad y'(e)=\frac{1+e^{e^2}}{e}$$

4. $y=\sqrt{x+\sqrt{y+\sqrt{x+\sqrt{y+\ldots}}}}\ \infty=\sqrt{x+\sqrt{y+y}}$

or $y^2=x+\sqrt{2y}$

Differentiating w.r.t. x, we get

or $$2y\cdot\frac{dy}{dx}=1+\frac{1}{\sqrt{2y}}\times\frac{dy}{dx}$$

or $$\frac{dy}{dx}\left[2y-\frac{1}{\sqrt{2y}}\right]=1$$

or $$\frac{dy}{dx}=\frac{\sqrt{2y}}{2y\sqrt{2y}-1}=\frac{y^2-x}{2y^3-2xy-1}$$

Exercise 3.4

1. $$\frac{dx}{dt}=\frac{(1+t^2)\,2-2t\times2t}{(1+t^2)^2}=\frac{2-2t^2}{(1+t^2)^2}$$

$$\frac{dy}{dt}=\frac{(1+t^2)(-2t)-(1-t^2)2t}{(1+t^2)^2}=\frac{-4t}{(1+t^2)^2}$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{-4t}{2-2t^2}=\frac{2t}{t^2-1}$$

or $$\left.\frac{dy}{dx}\right|_{t=2}=\frac{4}{3}$$

2. $x=\sqrt{a^{\sin^{-1}t}}$, $y=\sqrt{a^{\cos^{-1}t}}$

or $$x\cdot y=\sqrt{a^{\sin^{-1}t+\cos^{-1}t}}=\sqrt{a^{\pi/2}}$$

Differentiating w.r.t. x, we get

$$x\frac{dy}{dx}+y=0$$

or $$\frac{dy}{dx}=\frac{-y}{x}$$

3. The given equations are $x=\cos\theta-\cos 2\theta$

and $y=\sin\theta-\sin 2\theta$

Then, $$\frac{dx}{d\theta}=-\sin\theta-(-2\sin 2\theta)=2\sin 2\theta-\sin\theta$$

And $$\frac{dy}{d\theta}=\cos\theta-2\cos 2\theta$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{\cos\theta-2\cos 2\theta}{2\sin 2\theta-\sin\theta}$$

4. We have, $x=3\cos\theta-2\cos^3\theta$

and $y=3\sin\theta-2\sin^3\theta$

$$\therefore \quad \frac{dx}{d\theta}=-3\sin\theta-2\times3\cos^2\theta\frac{d}{d\theta}(\cos\theta)$$

$$=-3\sin\theta+6\cos^2\theta\sin\theta$$

And $$\frac{dy}{d\theta}=3\cos\theta-2\times3\sin^2\theta\frac{d}{d\theta}(\sin\theta)$$

$$=3\cos\theta-6\sin^2\theta.\cos\theta$$

Now, $$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{3\cos\theta-6\sin^2\theta\cos\theta}{-3\sin\theta+6\cos^2\theta\sin\theta}$$

$$=\frac{3\cos\theta(1-2\sin^2\theta)}{3\sin\theta(-1+2\cos^2\theta)}=\cot\theta.\frac{\cos 2\theta}{\cos 2\theta}=\cot\theta$$

5. $x=a\left[\cos t+\log\tan\frac{t}{2}\right]$ and $y=a\sin t$

Differentiating w.r.t. t, we get

$$\frac{dx}{dt}=a\left[-\sin t+\frac{1}{\tan t/2}\sec^2\frac{t}{2}\times\frac{1}{2}\right]$$

$$=a\left[-\sin t+\frac{1}{2\sin(t/2)\cos(t/2)}\right]$$

$$=a\left[-\sin t+\frac{1}{\sin t}\right] \text{ and } \frac{dy}{dt}=a\cos t$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{a\cos t}{\frac{a\cos^2 t}{\sin t}}=\tan t$$

At $t=\pi/4$, $\frac{dy}{dx}=1$

Exercise 3.5

1. Let $y=\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Taking logarithm on both sides, we get

$$\log y=\log\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

$$=\frac{1}{2}[\log(x-1)+\log(x-2)-\log(x-3)$$
$$-\log(x-4)-\log(x-5)]$$

Differentiating both sides with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx}=\left(\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right)$$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\times$$

$$\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right]$$

2. We have $x^y = e^{x-y}$

or $e^{y\log x} = e^{x-y}$ $\quad\left[\because x^y = e^{\log x^y} = e^{y\log x}\right]$

or $y\log x = x - y$

or $y = \dfrac{x}{1+\log x}$

On differentiating both the sides w.r.t. x, we get

$$\frac{dy}{dx}=\frac{(1+\log x)\times 1 - x\left(0+\frac{1}{x}\right)}{(1+\log x)^2}=\frac{\log x}{(1+\log x)^2}$$

3. The given function is $xy = e^{(x-y)}$.

Taking logarithm on both the sides, we obtain

$\log(xy) = \log(e^{x-y})$

$\log x + \log y = (x - y)$

Differentiating both sides with respect to x, we get

$$\frac{1}{x}+\frac{1}{y}\frac{dy}{dx}=1-\frac{dy}{dx}$$

or $$\left(1+\frac{1}{y}\right)\frac{dy}{dx}=1-\frac{1}{x}$$

$$\therefore \quad \frac{dy}{dx}=\frac{y(x-1)}{x(y+1)}$$

4. $y^x = x^y$

or $\log y^x = \log x^y$

or $x\log y = y\log x$

or $$x\frac{1}{y}\frac{dy}{dx}+\log y\times 1=\frac{y}{x}+\log x\frac{dy}{dx}$$

or $$\left(\frac{x}{y}-\log x\right)\frac{dy}{dx}=\frac{y}{x}-\log y$$

or $$\frac{dy}{dx}=\frac{\frac{y}{x}-\log y}{\frac{x}{y}-\log x}=\frac{y(y-x\log y)}{x(x-y\log x)}=\frac{y(x\log y-y)}{x(y\log x-x)}$$

5. Here, $x = e^{y+x}$

or $\log x = (y + x)$

or $y = \log x - x$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{x}-1=\frac{1-x}{x}$$

6. Let $y = x^x$. Then, $y = e^{x\log x}$.

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = e^{x\log x}\frac{d}{dx}(x\log x) \qquad [\because e^{x\log x} = x^x]$$

$$= x^x\left(\log x + x\frac{1}{x}\right)$$

$$= x^x(1 + \log x)$$

7. Let $y = (x\cos x)^x$.

Taking logarithm on both sides, we get

$$\log y = \log(x\cos x)^x$$

$$= x\log(x\cos x)$$

$$= x\log x + x\log\cos x$$

Differentiating both sides with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = x\cdot\frac{1}{x} + \log x + \log\cos x + x\cdot\left(\frac{-\sin x}{\cos x}\right)$$

or $$\frac{dy}{dx} = (x\cos x)^x\left[(\log x + 1) + \{\log\cos x - x\cot x\}\right]$$

8. Taking logarithm on both sides, we get

$$\log y = (\tan x)^{\tan x}\log\tan x$$

$\therefore$ $\log\log y = [\tan x\log\tan x] + \log\log\tan x$

Differentiating w.r.t. x, we get

$$\frac{1}{y\log y}\frac{dy}{dx} = \sec^2 x\log\tan x + \tan x\frac{\sec^2 x}{\tan x}$$

$$+\frac{1}{\log\tan x}\times\frac{\sec^2 x}{\tan x}$$

$$\therefore \quad \frac{dy}{dx} = y\log y\sec^2 x\,[\log\tan x + 1 + 1/(\tan x\log\tan x)]$$

Exercise 3.6

1. Let $y = \tan^{-1}\dfrac{2x}{1-x^2} = 2\tan^{-1}x$

and $z = \sin^{-1}\dfrac{2x}{1+x^2} = 2\tan^{-1}x$

$$\therefore \quad \frac{dy}{dz} = 1$$

2. Let $y = \sec^{-1}\left(\dfrac{1}{2x^2-1}\right)$ and $z = \sqrt{1-x^2}$

Put $x = \cos\theta$. Therefore,

$y = \sec^{-1}(\sec 2\theta) = 2\theta$ and $z = \sqrt{1-\cos^2\theta} = \sin\theta$

$$\therefore \quad \frac{dy}{d\theta} = 2;\ \frac{dz}{d\theta} = \cos\theta \text{ or } \frac{dy}{dz} = \frac{2}{\cos\theta} = \frac{2}{x}$$

At $x = \dfrac{1}{2}$, $\dfrac{dy}{dz} = 4$

3. Let $u = \dfrac{x}{\sin x}$ and $v = \sin x$

$$\therefore \quad \frac{du}{dx} = \frac{\sin x\cdot\frac{d}{dx}x - x\cdot\frac{d}{dx}\sin x}{(\sin x)^2}$$

$$= \frac{\sin x - x\cos x}{\sin^2 x} \qquad \ldots(i)$$

And $$\frac{dv}{dx} = \frac{d}{dx}(\sin x) = \cos x \qquad \ldots(ii)$$

$$\therefore \quad \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{\frac{\sin x - x\cos x}{\sin^2 x}}{\cos x}$$

$$= \frac{\tan x - x}{\sin^2 x}$$

4. We have $y = f(x^3)$

$$\therefore \quad \frac{dy}{dx} = f'(x^3)\,3x^2 = 3x^2 \tan x^3$$

Also, $z = g(x^5)$

$$\therefore \quad \frac{dz}{dx} = g'(x^5)\,5x^4 = 5x^4 \sec x^5$$

$$\therefore \quad \frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{3x^2 \tan x^3}{5x^4 \sec x^5}$$

$$= \frac{3}{5x^2} \times \frac{\tan x^3}{\sec x^5}$$

or $$\lim_{x\to 0} \frac{(dy/dz)}{x} = \lim_{x\to 0} \frac{3 \tan x^3}{5x^3 \sec x^5}$$

$$= \frac{3}{5}$$

Exercise 3.7

1. We have

$$f(x) = \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}$$

$$\therefore \quad f'(x) = \begin{vmatrix} 1 & 0 & 0 \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix} + \begin{vmatrix} x+a^2 & ab & ac \\ 0 & 1 & 0 \\ ac & bc & x+c^2 \end{vmatrix} + \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x+b^2 & bc \\ bc & x+c^2 \end{vmatrix} + \begin{vmatrix} x+a^2 & ac \\ ac & x+c^2 \end{vmatrix} + \begin{vmatrix} x+a^2 & ab \\ ab & x+b^2 \end{vmatrix}$$

$$= [(x+b^2)(x+c^2) - b^2c^2] + [(x+a^2)(x+c^2) - a^2c^2] + [(x+a^2)(x+b^2) - a^2b^2]$$

$$= 3x^2 + 2x(a^2+b^2+c^2)$$

2. Since in R_3, putting $x = 0$, all the elements are zero then

$$f'(0) = 0 + 0 + \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 2\cos 0 & 0 & 4(0)\cos 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{vmatrix} = 2$$

3. $$g(x) = \begin{vmatrix} f(x+c) & f(x+2c) & f(x+3c) \\ f(c) & f(2c) & f(3c) \\ f'(c) & f'(2c) & f'(3c) \end{vmatrix}$$

$$\therefore \quad g(0) = 0$$

$$\therefore \quad \lim_{x\to 0} \frac{g(x)}{x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{g'(x)}{1} \qquad \text{(using L'Hopital rule)}$$

$$= g'(0)$$

Now, $$g'(x) = \begin{vmatrix} f'(x+c) & f'(x+2c) & f'(x+3c) \\ f(c) & f(2c) & f(3c) \\ f'(c) & f'(2c) & f'(3c) \end{vmatrix}$$

$$\therefore \quad g'(0) = \begin{vmatrix} f'(c) & f'(2c) & f'(3c) \\ f(c) & f(2c) & f(3c) \\ f'(c) & f'(2c) & f'(3c) \end{vmatrix} = 0$$

$$\therefore \quad \lim_{x\to 0} \frac{g(x)}{x} = 0$$

Exercise 3.8

1. $f(x) = (1+x)^n, f(0) = 1$

$$\therefore \quad f'(x) = n(1+x)^{n-1}, f'(0) = n$$

$$\therefore \quad f''(x) = n(n-1)(1+x)^{n-2}, f''(0) = n(n-1)$$

Similarly, proceeding, we have

$f'''(0) = n(n-1)(n-2)$

$f''''(0) = n(n-1)(n-2)(n-3)$ and so on.

$f^n(0) = n(n-1)(n-2)(n-3)\cdots 1$

or $$f(0) + f'(0) + \frac{f''(0)}{2!} + \frac{f'''(0)}{3!} + \cdots + \frac{f^n(0)}{n!}$$

$$= 1 + n + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \cdots + \frac{n(n-1)(n-2)\ldots 1}{n!} + \cdots$$

$$= {}^nC_0 + {}^nC_1 + {}^nC_2 \cdots + {}^nC_{n-1} + {}^nC_n = 2^n$$

2. Given relation is $e^y(x+1) = 1$.

Differentiating both sides w.r.t. x, we get

$$(x+1)e^y \frac{dy}{dx} + e^y \cdot 1 = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{x+1} \qquad (1)$$

Differentiating again w.r.t. x both sides, we get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{1}{x+1}\right)$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{1}{(x+1)^2} \qquad (2)$$

From (1) and (2), we get $\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$.

3. $$\frac{d}{dx}\left[e^{2x} + e^{-2x}\right] = 2e^{2x} - 2e^{-2x} = 2^1\left[e^{2x} - e^{-2x}\right]$$

$$\frac{d^2}{dx^2}(e^{2x} + e^{-2x}) = \frac{d}{dx}2(e^{2x} - e^{-2x}) = 2^2(e^{2x} + e^{-2x})$$

$$\frac{d^3}{dx^3}(e^{2x} + e^{-2x}) = \frac{d}{dx}2^2(e^{2x} + e^{-2x}) = 2^3(e^{2x} - e^{-2x})$$

$$\because \quad \frac{d^n}{dx^n}[e^{2x} + e^{-2x}] = 2^n[e^{2x} + (-1)^n e^{-2x}]$$

4. $$\frac{dy}{dx} = \cos(\sin x)\cos x$$

$$\frac{d^2y}{dx^2} = -\cos(\sin x)\sin x + \cos x\,[-\sin(\sin x)]\cos x$$

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}\tan x = -\cos(\sin x)\sin x - \cos^2 x \sin(\sin x) + \cos(\sin x)\cos x \tan x$$
$$= -\cos^2 x \sin(\sin x)$$

$$\therefore \quad \frac{d^2y}{dx^2}+\frac{dy}{dx}\tan x + \cos^2 x \sin(\sin x) = 0$$

$$\therefore \quad f(x) = \cos^2 x \sin(\sin x)$$

5. $y = \log(1+\sin x)$ (1)

$$y_1 = \frac{\cos x}{1+\sin x} \quad (2)$$

$$y_2 = \frac{-\sin x(1+\sin x)-\cos x \cos x}{(1+\sin x)^2}$$

$$= \frac{-(1+\sin x)}{(1+\sin x)^2}$$

$$= -\frac{1}{(1+\sin x)} \quad (3)$$

$$y_3 = \frac{\cos x}{(1+\sin x)^2} = \frac{\cos x}{1+\sin x}\times\frac{1}{1+\sin x} = -y_1y_2 \quad (4)$$

$$\therefore \quad y_4 = -y_2^2 - y_1y_3$$

or $y_4 + y_3y_1 + y_2^2 = 0$

6. $$\frac{d^n}{dx^n}[f(x)] = \begin{vmatrix} \frac{d^n}{dx^n}(x^n) & n! & 2 \\ \frac{d^n}{dx^n}(\cos x) & \cos\frac{n\pi}{2} & 4 \\ \frac{d^n}{dx^n}(\sin x) & \sin\frac{n\pi}{2} & 8 \end{vmatrix}$$

$$= \begin{vmatrix} n! & n! & 2 \\ \cos\left(x+\frac{n\pi}{2}\right) & \cos\frac{n\pi}{2} & 4 \\ \sin\left(x+\frac{n\pi}{2}\right) & \sin\frac{n\pi}{2} & 8 \end{vmatrix}$$

or $$\frac{d^n}{dx^n}[f(x)]_{x=0} = 0$$

7. We have $y = b\sin\theta$, $x = a\cos\theta$.

Therefore $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\frac{b}{a}\cot\theta$$

or $$\frac{d^2y}{dx^2} = \frac{b}{a}\text{cosec}^2\theta\frac{d\theta}{dx} = -\frac{b}{a^2}\text{cosec}^3\theta$$

or $$\frac{d^3y}{dx^3} = -\frac{b}{a^2}3\text{cosec}^2\theta(-\text{cosec}\,\theta\cot\theta)\frac{d\theta}{dx}$$

$$= \frac{3b}{a^2}\text{cosec}^3\theta\cot\theta\times\frac{-1}{a\sin\theta} = -\frac{3b}{a^3}\text{cosec}^4\theta\cot\theta$$

8. $x = a\cos^3\theta$, $y = b\sin^3\theta$

$$y_1 = \frac{dy}{dx} = \frac{3b\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$$

$$= -\frac{b}{a}\tan\theta, \text{ if } \sin\theta\neq 0, \cos\theta\neq 0$$

Therefore, y_1 does not exist at $\theta = 0$.

Hence, y_2 and y_3 do not exist at $\theta = 0$.

Exercise 3.9

1. $f(x+y) = f(x)f(y)$ (1)

$$f'(5) = \lim_{h\to 0}\frac{f(5+h)-f(5)}{h}$$
$$= \lim_{h\to 0}\frac{f(5)f(h)-f(5)}{h}$$
$$= f(5)\lim_{h\to 0}\frac{f(h)-1}{h}$$
$$= f(5)\lim_{h\to 0}\frac{f(h)-1}{h}$$

In (1), replace x by 5 and y by 0. Then, $f(5+0) = f(5)\cdot f(0)$

or $f(0) = 1$

or $$f'(5) = f(5)\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$
$$= f(5)f'(0) = 2\times 3 = 6$$

2. $f(xy) = f(x)f(y)$ (1)

$$f'(2) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h\to 0}\frac{f\left(2\left(1+\frac{h}{2}\right)\right)-f(2)}{h}$$
$$= \lim_{h\to 0}\frac{f(2)f\left(1+\frac{h}{2}\right)-f(2)}{h}$$
$$= \frac{f(2)}{2}\lim_{h\to 0}\frac{f\left(1+\frac{h}{2}\right)-1}{\frac{h}{2}}$$

Replace x and y by 1 in equation (1). Then

$f(1) = (f(1))^2$

or $f(1) = 1$

$$\therefore \quad f'(2) = \frac{f(2)}{2}\lim_{h\to 0}\frac{f\left(1+\frac{h}{2}\right)-f(0)}{\frac{h}{2}} = \frac{f(2)f'(1)}{2} = \frac{3}{2}$$

3. $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{f(x)+f(h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{f(h)}{h}$$
$$= \lim_{h\to 0}\frac{(2h^2+3h)g(h)}{h}$$
$$= \lim_{h\to 0}(2h+3)\,g(h)$$
$$= (0+3)\,g(0)$$
$$= 3g(0)$$
$$= 3\times 3$$
$$= 9$$

4. $g(x) = g(y)\,g(x-y)$

Differentiating w.r.t. x, keeping y constant,

$g'(x) = g(y)[g'(x-y)]$

Put $y = x$. Then,

$g'(x) = g(x) \cdot g'(0) = a \,.\, g(x)$

or $g(x) = ae^x$ $[\because g(0) = 1]$

or $g'(x) = ae^x, g'(3) = ae^3 = b$

or $g'(-3) = ae^{-3} = \dfrac{a^2}{b}$

5. For any $x \in R^+$, we have

$f(x^m y^n) = mf(x) + nf(y)$ (1)

$\therefore$ $f(1) = f(1) + f(1)$ [Putting $x = y = m = n = 1$]

or $f(1) = 0$

or $\lim\limits_{x\to 0} \dfrac{f(1+x)}{x} = \lim\limits_{x\to 0} \dfrac{f'(1+x)}{1}$ (using L'Hopital's rule)

$= f'(1) = e$

6. Given $f\left(\dfrac{x+2y}{3}\right) = \dfrac{f(x)+2f(y)}{3}$ $\forall x, y \in R$, which satisfies section formula for abscissa on L.H.S. and ordinate on R.H.S. Hence, $f(x)$ must be the linear function (as only straight line satisfies such section formula).

Hence, $f(x) = ax + b$.

But $f(0) = 2 \Rightarrow b = 2$ and $f'(0) = 1 \Rightarrow a = 1$.

Thus, $f(x) = x + 2$.

7. $f''(x) = \lim\limits_{h\to 0} \dfrac{f'(x+h) - f'(x)}{h}$

$= \lim\limits_{h\to 0} \dfrac{\lim\limits_{k\to 0}\left[\dfrac{f(x+h+k)-f(x+h)}{k} - \dfrac{f(x+k)-f(x)}{k}\right]}{h}$

Let $k = -h$. Then,

$f''(x) = -\lim\limits_{h\to 0} \dfrac{f(x) - f(x+h) - f(x-h) + f(x)}{h^2}$

$= \lim\limits_{h\to 0} \dfrac{f(x+h) + f(x-h) - 2f(x)}{h^2}$

Exercises

Single Correct Answer Type

1. (4) $y = a \sin x + b \cos x$

Differentiating with respect to x, we get

$\dfrac{dy}{dx} = a\cos x - b\sin x$

Now, $\left(\dfrac{dy}{dx}\right)^2 = (a\cos x - b\sin x)^2$

$= a^2 \cos^2 x + b^2 \sin^2 x - 2ab \sin x \cos x$

and $y^2 = (a \sin x + b \cos x)^2$

$= a^2 \sin^2 x + b^2 \cos^2 x + 2ab \sin x \cos x$

So, $\left(\dfrac{dy}{dx}\right)^2 + y^2 = a^2(\sin^2 x + \cos^2 x) + b^2(\sin^2 x + \cos^2 x)$

$= (a^2 + b^2) = \text{constant}$

2. (2) $y = \sqrt{\dfrac{1-\sin 2x}{1+\sin 2x}} = \dfrac{\cos x - \sin x}{\cos x + \sin x} = \dfrac{1-\tan x}{1+\tan x} = \tan\left(\dfrac{\pi}{4} - x\right)$

$\therefore$ $\dfrac{dy}{dx} = -\sec^2\left(\dfrac{\pi}{4} - x\right)$

3. (3) In neighborhood of $x = \dfrac{2\pi}{3}$, $|\cos x| = -\cos x$ and $|\sin x| = \sin x$

or $y = -\cos x + \sin x$

$\therefore$ $\dfrac{dy}{dx} = \sin x + \cos x$

Thus, at $x = \dfrac{2\pi}{3}$, we get $\dfrac{dy}{dx} = \sin\dfrac{2\pi}{3} + \cos\dfrac{2\pi}{3} = \dfrac{\sqrt{3}-1}{2}$.

4. (2) For $x > 1$, we have $f(x) = |\log|x|| = \log x$

or $f'(x) = \dfrac{1}{x}$

For $x < -1$, we have $f(x) = |\log|x|| = \log(-x)$

or $f'(x) = \dfrac{1}{x}$

For $0 < x < 1$, we have $f(x) = |\log|x|| = -\log x$

or $f'(x) = \dfrac{-1}{x}$

For $-1 < x < 0$, we have $f(x) = -\log(-x)$

or $f'(x) = \dfrac{-1}{x}$.

Hence, $f'(x) = \begin{cases} \dfrac{1}{x}, & |x| > 1 \\ -\dfrac{1}{x}, & |x| < 1 \end{cases}$

5. (3) $f(x) = \sqrt{1-\sin 2x} = \sqrt{(\cos x - \sin x)^2}$

$= |\cos x - \sin x|$

$= \begin{cases} \cos x - \sin x, & \text{for } 0 \le x \le \pi/4 \\ -(\cos x - \sin x), & \text{for } \pi/4 < x \le \pi/2 \end{cases}$

$\therefore$ $f'(x) = \begin{cases} -(\cos x + \sin x), & \text{for } 0 < x < \pi/4 \\ (\cos x + \sin x), & \text{for } \pi/4 < x < \pi/2. \end{cases}$

6. (3) $D^*(x) = \lim\limits_{h\to 0} \dfrac{f^2(x+h) - f^2(x)}{h}$

$= \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}(f(x+h) + f(x))$

$= 2f(x) \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$= 2f(x) \cdot f'(x)$

or $D^*(x \log x) = 2x \log x (1 + \log x)$

or $D^* f(x)|_{x=e} = 4e$

7. (1) $y = \cot^{-1}\left[\dfrac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right]$

$= \cot^{-1}\left[\dfrac{2+2\cos x}{2\sin x}\right] = \cot^{-1}\left[\dfrac{1+\cos x}{\sin x}\right]$

$= \cot^{-1}\left[\cot\dfrac{x}{2}\right] = \dfrac{x}{2}$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{1}{2}$

8. (2) $y=\dfrac{(a-x)^{3/2}+(x-b)^{3/2}}{\sqrt{a-x}+\sqrt{x-b}}$

$=\dfrac{\left(\sqrt{a-x}+\sqrt{x-b}\right)\left(a-x-\sqrt{a-x}\sqrt{x-b}+x-b\right)}{\sqrt{a-x}+\sqrt{x-b}}$

$=a-b-\sqrt{a-x}\sqrt{x-b}$

$\therefore \quad \dfrac{dy}{dx}=\dfrac{1}{2\sqrt{a-x}}\sqrt{x-b}-\dfrac{1}{2\sqrt{x-b}}\sqrt{a-x}$

$=\dfrac{2x-a-b}{2\sqrt{a-x}\sqrt{x-b}}$

9. (2) $\dfrac{dy}{dx}=-[(2-x)(3-x)\cdots(n-x)+(1-x)(3-x)\cdots$

$(n-x)+\cdots(1-x)(2-x)\cdots(n-1-x)]$

At $x=1$,

$\dfrac{dy}{dx}=-[(n-1)!+0+\cdots+0]=(-1)(n-1)!$

10. (3) We have $y=\sqrt{\dfrac{1-x}{1+x}}$

Differentiating w.r.t. x, we get

$\dfrac{dy}{dx}=\dfrac{1}{2}\left(\dfrac{1-x}{1+x}\right)^{1/2-1}\dfrac{d}{dx}\left(\dfrac{1-x}{1+x}\right)$

$=\dfrac{1}{2}\sqrt{\dfrac{1+x}{1-x}}\cdot\dfrac{(1+x)(-1)-(1-x)(1)}{(1+x)^2}$

$=-\sqrt{\dfrac{1+x}{1-x}}\dfrac{1}{(1+x)^2}$

or $(1-x^2)\dfrac{dy}{dx}=-\sqrt{\dfrac{1+x}{1-x}}\dfrac{1}{(1+x)^2}(1-x^2)$

or $(1-x)^2\dfrac{dy}{dx}=-\sqrt{\dfrac{1-x}{1+x}}$

or $(1-x^2)\dfrac{dy}{dx}=-y$

or $\left(1-x^2\right)\dfrac{dy}{dx}+y=0$

11. (1) $y=\dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}$

$=\dfrac{\left(\sqrt{a+x}-\sqrt{a-x}\right)^2}{(a+x)-(a-x)}$

$=\dfrac{(a+x)+(a-x)-2\left(\sqrt{a^2-x^2}\right)}{2x}$

$=\dfrac{2a-2\sqrt{a^2-x^2}}{2x}=\dfrac{a-\sqrt{a^2-x^2}}{x}$ (1)

Differentiating w.r.t. x, we get

$\dfrac{dy}{dx}=\dfrac{x\left[-\dfrac{1}{2\sqrt{a^2-x^2}}(-2x)\right]-\left(a-\sqrt{a^2-x^2}\right)}{x^2}$

$=\dfrac{x^2-a\sqrt{a^2-x^2}+a^2-x^2}{x^2\sqrt{a^2-x^2}}=\dfrac{a\left(a-\sqrt{a^2-x^2}\right)}{x^2\sqrt{a^2-x^2}}$

$=\dfrac{a}{x\sqrt{a^2-x^2}}\left[\dfrac{a-\sqrt{a^2-x^2}}{x}\right]=\dfrac{ay}{x\sqrt{a^2-x^2}}$ [by (1)]

12. (1) $u(x)=7v(x)$ or $u'(x)=7v'(x)$ or $p=7$ (given)

Again $\dfrac{u(x)}{v(x)}=7$ or $\left(\dfrac{u(x)}{v(x)}\right)'=0$ or $q=0$

Now, $\dfrac{p+q}{p-q}=\dfrac{7+0}{7-0}=1$

13. (4) We have $\sin^{-1}\left(\dfrac{x^2-y^2}{x^2+y^2}\right)=\log a$

or $\dfrac{x^2-y^2}{x^2+y^2}=\sin(\log a)$

or $\dfrac{1-\tan^2\theta}{1+\tan^2\theta}=\sin(\log a)$ (on putting $y=x\tan\theta$)

or $\cos 2\theta=\sin(\log a)$

or $2\theta=\cos^{-1}(\sin(\log a))$

or $\theta=\dfrac{1}{2}\cos^{-1}(\sin(\log a))$

or $\tan^{-1}\left(\dfrac{y}{x}\right)=\dfrac{1}{2}\cos^{-1}(\sin(\log a))$

or $\dfrac{y}{x}=\tan\left(\dfrac{1}{2}\cos^{-1}(\sin(\log a))\right)$

Differentiating w.r.t. x, we get

$\dfrac{x\dfrac{dy}{dx}-y}{x^2}=0$

or $x\dfrac{dy}{dx}-y=0$

or $\dfrac{dy}{dx}=\dfrac{y}{x}$

14. (3) $f'(x)=(kx+e^x)h'(x)+h(x)(k+e^x)$

$f'(0)=h'(0)+h(0)(k+1)$

or $18=-2+5(k+1)$ or $k=3$

15. (3) $\lim\limits_{t\to x}\dfrac{e^t f(x)-e^x f(t)}{(t-x)(f(x))^2}=2$

or $\lim\limits_{t\to x}\dfrac{e^t f(x)-e^x f'(t)}{1\cdot(f(x))^2}=2$ (Using L' Hospital Rule)

or $\dfrac{e^x f(x)-e^x f'(x)}{1\cdot(f(x))^2}=2$

or $\dfrac{d}{dx}\left(\dfrac{e^x}{f(x)}\right)=2$

or $\dfrac{e^x}{f(x)}=2x+c$

given $f(0)=2$

or $\dfrac{1}{f(0)}=c$

or $c = 2$

or $f(x) = \dfrac{e^x}{2x+2}$

$\therefore \quad f'(x) = \dfrac{e^x(2x+2) - 2e^x}{(2x+2)^2}$

Hence, $(f'(0) = 0$

16. (3) $y'(x) = f'(f(f(f(x)))) f'(f(f(x))) f'(f(x)) f'(x)$

$\therefore \quad y'(0) = f'(f(f(f(0)))) f'(f(f(0))) f'(f(0)) f'(0)$

$= f'(f(f(0))) f'(f(0)) f'(0) f'(0)$

$= f'(f(0)) f'(0) f'(0) f'(0)$

$= f'(0) f'(0) f'(0) f'(0)$

$= (f'(0))^4 = 2^4 = 16$

17. (2) $f(x) = \sqrt{1+\cos^2(x^2)}$

$\therefore \quad f'(x) = \dfrac{1}{2\sqrt{1+\cos^2(x)^2}} (2\cos x^2)(-\sin x^2)(2x)$

$= \dfrac{-x\sin 2x^2}{\sqrt{1+\cos^2(x^2)}}$

or $f'\left(\dfrac{\sqrt{\pi}}{2}\right) = \dfrac{-\dfrac{\sqrt{\pi}}{2}\sin\dfrac{2\pi}{4}}{\sqrt{1+\cos^2\dfrac{\pi}{4}}} = \dfrac{-\dfrac{\sqrt{\pi}}{2}1}{\sqrt{\dfrac{3}{2}}}$

$\therefore \quad f'\left(\dfrac{\sqrt{\pi}}{2}\right) = -\sqrt{\dfrac{\pi}{6}}$

18. (1) $\dfrac{d}{dx}\cos^{-1}\sqrt{\cos x} = \dfrac{\sin x}{2\sqrt{\cos x}\sqrt{1-\cos x}}$

$= \dfrac{\sqrt{1-\cos^2 x}}{2\sqrt{\cos x}\sqrt{1-\cos x}} = \dfrac{1}{2}\sqrt{\dfrac{1+\cos x}{\cos x}}$

19. (4) $y = \tan^{-1}\left(\dfrac{2^{x+1}-2^x}{1+2^x\cdot 2^{x+1}}\right) = \tan^{-1}2^{(x+1)} - \tan^{-1}2^x$

$\therefore \quad y' = \dfrac{2^{x+1}\ln 2}{1+(2^{x+1})^2} - \dfrac{2^x \ln 2}{1+(2^x)^2}$

or $y'(0) = -\dfrac{1}{10}\ln 2$

20. (1) $\dfrac{dy}{dx} = \dfrac{d}{dx}\left[\left(x+\sqrt{x^2+a^2}\right)^n\right]$

$= n\left(x+\sqrt{x^2+a^2}\right)^{n-1}\cdot\dfrac{d}{dx}\left(x+\sqrt{x^2+a^2}\right)$

$= n(x+\sqrt{x^2+a^2})^{n-1}\left(\dfrac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}\right)$

$= \dfrac{n(x+\sqrt{x^2+a^2})^n}{\sqrt{x^2+a^2}}$

$= \dfrac{ny}{\sqrt{x^2+a^2}}$

21. (3) $y = \dfrac{\log\tan x}{\log\sin x}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{(\log\sin x)\left(\dfrac{\sec^2 x}{\tan x}\right) - (\log\tan x)(\cot x)}{(\log\sin x)^2}$

or $\left(\dfrac{dy}{dx}\right)_{\pi/4} = \dfrac{-4}{\log 2}$ (On simplification)

22. (2) Let $y = \sin^2\cot^{-1}\left\{\sqrt{\dfrac{1-x}{1+x}}\right\}$

Put $x = \cos\theta$ or $\theta = \cos^{-1}x$

$\therefore \quad y = \sin^2\cot^{-1}\left\{\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}\right\} = \sin^2\cot^{-1}\left(\tan\dfrac{\theta}{2}\right)$

$= \sin^2\left(\dfrac{\pi}{2} - \dfrac{\theta}{2}\right)$

$= \cos^2\left(\dfrac{\theta}{2}\right)$

$= \dfrac{1+\cos\theta}{2}$

$= \dfrac{1+x}{2}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{2}$

23. (3) $f(\log_e x) = \log_e(\log_e x)$

$\therefore \quad \dfrac{df(\log_e x)}{dx} = \dfrac{1}{\log_e x}\times\dfrac{1}{x}$

24. (1) $y = f(x^2)$

or $\dfrac{dy}{dx} = f'(x^2)2x = 2x\sqrt{2(x^2)^2 - 1}$

At $x = 1$, $\dfrac{dy}{dx} = 2\times1\times\sqrt{2-1} = 2$

25. (1) $\dfrac{du}{dv} = \dfrac{\dfrac{du}{dx}}{\dfrac{dv}{dx}} = \dfrac{f'(x^3)3x^2}{g'(x^2)2x} = \dfrac{\cos x^3(3x^2)}{\sin x^2(2x)} = \dfrac{3}{2}x\cos x^3\operatorname{cosec} x^2$

26. (2) $2xf'(x^2) = 3x^2$ or $4f'(4) = 12$ or $f'(4) = 3$.

27. (4) $g'(x) = 2f(2f^2(x)+2)\times f'(2f^2(x)+2)\times 4f(x)\cdot f'(x)$

$\therefore \quad g'(0) = 2f(2f^2(0)+2)\times f'(2f^2(0)+2)\times 4f(0)\cdot f'(0)$

$= 2f(4)\cdot f'(4)\cdot 4f(0)f'(0)$

$= 2\times 4(-1)(1)$

$= -8$

28. (2) Let $g(x) = f^{-1}(x) \quad \therefore \quad f(g(x)) = x$

or $f'(g(x))g'(x) = 1$

or $(e^{g(x)} + 1)\, g'(x) = 1$

or $(e^{g(f(\log 2))} + 1)\, g'(f(\log 2)) = 1$

or $(e^{\log 2} + 1)\, g'(f(\log 2)) = 1$

or $g'(f(\log 2)) = 1/3$

29. (3) $f(x) = x + \tan x$

$\therefore \quad f(f^{-1}(y)) = f^{-1}(y) + \tan f^{-1}(y)$

or $y = g(y) + \tan g(y)$

or $x = g(x) + \tan g(x)$

Differentiating, we get $1 = g'(x) + \sec^2 g(x)\, g'(x)$

or $g'(x) = \dfrac{1}{1+\sec^2 g(x)} = \dfrac{1}{2+[g(x)-x]^2}$

30. (3) $\dfrac{d}{dx}\left(\dfrac{g(x)}{g(g(x))}\right) = \dfrac{g(g(x))\cdot g'(x) - g(x)\cdot g'(g(x))\cdot g'(x)}{(g(g(x)))^2}$

Now, $f(0) = 4$

$\Rightarrow \quad f^{-1}(4) = g(4) = 0$

and $g'(4) = \dfrac{1}{f'(0)} = \dfrac{1}{3}$.

Also, $f(-1) = 0$

$\Rightarrow \quad f^{-1}(0) = g(0) = -1$

$\therefore \quad \dfrac{d}{dx}\left(\dfrac{g(x)}{g(g(x))}\right)_{x=4}$

$= \dfrac{g(g(4))\, g'(4) - g(4)\, g'(g(4))\, g'(4)}{(g(4))^2}$

$= \dfrac{g(0)\cdot\dfrac{1}{f'(0)} - 0}{(g(0))^2}$

$= \dfrac{-1\times\dfrac{1}{3}}{1}$

$= \dfrac{-1}{3}$

31. (2) $y = \dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{\sqrt{1-x^2}\,\dfrac{1}{\sqrt{1-x^2}} - (\sin^{-1} x)\dfrac{1}{2}\dfrac{(-2x)}{\sqrt{1-x^2}}}{1-x^2}$

or $(1-x^2)\dfrac{dy}{dx} = 1 + x\left(\dfrac{\sin^{-1} x}{\sqrt{1-x^2}}\right) = 1 + xy$

32. (3) $y = \sqrt{\log x + y}$

or $y^2 = \log x + y$

or $2y\dfrac{dy}{dx} = \dfrac{1}{x} + \dfrac{dy}{dx}$ or $\dfrac{dy}{dx} = \dfrac{1}{x(2y-1)}$

33. (4) $\dfrac{d}{dx}\left(\tan^{-1}\dfrac{\left(\sqrt{x}(3-x)\right)}{1-3x}\right)$

$= \dfrac{d}{dx}\left(\tan^{-1}\dfrac{\left(\tan\theta\left(3-\tan^2\theta\right)\right)}{1-3\tan^2\theta}\right)$

(putting $\sqrt{x} = \tan\theta$ or $\theta = \tan^{-1}\sqrt{x}$)

$= \dfrac{d}{dx}\left(\tan^{-1}\dfrac{\left(3\tan\theta - \tan^3\theta\right)}{1-3\tan^2\theta}\right)$

$= \dfrac{d}{dx}\left[\tan^{-1}(\tan 3\theta)\right] = \dfrac{d}{dx}(3\theta)$

$= \dfrac{d}{dx}\left[3\tan^{-1}\sqrt{x}\right] = \dfrac{3}{2\sqrt{x}\,(1+x)}$

34. (1) $y = f(x) - f(2x)$ or $y' = f'(x) - 2f'(2x)$

or $y'(1) = f'(1) - 2f'(2) = 5$, and (1)

$y'(2) = f'(2) - 2f'(4) = 7$ (2)

Now, let $y = f(x) - f(4x)$

$\therefore \quad y' = f'(x) - 4f'(4x)$

or $y'(1) = f'(1) - 4f'(4)$ (3)

Substituting the value of $f'(2) = 7 + 2f'(4)$ in (1), we get

$f'(1) - 2(7 + 2f'(4)) = 5$

$f'(1) - 4f'(4) = 19$

35. (1) $y\sqrt{x^2+1} = \log\left\{\sqrt{x^2+1} - x\right\}$

Differentiating both sides w.r.t. x., we get

$$\frac{dy}{dx}\sqrt{x^2+1} + y\frac{1}{2\sqrt{x^2+1}}2x = \frac{1}{\sqrt{x^2+1}-x}\times\left\{\frac{1}{2}\frac{2x}{\sqrt{x^2+1}} - 1\right\}$$

or $(x^2+1)\dfrac{dy}{dx} + xy = \sqrt{x^2+1}\,\dfrac{-1}{\sqrt{x^2+1}}$

or $(x^2+1)\dfrac{dy}{dx} + xy + 1 = 0$

36. (3) $e^y = \dfrac{1+\sqrt{\dfrac{1-\alpha}{1+\alpha}}}{1-\sqrt{\dfrac{1-\alpha}{1+\alpha}}} = \dfrac{1+\tan\dfrac{x}{2}}{1-\tan\dfrac{x}{2}} = \tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$

$\therefore \quad e^y \cdot y' = \dfrac{1}{2}\sec^2\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$ (Diff. w.r.t. x)

$= \dfrac{1}{2}\left[1+\tan^2\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)\right]$

$= \dfrac{1}{2}\left[1+e^{2y}\right]$

For $x = \dfrac{\pi}{3}$, $e^y = \tan\left(\dfrac{\pi}{4}+\dfrac{\pi}{6}\right) = 2+\sqrt{3}$

$\Rightarrow \quad y' = \dfrac{1}{2(2+\sqrt{3})}\left[1+(2+\sqrt{3})^2\right]$

$= \dfrac{1}{2(2+\sqrt{3})}\left[8+4\sqrt{3}\right]$

$= 2$

37. (2) Let $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ and $z = \tan^{-1}\left(\dfrac{2x\sqrt{1-x^2}}{1-2x^2}\right)$

Putting $x = \tan\theta$ in y, we get

$y = \tan^{-1}\left(\dfrac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\tan\dfrac{\theta}{2}\right) = \dfrac{1}{2}\tan^{-1} x$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{2(1+x^2)}$

Putting $x = \sin\theta$ in z, we get

$z = \tan^{-1}\left(\dfrac{2\sin\theta\cos\theta}{\cos 2\theta}\right) = \tan^{-1}(\tan 2\theta) = 2\theta = 2\sin^{-1} x$

$\therefore \quad \dfrac{dz}{dx} = \dfrac{2}{\sqrt{1-x^2}}$

Thus, $\frac{dy}{dz}=\frac{\frac{dy}{dx}}{\frac{dz}{dx}}=\frac{1}{4(1+x^2)}\sqrt{1-x^2}$ or $\left(\frac{dy}{dz}\right)_{x=0}=\frac{1}{4}$

38. (3) We have, $(e-1)e^{xy}+x^2=e^{x^2+y^2}$

Differentiating both sides w.r.t. x, we get

$$(e-1)\cdot e^{xy}\cdot\left(x\cdot\frac{dy}{dx}+y\right)+2x=e^{x^2+y^2}\cdot\left(2x+2y\frac{dy}{dx}\right)$$

Putting $x=1$ and $y=0$; we get

$$(e-1)\left(\frac{dy}{dx}\right)+2=e(2+0)$$

$$\Rightarrow\quad\left(\frac{dy}{dx}\right)_{(1,0)}=2$$

39. (3) $y=x^{(x^x)}$

or $\log y=x^x\log x$

or $\frac{1}{y}\frac{dy}{dx}=\frac{dz}{dx}\log x+\frac{1}{x}z\ (\text{where } x^x=z)$

or $\frac{dy}{dx}=x^{(x^x)}\left[x^x(\log ex)\log x+x^{x-1}\right]\ \left(\because\frac{dz}{dx}=x^x\log ex\right)$

40. (1) $f(x)=\cos^{-1}\left[\cos\left(\frac{\pi}{2}-\sqrt{\frac{1+x}{2}}\right)\right]+x^x=\frac{\pi}{2}-\sqrt{\frac{1+x}{2}}+x^x$

$\therefore\quad f'(x)=-\frac{1}{\sqrt{2}}\times\frac{1}{2\sqrt{1+x}}+x^x(1+\log x)$

or $f'(1)=-\frac{1}{4}+1=\frac{3}{4}$

41. (3) $f(x)=x^x$, $g(x)$ is inverse of $f(x)$

$\therefore\quad f(g(x))=x$

$\Rightarrow\quad f'(g(x))\cdot g'(x)=1$

$\Rightarrow\quad g'(x)=\frac{1}{f'(g(x))}$

Now $\log_e f(x)=x\log_e x$

$\therefore\quad\frac{f'(x)}{f(x)}=x\frac{1}{x}+\log_e x=1+\log_e x$

$\therefore\quad f'(x)=x^x(1+\log_e x)$

$\therefore\quad f'(g(x))=(g(x))^{g(x)}(1+\log g(x))$

Now $f(g(x))=g(x)^{g(x)}=x$

$\therefore\quad f'(g(x))=x(1+\log g(x))$

$\therefore\quad g'(x)=\frac{1}{x(1+\log g(x))}$

42. (2) $y=ax^{n+1}+bx^{-n}$

or $\frac{dy}{dx}=(n+1)ax^n-nbx^{-n-1}$

or $\frac{d^2y}{dx^2}=n(n+1)ax^{n-1}+n(n+1)bx^{-n-2}$

or $x^2\frac{d^2y}{dx^2}=n(n+1)y$

43. (3) $y=ae^{mx}+be^{-mx}$

$\therefore\quad\frac{dy}{dx}=ame^{mx}-mbe^{-mx}$

Again $\frac{d^2y}{dx^2}=am^2e^{mx}+m^2be^{-mx}$

$=m^2\left(ae^{mx}+be^{-mx}\right)=m^2y$

or $\frac{d^2y}{dx^2}-m^2y=0$

44. (3) $(a^2-2a-15)e^{ax}+(b^2-2b-15)e^{bx}=0$

or $(a^2-2a-15)=0$ and $b^2-2b-15=0$

or $(a-5)(a+3)=0$ and $(b-5)(b+3)=0$

i.e., $a=5$ or -3 and $b=5$ or -3

$\therefore\quad a\neq b$

Hence, $a=5$ and $b=-3$ or $a=-3$ and $b=5$

or $ab=-15$

45. (2) $y=2\cos x\cos 3x=\cos 4x+\cos 2x$

$\therefore\quad\frac{d^{20}y}{dx^{20}}=4^{20}\cos 4x+2^{20}\cos 2x$

46. (4) Let $y=\log x$. Then,

$$y_1=\frac{1}{x},\ y_2=\frac{-1}{x^2},\ y_3=\frac{2}{x^3},\ \cdots,\ y_n=\frac{(-1)^{n-1}(n-1)!}{x^n}.$$

47. (2) $f(x)=1+x^2+x^4+x^6+\cdots\infty$, where $|x|\le 1$

$\therefore\quad f^n(0)=n!$, where n is even

48. (2) From the given relation, $\frac{y}{x}=\log x-\log(a+bx)$

Differentiating w.r.t. x, we get

$$\frac{\left(x\frac{dy}{dx}\right)-y}{x^2}=\frac{1}{x}-\frac{b}{a+bx}=\frac{a}{x(a+bx)}$$

$$\therefore\quad x\frac{dy}{dx}-y=\frac{ax}{a+bx}\qquad(1)$$

Differentiating again w.r.t. x, we get

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}-\frac{dy}{dx}=\frac{(a+bx)a-ax.b}{(a+bx)^2}$$

or $x\frac{d^2y}{dx^2}=\frac{a^2}{(a+bx)^2}$

or $x^3\frac{d^2y}{dx^2}=\frac{a^2x^2}{(a+bx)^2}=\left(x\frac{dy}{dx}-y\right)^2$ [by (1)]

49. (1) $ax^2+2hxy+by^2=1$

Differentiating both sides w.r.t. x, we get

$$2ax+2hx\frac{dy}{dx}+2hy+2by\frac{dy}{dx}=0$$

or $\frac{dy}{dx}=-\frac{ax+hy}{hx+by}$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=-\left[\frac{(hx+by)\left(a+h\frac{dy}{dx}\right)-(ax+hy)\left(h+b\frac{dy}{dx}\right)}{(hx+by)^2}\right]$$

$$=-\frac{\left[y(ab-h^2)+\frac{dy}{dx}\left(h^2x-abx\right)\right]}{(hx+by)^2}$$

$$= \frac{(h^2 - ab)\left(y - x\dfrac{dy}{dx}\right)}{(hx + by)^2}$$

$$= \frac{(h^2 - ab)}{(hx + by)^2}\left[y + x\frac{ax + hy}{hx + by}\right]$$

$$= \frac{h^2 - ab}{(hx + by)^3}$$

50. (1) We have

$$y^{1/m} = \left(x + \sqrt{1 + x^2}\right)$$

or $$y = \left(x + \sqrt{1 + x^2}\right)^m$$

or $$\frac{dy}{dx} = m\left(x + \sqrt{1 + x^2}\right)^{m-1}\left(1 + \frac{x}{\sqrt{x^2 + 1}}\right)$$

$$= m\frac{\left(x + \sqrt{1 + x^2}\right)^m}{\sqrt{1 + x^2}}$$

$$= \frac{my}{\sqrt{1 + x^2}}$$

or $$y_1^2(1 + x^2) = m^2y^2$$

or $$2y_1y_2(1 + x^2) + 2xy_1^2 = 2m^2yy_1$$

or $$y_2(1 + x^2) + xy_1 = m^2y$$

51. (1) $(\sin x)(\cos y) = \dfrac{1}{2}$

$\therefore$ $$(\cos x)(\cos y) - \sin y \sin x \frac{dy}{dx} = 0$$

or $$\frac{dy}{dx} = (\cot x)(\cot y)$$

or $$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2 x \cdot \cot y - \operatorname{cosec}^2 y \cot x \cdot \frac{dy}{dx}$$

Now, $$\left(\frac{dy}{dx}\right)_{(\pi/4,\, \pi/4)} = 1$$

or $$\left(\frac{d^2y}{dx^2}\right)_{(\pi/4,\, \pi/4)} = -(2)(1) - (2)(1)(1) = -4$$

52. (1) Given $f = f' + f'' + f''' + \cdots \infty$

or $f' = f'' + f''' + f'''' + \cdots \infty$

or $f - f' = f'$

or $f = 2f'$

Hence, $\dfrac{f'}{f} = 1/2$ or $\displaystyle\int \frac{f'}{f}dx = \int \frac{1}{2}dx$

or $\log f(x) = x/2 + c$

or $f(x) = e^{x/2 + c}$

Also, $f(0) = 1$ or $c = 0$ or $f(x) = e^{x/2}$

53. (2) Let $f(x) = a(x - 3)^3 + b(x - 3)^2 + c(x - 3) + d$

$f(3) = 1 \Rightarrow d = 1$

$f'(3) = -1 \Rightarrow c = -1$

$f''(3) = 0 \Rightarrow b = 0$

$f'''(3) = 12 \Rightarrow a = 2$

$\therefore$ $f'(x) = 3a(x - 3)^2 + 2b(x - 3) + c$

$= 6(x - 3)^2 - 1$

or $f'(1) = 23$

54. (1) $y^2 = ax^2 + bx + c$

or $$2y\frac{dy}{dx} = 2ax + b$$

or $$2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 2a$$

or $$y\frac{d^2y}{dx^2} = a - \left(\frac{dy}{dx}\right)^2$$

$$= a - \left(\frac{2ax + b}{2y}\right)^2$$

$$= \frac{4ay^2 - (2ax + b)^2}{4y^2}$$

or $$4y^3\frac{d^2y}{dx^2} = 4a\left(ax^2 + bx + c\right) - \left(4a^2x^2 + 4abx + b^2\right)$$

or $$4y^3\frac{d^2y}{dx^2} = 4ac - b^2 = \text{constant}$$

55. (3) $y = \sin x + e^x$ or $\dfrac{dy}{dx} = \cos x + e^x$

or $$\frac{dx}{dy} = (\cos x + e^x)^{-1} \qquad (1)$$

$\therefore$ $$\frac{d^2x}{dy^2} = -(\cos x + e^x)^{-2}(-\sin x + e^x)\frac{dx}{dy}$$

Substituting the value of $\dfrac{dy}{dx}$ from (1),

$$\frac{d^2x}{dy^2} = \frac{\left(\sin x - e^x\right)}{\left(\cos x + e^x\right)^2}\left(\cos x + e^x\right)^{-1} = \frac{\sin x - e^x}{\left(\cos x + e^x\right)^3}$$

56. (2) $$D = \begin{vmatrix} \sin px & p\cos px & -p^2\sin px \\ -p^3\cos px & p^4\sin px & p^5\cos px \\ -p^6\sin px & -p^7\cos px & p^8\sin px \end{vmatrix}$$

$$= p^9\begin{vmatrix} \sin px & p\cos px & -p^2\sin px \\ -\cos px & p\sin px & p^2\cos px \\ -\sin px & -p\cos px & p^2\sin px \end{vmatrix}$$

$$= -p^9\begin{vmatrix} \sin px & p\cos px & -p^2\sin px \\ \cos px & p\sin px & p^2\cos px \\ \sin px & p\cos px & -p^2\sin px \end{vmatrix} = 0$$

57. (1) $$F'(x) = \left[f\left(\frac{x}{2}\right)f'\left(\frac{x}{2}\right) + g\left(\frac{x}{2}\right)g'\left(\frac{x}{2}\right)\right]$$

Here, $g(x) = f'(x)$

and $g'(x) = f''(x) = -f(x)$

So, $$F'(x) = f\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right) - f\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right) = 0$$

Hence, $F(x)$ is a constant function.

Therefore, $F(10) = 5$.

58. (1) $y = 2\ln(1 + \cos x)$

$$\frac{dy}{dx} = \frac{-2\sin x}{1 + \cos x}$$

$$\frac{d^2y}{dx^2} = -2\left[\frac{(1+\cos x)\cos x - \sin x(-\sin x)}{(1+\cos x)^2}\right]$$

$$= -2\left[\frac{\cos x + 1}{(1+\cos x)^2}\right] = \frac{-2}{(1+\cos x)}$$

Now, $2e^{-y/2} = 2 \cdot e^{-\frac{\ln(1+\cos x)^2}{2}} = \frac{2}{(1+\cos x)}$

$\therefore \quad \frac{d^2y}{dx^2} + \frac{2}{e^{y/2}} = 0$

59. (2) $\frac{dx}{dy} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}} = \frac{\cos t - t\sin t}{1+\cos t}$

$$\therefore \quad \frac{d^2x}{dy^2} = \frac{\frac{d}{dt}\left(\frac{dx}{dy}\right)}{\frac{dy}{dt}}$$

$$= \frac{\frac{(-2\sin t - t\cos t)(1+\cos t) - (\cos t - t\sin t)(-\sin t)}{(1+\cos t)^2}}{1+\cos t}$$

Now, put $t = \pi/2$.

60. (4) $f(x) = (x-1)^4(x-2)^3(x-3)^2(x-4)$

$f'''(1) = f''(2) = f'(3) = 0$

$f'(x) = (x-1)^4(x-2)^3(x-3)^2 \cdot 1 + (x-1)^4(x-2)^3 2(x-3)$
$(x-4) + (x-1)^3 \cdot 3(x-2)^2 + (x-3)^2(x-4)$
$+ 4(x-1)^3(x-2)^3(x-3)^2(x-4)$

$f'(4) = 3^4 \times 2^3 \times 1 = 3 \times 6^3$

61. (3) Here, $y = t^{10} + 1$ and $x = t^8 + 1$

$\therefore \quad t^8 = x - 1$ or $t^2 = (x-1)^{1/4}$

So, $y = (x-1)^{5/4} + 1$

Differentiating both sides w.r.t. x, we get $\frac{dy}{dx} = \frac{5}{4}(x-1)^{1/4}$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{5}{16}(x-1)^{-3/4} = \frac{5}{16(x-1)^{3/4}} = \frac{5}{16\left(t^2\right)^3} = \frac{5}{16t^6}$$

62. (3) $\frac{dy}{dx} = \frac{-\frac{1}{p^2}}{\frac{1}{p}} = -\frac{1}{p} = -y$

or $\frac{d^2y}{dx^2} = -\frac{dy}{dx}$

or $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

63. (2) We have $x = \phi(t)$, $y = \psi(t)$. Therefore,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\psi'}{\phi'}$$

or $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\psi'}{\phi'}\right) = \frac{d}{dt}\left(\frac{\psi'}{\phi'}\right)\frac{dt}{dx}$

$$= \frac{\phi'\psi'' - \psi'\phi''}{\phi'^2}\frac{1}{\phi'} = \frac{\phi'\psi'' - \psi'\phi''}{\phi'^3}$$

64. (1) As $f(x) = x^4\tan(x^3) - x\ln(1+x^2)$ is odd, $\frac{d^3f(x)}{dx^3}$ is even

Hence, $\frac{d^4f(x)}{dx^4} = 0$ at $x = 0$.

65. (1) $y = f(x)$ is an even function and $y = g(x)$ is an odd function.

or $h(x) = f(x)\,g(x)$ is an odd function.

or $h(x) = -h(-x)$

$\therefore \quad h'(x) = h'(-x)$

or $h''(x) = -h''(-x)$

or $h'''(x) = h'''(-x)$

Now, we cannot determine the value of $h'''(0)$.

66. (1) Given that $g^{-1}(x) = f(x)$

or $x = g(f(x))$

or $g'(f(x))\,f'(x) = 1$

or $g'(f(x)) = \frac{1}{f'(x)}$

or $g''(f(x))\,f'(x) = \frac{-f''(x)}{[f'(x)]^2}$

or $g''(f(x)) = \frac{-f''(x)}{[f'(x)]^3}$.

67. (3) $f(x) = e^x - e^{-x} - 2\sin x - \frac{2}{3}x^3$

$f^{\mathrm{I}}(x) = e^x + e^{-x} - 2\cos x - 2x^2$

$f^{\mathrm{II}}(x) = e^x - e^{-x} + 2\sin x - 4x$

$f^{\mathrm{III}}(x) = e^x + e^{-x} + 2\cos x - 4$

$f^{\mathrm{IV}}(x) = e^x - e^{-x} - 2\sin x$

$f^{\mathrm{V}}(x) = e^x + e^{-x} - 2\cos x$

$f^{\mathrm{VI}}(x) = e^x - e^{-x} + 2\sin x$

$f^{\mathrm{VII}}(x) = e^x + e^{-x} + 2\cos x$

Clearly, $f^{\mathrm{VII}}(0)$ is nonzero.

68. (2) We have $\frac{dy}{dx} = f'(x)$ and $\frac{dx}{dz} = -\frac{1}{z^2}$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = -\frac{dy}{dz}\cdot z^2$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{d}{dz}\left[\frac{dy}{dz}\cdot z^2\right]\frac{dz}{dx}$$

$$= \left[z^2\frac{d^2y}{dz^2} + \frac{dy}{dz}\cdot 2z\right]z^2$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = z^4\frac{d^2y}{dz^2} + 2z^3\frac{dy}{dz}$$

$$\Rightarrow \quad \lambda = 2$$

69. (2) We have $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{g'(t)}{f'(t)}\right)$$

$$= \frac{d}{dt}\left(\frac{g'(t)}{f'(t)}\right)\cdot\frac{dt}{dx}$$

$$= \frac{f'(t)\,g''(t) - g'(t)\,f''(t)}{(f'(t))^2}\cdot\frac{dt}{dx}$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{t=0} = \frac{f'(0)\,g''(0) - g'(0)\,f''(0)}{(f'(0))^3}$$

$$= \frac{2\times 6 - 2\times 2}{2^3}$$

$$= 1$$

(11) Given $f\left(\frac{5x-3y}{2}\right) = \frac{5f(x)-3f(y)}{2}$

or $f\left(\frac{5x-3y}{5-3}\right) = \frac{5f(x)-3f(y)}{5-3}$

which satisfies section formula for abscissa on L.H.S. and for ordinate on R.H.S. Hence, $f(x)$ must be the linear function (as only straight line satisfies such section formula).

Hence, $f(x) = ax + b$

But $f(0) = 3$ or $b = 3, f'(0) = 2$ or $a = 2$.

Thus, $f(x) = 2x + 3$ or Period of $\sin(f(x)) = \sin(2x + 3)$ is π.

(12) We have $\frac{f(2x+2y)}{f(2x-2y)} = \frac{\sin(x+y)}{\sin(x-y)}$

or $\frac{f(\alpha)}{\sin\frac{\alpha}{2}} = \frac{f(\beta)}{\sin\frac{\beta}{2}} = K$

or $f(x) = K\sin\frac{x}{2}$

$\therefore\ f'(x) = \frac{K}{2}\cos\frac{x}{2}$

and $f''(x) = \frac{-K}{4}\sin\frac{x}{2}$

or $4f''(x) + f(x) = 0$

Multiple Correct Answers Type

1. **(1), (3)**

$$\frac{dy}{dx} = \frac{e^{\sqrt{x}}}{2\sqrt{x}} - \frac{e^{-\sqrt{x}}}{2\sqrt{x}} = \frac{e^{\sqrt{x}} - e^{-\sqrt{x}}}{2\sqrt{x}}$$

$$= \frac{\sqrt{\left(e^{\sqrt{x}} + e^{-\sqrt{x}}\right)^2 - 4}}{2\sqrt{x}}$$

$$= \frac{\sqrt{y^2-4}}{2\sqrt{x}}$$

2. **(1), (3), (4)**

$$y^2 = x + y \text{ or } \frac{dy}{dx} = \frac{1}{2y-1}$$

Also, $y = \frac{x}{y} + 1$ or $\frac{dy}{dx} = \frac{y}{2x+y}$

Also, $y^2 - y - x = 0$

or $y = \frac{1+\sqrt{1+4x}}{2}$ (as $y > 0$)

or $y' = \frac{1}{4}\frac{4}{\sqrt{1+4x}} = \frac{1}{\sqrt{1+4x}}$

3. **(2), (3), (4)**

$$f(\theta) = \tan\left(\sin^{-1}\sqrt{\frac{2}{3+\cos 2\theta}}\right)$$

$$= \tan\left(\tan^{-1}\sqrt{\frac{2}{1+\cos 2\theta}}\right)$$

$$= \sqrt{\frac{2}{1+\cos 2\theta}} = \sec\theta$$

$$\Rightarrow\ f\left(\frac{\pi}{4}\right) = \sqrt{2}$$

$f'(\theta) = \sec\theta\tan\theta$

$$\Rightarrow\ f'\left(\frac{\pi}{4}\right) = \sqrt{2}$$

$$\left.\frac{d(f(\theta))}{d(\cos\theta)}\right|_{\theta=\frac{\pi}{4}} = \left(\frac{-1}{\cos^2\theta}\right)_{\theta=\frac{\pi}{4}} = -2$$

4. **(1), (2), (3)**

$$f(x) = |x^2 - 3|x| + 2|$$

$$= \begin{cases} |x^2-3x+2|, & x \ge 0 \\ |x^2+3x+2|, & x < 0 \end{cases}$$

$$= \begin{cases} x^2-3x+2, & x^2-3x+2 \ge 0,\ x \ge 0 \\ -x^2+3x-2, & x^2-3x+2 < 0,\ x \ge 0 \\ x^2+3x+2, & x^2+3x+2 \ge 0,\ x < 0 \\ -x^2-3x-2, & x^2+3x+2 < 0,\ x < 0 \end{cases}$$

$$= \begin{cases} x^2-3x+2, & x \in [0,1] \cup [2,\infty) \\ -x^2+3x-2, & x \in (1,2) \\ x^2+3x+2, & x \in (-\infty,-2] \cup [-1,0) \\ -x^2-3x+2, & x \in (-2,-1) \end{cases}$$

or $f'(x) = \begin{cases} 2x-3, & x \in (0,1) \cup (2,\infty) \\ -2x+3, & x \in (1,2) \\ 2x+3, & x \in (-\infty,-2) \cup (-1,0) \\ -2x-3, & x \in (-2,-1) \end{cases}$

5. **(1), (2), (4)**

$$f(x) = \frac{\sqrt{\left(\sqrt{x-1}\right)^2 + 1 - 2\sqrt{x-1}}}{\sqrt{x-1}-1}x$$

$$= \frac{\left|\sqrt{x-1}-1\right|}{\sqrt{x-1}-1}x$$

$$= \begin{cases} -x, & \text{if } x \in [1,\ 2) \\ x, & \text{if } x \in (2,\infty) \end{cases}$$

6. **(1), (3)**

$$x^3 - 2x^2y^2 + 5x + y - 5 = 0$$

Differentiating w.r.t. x, we get

$$3x^2 - 4xy^2 - 4x^2y\frac{dy}{dx} + 5 + \frac{dy}{dx} = 0$$

or $y' = \frac{dy}{dx} = \frac{3x^2 - 4xy^2 + 5}{4x^2y - 1}$

$$y'(1) = \frac{3-4+5}{4-1} = \frac{4}{3}$$

Also, y''

$$= \frac{(6x-4y^2-8xyy')(4x^2y-1)-(8xy+4x^2y')(3x^2-4xy^2+5)}{(4x^2y-1)^2}$$

or $$y''(1) = \frac{(6-4-8\times\frac{4}{3})(4-1)-(8+4\times\frac{4}{3})(3-4+5)}{(4-1)^2}$$

$$= -8\frac{22}{27}$$

7. (2), (4)

$$y = x^{(\log x)^{\log(\log x)}}$$

$$\therefore \quad \log y = (\log x)\,(\log x)^{\log(\log x)} \qquad (1)$$

Taking log of both sides, we get

$$\log(\log y) = \log(\log x) + \log(\log x)\log(\log x)$$

Differentiating w.r.t. x, we get

$$\frac{1}{\log y}\cdot\frac{1}{y}\frac{dy}{dx} = \frac{1}{x\log x} + \frac{2\log(\log x)}{\log x}\frac{1}{x}$$

$$= \frac{2\log(\log x)+1}{x\log x}$$

or $$\frac{dy}{dx} = \frac{y}{x}\cdot\frac{\log y}{\log x}\left(2\log(\log x)+1\right)$$

Substituting the value of $\log y$ from (1), we get

$$\frac{dy}{dx} = \frac{y}{x}(\log x)^{\log(\log x)}(2\log(\log x)+1)$$

8. (1), (3)

$f(x-y)$, $f(x)f(y)$, and $f(x+y)$ are in A.P. Therefore,

$f(x+y)+f(x-y) = 2f(x)f(y)$ for all x, y

Putting $x = 0, y = 0$ in (1), we get

$f(0)+f(0) = 2f(0)f(0)$

or $f(0) = 1 \qquad [\because f(0) \neq 0]$

Putting $x = 0, y = x$, we get

$f(x)+f(-x) = 2f(0)f(x)$

or $f(x) = f(-x) \qquad (1)$

or $f(4) = f(-4), f(3) = f(-3)$

Differentiating (1) w.r.t. x, $f'(x)+f'(-x) = 0$

or $f'(4)+f'(-4) = 0$

9. (2), (3)

$$y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$$

or $$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\frac{4x^2}{(1+x^2)^2}}}\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

$$= -\frac{1+x^2}{\sqrt{(1-x^2)^2}}\frac{2(1+x^2)-4x^2}{(1+x^2)^2}$$

$$= -2\frac{(1+x^2)}{|1-x^2|}\frac{1-x^2}{(1+x^2)^2}$$

$$= -2\left(\frac{1-x^2}{|1-x^2|}\right)\left(\frac{1}{1+x^2}\right)$$

$$= \begin{cases} \dfrac{2}{1+x^2}, & \text{if } |x|>1 \\ \dfrac{-2}{1+x^2}, & \text{if } |x|<1 \end{cases}$$

10. (1), (3)

$$\frac{d}{dx}\{f_n(x)\} = \frac{d}{dx}\{e^{f_{n-1}(x)}\}$$

$$= e^{f_{n-1}(x)}\frac{d}{dx}\{f_{n-1}(x)\} = f_n(x)\frac{d}{dx}\{f_{n-1}(x)\}$$

$$= f_n(x)\cdot\frac{d}{dx}\{e^{f_{n-2}(x)}\} = f_n(x)\cdot e^{f_{n-2}(x)}\frac{d}{dx}\{f_{n-2}(x)\}$$

$$= f_n(x)f_{n-1}(x)\frac{d}{dx}\{f_{n-2}(x)\}$$

$$\cdots$$

$$= f_n(x)f_{n-1}(x)\cdots f_2(x)\frac{d}{dx}\{f_1(x)\}$$

$$= f_n(x)f_{n-1}(x)\cdots f_2(x)\frac{d}{dx}\{e^{f_0(x)}\}$$

$$= f_n(x)f_{n-1}(x)\cdots f_2(x)\,e^{f_0(x)}\frac{d}{dx}\{f_0(x)\}$$

Use $e^{f_0(x)} = f_1(x)$ and $f_0(x) = x$

11. (2), (4)

We have $g = \frac{1}{f}$

$$\therefore \quad g' = \frac{-1}{f^2}f'$$

or $$g'' = -\left[-\frac{2}{f^3}f'^2+\frac{1}{f^2}f''\right]$$

$$= \frac{2}{f^3}f'^2 - \frac{f''}{f^2}$$

or $$\frac{f''}{f'} - \frac{g''}{g'} = \frac{f''}{f'} - \frac{\frac{2}{f^3}f'^2 - \frac{f''}{f^2}}{-\frac{1}{f^2}f'}$$

$$= \frac{f''}{f'} - \left(\frac{-2f'}{f}+\frac{f''}{f'}\right) = \frac{2f'}{f}$$

Also, $g\cdot f = 1$

or $g'f+gf' = 0$

$$\therefore \quad \frac{f''}{f'} - \frac{g''}{g'} = -\frac{2g'}{g}$$

12. (1), (2)

$$y = e^{-x}\cos x$$

$$y_1 = -e^{-x}\cos x - e^{-x}\sin x = -\sqrt{2}\,e^{-x}\cos\left(x-\frac{\pi}{4}\right)$$

$$y_2 = (-\sqrt{2})^2 e^{-x}\cos\left(x-\frac{\pi}{2}\right)$$

$$y_3 = (-\sqrt{2})^3 e^{-x}\cos\left(x-\frac{3\pi}{4}\right)$$

$$y_4 = (-\sqrt{2})^4 e^{-x}\cos(x-\pi) = -4e^{-x}\cos x$$

or $y_4+4y = 0$ or $k_4 = 4$

Differentiating it again four times, we get

$y_8+4y_4 = 0$

or $y_8-16y = 0$

or $k_8 = -16$

Further, $y_{12}+4y_8 = 0$

or $y_{12}+64y = 0$

or $k_{12} = 64$

Similarly, $k_{16} = -256$

13. (2), (4)

$$x=\frac{1+\log_e t}{t^2};\ y=\frac{3+2\log_e t}{t}$$

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$$

$$=\frac{t\left(\frac{2}{t}\right)-(3+2\log_e t)}{t^2}$$
$$\frac{t^2\left(\frac{1}{t}\right)-(1+\log_e t)2t}{t^4}$$

$$=\left(\frac{-1-2\log_e t}{-1-2\log_e t}\right).t=t$$

Eliminating $\log_e t$ term from y, we get

$$y=\frac{1+2t^2x}{t}=\frac{1+2(y')^2x}{y'}$$

or $yy'=1+2x(y')^2$ (Differentiating w.r.t x)

or $yy''+(y')^2=4xy'\cdot y''+2(y')^2$

or $yy''=4xy\cdot y''+(y')^2$

14. (1), (3)

$$y=\frac{x^2}{2}+\frac{1}{2}x\sqrt{x^2+1}+\frac{1}{2}\log_e\left(x+\sqrt{x^2+1}\right)$$

$$\Rightarrow\quad y'=x+\frac{1}{2}\left[\frac{x^2}{\sqrt{x^2+1}}+\sqrt{x^2+1}\right]+\frac{1}{2\sqrt{x^2+1}}$$

$$=x+\frac{1}{2}\left[\frac{x^2+1}{\sqrt{x^2+1}}\right]+\frac{\sqrt{x^2+1}}{2}$$

$$=x+\sqrt{x^2+1}$$

Also, $2y=x^2+x\sqrt{x^2+1}+\log_e(x+\sqrt{x^2+1})$

$$=x\left(x+\sqrt{x^2+1}\right)+\log_e\left(x+\sqrt{x^2+1}\right)$$

$$=xy'+\log_e y'$$

15. (1), (4)

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{2t}{1+3t^2}=\frac{1}{2}\ \text{(given)}$$

$$\Rightarrow\quad 3t^2-4t+1=0$$

$$\Rightarrow\quad (3t-1)(t-1)=0$$

$$\Rightarrow\quad t=\frac{1}{3},t=1$$

16. (1), (2), (4)

We have $e^{\sin(x^2+y^2)}=\tan\frac{y^2}{4}+\sin^{-1}x$ (1)

When $x=0$,

$$e^{\sin y^2}=\tan\frac{y^2}{4}$$

$$\Rightarrow\quad y=\pm\sqrt{\pi},\pm\sqrt{5\pi},\ldots$$

Now differentiating (1), w.r.t. x, we get

$$e^{\sin(x^2+y^2)}\left[\cos\left(x^2+y^2\right)\left(2x+2y\frac{dy}{dx}\right)\right]$$

$$=\frac{2y}{4}\sec^2\frac{y^2}{4}\frac{dy}{dx}+\frac{1}{\sqrt{1-x^2}}$$

Putting $x=0$, we get

$$\Rightarrow\quad e^{\sin y^2}\left[\cos y^2\left(2y\frac{dy}{dx}\right)\right]=\frac{y}{2}\sec^2\frac{y^2}{4}\frac{dy}{dx}+1$$

Now, at $x=0,\ y=\sqrt{\pi},\ \frac{dy}{dx}=\frac{-1}{3\sqrt{\pi}}$

at $x=0,\ y=-\sqrt{\pi},\ \frac{dy}{dx}=\frac{1}{3\sqrt{\pi}}$

at $x=0,\ y=\sqrt{5\pi},\ \frac{dy}{dx}=\frac{-1}{3\sqrt{5\pi}}$

at $x=0,\ y=-\sqrt{5\pi},\ \frac{dy}{dx}=\frac{1}{3\sqrt{5\pi}}$

Linked Comprehension Type

For Problems 1–3

1. (2), 2. (1), 3. (4)

Suppose degree of $f(x)=n$. Then degree of $f'=n-1$, and degree of $f''=n-2$. So,

$$n=n-1+n-2$$

Hence, $n=3$.

So, put $f(x)=ax^3+bx^2+cx+d$ (where $a\neq 0$).

From $f(2x)=f'(x).f''(x)$, we have

$$8ax^3+4bx^2+2cx+d=(3ax^2+2bx+c)(6ax+2b)$$
$$=18a^2x^3+18abx^2+(6ac+4b^2)x+2bc$$

Comparing coefficients of terms, we have

$18a^2=8a \Rightarrow a=4/9$

$18ab=4b \Rightarrow b=0$

$2c=6ac+4b^2 \Rightarrow c=0$

$d=2bc \Rightarrow d=0$

or $f(x)=\frac{4x^3}{9}$, which is clearly one-one and onto

or $f(3)=12$

Also, $\frac{4x^3}{9}=x$ or $x=0,x=\pm 3/2$.

Hence, sum of roots of equation is zero.

For Problems 4–6

4. (4), 5. (3), 6. (4)

Here, $f(x)=x^3+x^2f'(1)+xf''(2)+f'''(3)$

Put $f'(1)=a,f''(2)=b,\ f'''(3)=c$ (1)

$\therefore\quad f(x)=x^3+ax^2+bx+c$

or $f'(x)=3x^2+2ax+b$

or $f'(1)=3+2a+b$ (2)

or $f''(x)=6x+2a$

Also, $f''(2)=12+2a$ (3)

or $f'''(x)=6$

Further $f'''(3)=6$ (4)

From (1) and (4), $c=6$.

From (1), (2), and (3), we have

$a=-5,\ b=2$

$\therefore f(x)=x^3-5x^2+2x+6$

or $f'(x)=3x^2-10x+2$

For Problems 7–9

7. (2) From the given information, we have $f(x) = (x-c)^m g(x)$, where $g(x)$ is polynomial of degree $n-m$.

Then $x = c$ is common root for the equations $f(x) = 0, f^1(x) = 0, f^2(x) = 0, \ldots, f^{m-1}(x) = 0$, where $f^r(x)$ represent rth derivative of $f(x)$ w.r.t. x.

8. (1) Let $f(x) = a_1x^3 + b_1x^2 + c_1x + d_1 = 0$ has roots α, α, β.

Then $g(x) = a_2x^3 + b_2x^2 + c_2x + d_2 = 0$ must have roots α, α, γ. Then

$$a_1\alpha^3 + b_1\alpha^2 + c_1\alpha + d_1 = 0 \quad (1)$$

and $$a_2\alpha^3 + b_2\alpha^2 + c_2\alpha + d_2 = 0 \quad (2)$$

α is also a root of equations $f'(x) = 3a_1x^2 + 2b_1x + c_1 = 0$ and $g'(x) = 3a_2x^2 + 2b_2x + c_2 = 0$. Then

$$3a_1\alpha^2 + 2b_1\alpha + c_1 = 0 \quad (3)$$

and $$3a_2\alpha^2 + 2b_2\alpha + c_2 = 0 \quad (4)$$

Also, from $a_2(1) - a_1(2)$, we have

$$(a_2b_1 - a_1b_2)\,\alpha^2 + (c_1a_2 - c_2a_1)\,\alpha + d_1a_2 - d_2a_1 = 0 \quad (5)$$

Eliminating α from (3), (4), and (5) we have

$$\begin{vmatrix} 3a_1 & 2b_1 & c_1 \\ 3a_2 & 2b_2 & c_2 \\ a_2b_1 - a_1b_2 & c_1a_2 - c_2a_1 & d_1a_2 - d_2a_1 \end{vmatrix} = 0.$$

9. (4)

For Problems 10–12

10. (2) Since $1, a_1, a_2, \ldots, a_{n-1}$ are roots of $x^n - 1 = 0$, we have

$$x^n - 1 = (x-1)(x-a_1)(x-a_2)\cdots(x-a_{n-1}) \quad (1)$$

or $$\frac{x^n-1}{x-1} = (x-a_1)(x-a_2)\cdots(x-a_{n-1})$$

or $$\lim_{x\to 1}\frac{x^n-1}{x-1} = \lim_{x\to 1}\,[(x-a_1)(x-a_2)\cdots(x-a_{n-1})]$$

or $$(1-a_1)(1-a_2)\cdots(1-a_{n-1}) = n$$

11. (1) From (1), $\log(x^n-1) = \log(x-1) + \log(x-a_1) + \cdots + \log(x-a_{n-1})$

Differentiating w.r.t. x, we get

$$\frac{nx^{n-1}}{x^n-1} = \frac{1}{x-1} + \frac{1}{x-a_1} + \frac{1}{x-a_2} + \cdots + \frac{1}{x-a_{n-1}} \quad (2)$$

Putting $x = 2$ in (2), we get

$$\frac{n2^{n-1}}{2^n-1} = 1 + \frac{1}{2-a_1} + \frac{1}{2-a_2} + \cdots + \frac{1}{2-a_{n-1}}$$

or $$\frac{1}{2-a_1} + \frac{1}{2-a_2} + \cdots + \frac{1}{2-a_{n-1}} = \frac{n2^{n-1}}{2^n-1} - 1$$

$$= \frac{n2^{n-1} - 2^n + 1}{2^n-1}$$

$$= \frac{2^{n-1}(n-2)+1}{2^n-1}$$

12. (3) From (2), $$\frac{nx^{n-1}}{x^n-1} - \frac{1}{x-1} = \frac{1}{x-a_1} + \frac{1}{x-a_2} + \cdots + \frac{1}{x-a_{n-1}}$$

or $$\frac{nx^{n-1} - 1(1+x+x^2+\cdots+x^{n-1})}{x^n-1}$$

$$= \frac{1}{x-a_1} + \frac{1}{x-a_2} + \cdots + \frac{1}{x-a_{n-1}}$$

or $$\lim_{x\to 1}\frac{nx^{n-1} - 1(1+x+x^2+\cdots+x^{n-1})}{x^n-1}$$

$$= \lim_{x\to 1}\left(\frac{1}{x-a_1} + \frac{1}{x-a_2} + \cdots + \frac{1}{x-a_{n-1}}\right)$$

or $$\lim_{x\to 1}\frac{n(n-1)x^{n-2} - (1+2x+\cdots+(n-1)x^{n-2})}{nx^{n-1}}$$

$$= \frac{1}{1-a_1} + \frac{1}{1-a_2} + \cdots + \frac{1}{1-a_{n-1}}$$

(applying L Hopital's rule on L.H.S.)

or $$\frac{n(n-1) - (1+2+\cdots+(n-1))}{n}$$

$$= \frac{1}{1-a_1} + \frac{1}{1-a_2} + \cdots + \frac{1}{1-a_{n-1}}$$

or $$\frac{1}{1-a_1} + \frac{1}{1-a_2} + \cdots + \frac{1}{1-a_{n-1}} = \frac{n-1}{2}$$

For Problems 13–15

13. (2), 14. (3), 15. (3)

Here put $g'(1) = a$, $g''(2) = b$ (1)

Then,

$f(x) = x^2 + ax + b$,

$f(1) = 1 + a + b$ or $f'(x) = 2x + a, f''(x) = 2$. Therefore,

$g(x) = (1 + a + b)x^2 + (2x + a)\,x + 2$

$= x^2(3 + a + b) + ax + 2$

or $g'(x) = 2x\,(3 + a + b) + a$ and $g''(x) = 2(3 + a + b)$.

Hence, $g'(1) = 2(3 + a + b) + a$ (2)

$g''(2) = 2(3 + a + b)$ (3)

From (1), (2), and (3), we have

$a = 2(3 + a + b) + a$ and $b = 2(3 + a + b)$

or $3 + a + b = 0$ and $b + 2a + 6 = 0$

Hence, $b = 0$ and $a = -3$. So, $f(x) = x^2 - 3x$ and $g(x) = -3x + 2$.

$$\sqrt{\frac{f(x)}{g(x)}} = \sqrt{\frac{x^2-3x}{-3x+2}} \text{ is defined if } \frac{x^2-3x}{-3x+2} \geq 0$$

or $$\frac{x(x-3)}{(x-2/3)} \leq 0$$

$\Rightarrow$ $x \in (-\infty, 0] \cup (2/3, 3]$

For Problems 16–18

16. (4), 17. (3), 18. (3)

$$g(x+y) = g(x) + g(y) + 3x^2y + 3xy^2 \quad (1)$$

or $g'(x+y) = g'(x) + 6yx + 3y^2$ (differentiating w.r.t. x keeping y as constant)

Put $x = 0$. Then,

or $g'(y) = g'(0) + 3y^2 = -4 + 3y^2$

or $g'(x) = -4 + 3x^2$

or $g(x) = -4x + x^3 + c$

Now, put $x = y = 0$ in (1). Then $g(0) = g(0) + g(0) + 0$

or $g(0) = 0$

or $g(x) = x^3 - 4x$

$g(x) = 0 \Rightarrow x^3 - 4x = 0 \Rightarrow x = 0, 2, -2.$

Hence, three roots.

$\sqrt{g(x)} = \sqrt{x^3 - 4x}$ is defined if $x^3 - 4x \geq 0$

or $x \in [-2, 0] \cup [2, \infty)$.

Also, $g'(x) = 3x^2 - 4 \Rightarrow g'(1) = -1$.

For Problems 19–21

$x = f(t) = a^{\ln(b^t)} = a^{t \ln b}$ (1)

$y = g(t) = b^{-\ln(a^t)} = (b^{\ln a})^{-t} = (a^{\ln b})^{-t} = a^{-t \ln b}$

$\therefore \quad y = g(t) = a^{\ln(b^{-t})} = f(-t)$ (2)

From equations (1) and (2),

$xy = 1$

19. (4) $y = \dfrac{1}{x}$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{1}{x^2} = -\dfrac{1}{f^2(t)}$

Also, $xy = 1 \quad \Rightarrow \quad -\dfrac{1}{f^2(t)} = -g^2(t)$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{1}{x^2} = -\dfrac{y^2}{1}$

Also, $xy = 1 \quad \Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{y}{x} = -\dfrac{g(t)}{f(t)}$

20. (4) $f(t) = g(t) \quad \Rightarrow \quad f(t) = f(-t) \quad \Rightarrow \quad t = 0$

[$\because f(t)$ is one-one function]

At $t = 0, x = y = 1$

$\because \quad xy = 1, \quad \dfrac{dy}{dx} = \dfrac{-1}{x^2}$ and $\dfrac{d^2y}{dx^2} = \dfrac{2}{x^3}$ At $x = 1, \dfrac{d^2y}{dx^2} = 2$

21. (2) $\because \quad xy = 1$

$fg = 1$

$\therefore \quad fg' + gf' = 0$

$\therefore \quad fg'' + g'f' + g'f' + gf'' = 0$

or $fg'' + gf'' + 2g'f' = 0$

or $\dfrac{f}{f'}\dfrac{g''}{g'} + \dfrac{gf''}{g'f'} = -2$ (3)

From equation (2), $g(t) = f(-t)$

$\therefore \quad g'(t) = -f'(-t)$

and $g''(t) = f''(-t)$

Substituting in equation (3), we get

$$\frac{f(t)}{f'(t)}\cdot\frac{f''(-t)}{-f'(-t)} + \frac{f(-t)}{-f'(-t)}\cdot\frac{f''(t)}{f'(t)} = -2$$

$$\frac{f(t)}{f'(t)}\cdot\frac{f''(-t)}{f'(-t)} + \frac{f(-t)}{f'(-t)}\cdot\frac{f''(t)}{f'(t)} = 2$$

For Problems 22 and 23

22. (2), 23. (3)

$f(0) = 0$

So, $\lim\limits_{x\to 0}\dfrac{f(x)}{x} = f'(0) = 1$ (Using L'Hopital's rule)

$f(x + y) = f(x) + f(y) + x^2y + xy^2$

Differentiating w.r.t. x, keeping y constant, we get

$f'(x + y) = f'(x) + 2xy + y^2$

Put $x = 0$. Then

$f'(y) = y^2 + 1$ or $f'(x) = x^2 + 1$

$\therefore \quad f(x) = \dfrac{x^3}{3} + x + c$

$f(0) = 0 = c$

$\therefore \quad f(x) = \dfrac{x^3}{3} + x$

$f'(3) = 10$

and $f(9) = 243 + 9 = 252$

Matrix Match Type

1. a → p; b → q, r; c → s, r; d → q, r.

a. $f(1 - x) = f(1 + x)$

$\therefore \quad -f'(1 - x) = f'(1 + x)$

Hence, graph of $f(x)$ is symmetrical about point (1, 0)

[as if $f(x) = -f(-x)$, then $f(x)$ is odd and its graph is symmetrical about (0, 0). Now shift the graph at (1, 0)].

b. $f(2 - x) + f(x) = 0$

Replace x by $1 + x$. Then $f(2 - (1 + x)) + f(1 + x) = 0$

or $f(1 - x) + f(1 + x) = 0$

or $-f'(1 - x) + f'(1 + x) = 0$

or $f'(1 - x) = f'(1 + x)$ (1)

Therefore, graph of $f'(x)$ is symmetrical about line $x = 1$.

Also, put $x = 2$ in (1). Then $f'(-1) = f'(3)$.

c. $f(x + 2) + f(x) = 0$ (1)

Replace x by $x + 2$. Then $f(x + 4) + f(x + 2) = 0$ (2)

From (1) and (2), we have $f(x) = f(x + 4)$

Hence, $f(x)$ is periodic with period 4.

Also, $f'(x) = f'(x + 4)$.

Hence, $f'(x)$ is periodic with period 4.

Put $x = -1$ in $f'(x) = f'(x + 4)$. Then $f'(-1) = f'(3)$.

d. Putting $x = 0, y = 0$, we get $2f(0) + \{f(0)\}^2 = 1$

or $f(0) = \sqrt{2} - 1$ [$\because f(0) > 0$]

Putting $y = x$, $2f(x) + \{f(x)\}^2 = 1$

Differentiating w.r.t. x, we get

$2f'(x) + 2f(x) \cdot f'(x) = 0$

or $f'(x)\{1 + f(x)\} = 0$

or $f'(x) = 0$, because $f(x) > 0$.

2. a → q; b → r; c → s; d → p.

a. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{12t^2 - 6t - 18}{5t^4 - 15t^2 - 20}$

or $\left.\dfrac{dy}{dx}\right|_{t=1} = \dfrac{12 - 6 - 18}{5 - 15 - 20} = \dfrac{2}{5}$

or $-5\left.\dfrac{dy}{dx}\right|_{t=1} = -2$ at $t = 1$

b. Let us take

$P(x) = a(x - 2)^4 + b(x - 2)^3 + c(x - 2)^2 + d(x - 2) - 1$

$\therefore \quad P(2) = -1$

$0 = P'(2) = d$

$2 = P''(2) = 2c$ or $c = 1$

$-12 = P'''(2) = 6b$ or $b = -2$

$24 = P^{iv}(2) = 24a$ or $a = 1$

Thus, $P''(x) = 12(x - 2)^2 - 12(x - 2) + 2$

or $P''(3) = 12 - 12(1) + 2 = 2$

c. Here, $\sqrt{(1 + y^4)} = \sqrt{\left(1 + \dfrac{1}{x^4}\right)} = \dfrac{\sqrt{1 + x^4}}{x^2} \quad \left(\because y = \dfrac{1}{x}\right)$

or $\dfrac{\sqrt{1 + y^4}}{\sqrt{1 + x^4}} = \dfrac{1}{x^2}$ (1)

But $y = \dfrac{1}{x}$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{1}{x^2}$ (2)

From (1) and (2), $\frac{\sqrt{1+y^4}}{\sqrt{1+x^4}} = -\frac{dy}{dx}$

or $\frac{\frac{dy}{\sqrt{1+y^4}}}{\frac{dx}{\sqrt{1+x^2}}} = -1$

d. Obviously, $f(x)$ is a linear function.
Also, from $f'(0) = p$ and $f(0) = q$, $f(x) = px + q$
or $f''(0) = 0$

3. **a → q, r; b → p, r, s; c → q, s; d → q, r.**

a. We know that

$$2\tan^{-1}x = \begin{cases} \sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{if } -1 \le x \le 1 \\ \pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{if } x > 1 \\ -\pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{if } x < -1 \end{cases}$$

or $\frac{dy}{dx} = -\frac{2}{1+x^2}$ if $x < -1$ or $x > 1$

b. $\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \begin{cases} \tan^{-1}x, & x \ge 0 \\ -\tan^{-1}x, & x < 0 \end{cases}$

or $\frac{dy}{dx} = -\frac{1}{1+x^2}$ if $x < 0$

c. $y = |e^{|x|} - e| = \begin{cases} |e^x - e|, & x \ge 0 \\ |e^{-x} - e|, & x < 0 \end{cases} = \begin{cases} e^x - e, & x \ge 1 \\ e - e^x, & 0 \le x < 1 \\ e - e^{-x}, & -1 \le x < 0 \\ e^{-x} - e, & x < -1 \end{cases}$

or $\frac{dy}{dx} > 0$ if $x > 1$ or $-1 < x < 0$.

d. $u = \log|2x|$, $v = |\tan^{-1}x|$

or $\frac{du}{dx} = \frac{1}{x}$ and $\frac{dv}{dx} = \begin{cases} \frac{1}{1+x^2}, & x > 0 \\ -\frac{1}{1+x^2}, & x < 0 \end{cases}$

$$\therefore \quad \frac{du}{dv} = \begin{cases} \frac{1+x^2}{x}, & x > 0 \\ -\frac{1+x^2}{x}, & x < 0 \end{cases}$$

Now, we know that

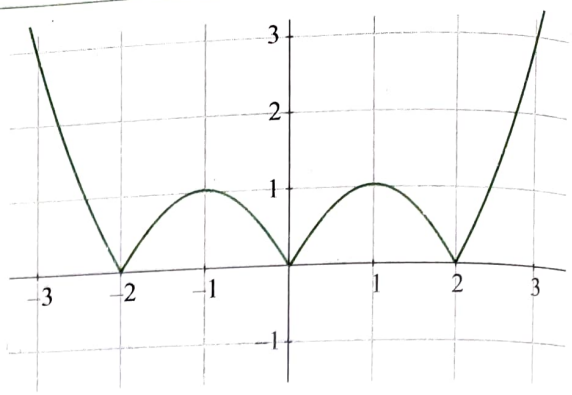
$$\frac{1+x^2}{x} = x + \frac{1}{x} > 2 \text{ if } x > 1 \text{ and } < -2 \text{ if } x < -1$$

$$\therefore \quad \frac{du}{dv} > 2 \text{ if } x < -1 \text{ or } x > 1$$

4. **a. → p, q, r; b → q, s; c → q, r; d → r.**

a. **p, q, r**

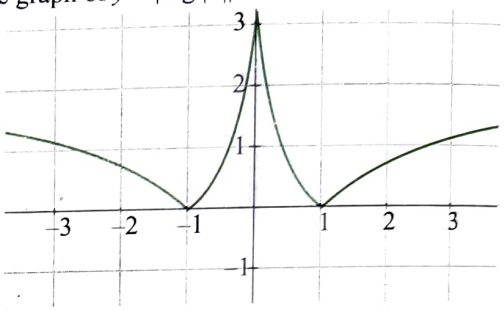
The graph of $y = |x^2 - 2|x||$:



From the graph, dy/dx is negative for (p), (q), (r)

b. **q, s**

The graph of $y = |\log|x||$:



From the graph, dy/dx is negative for (q), (s).

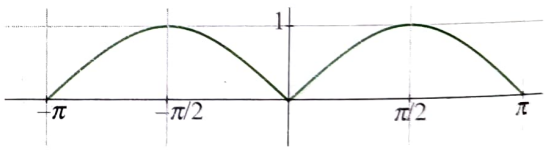
c. **q, r**

$$y = x[x/2] = \begin{cases} -2x, & -4 \le x < -2 \\ -x, & -2 \le x < 0 \\ 0, & 0 \le x < 2 \\ x, & 2 \le x < 4 \end{cases}$$

Hence, dy/dx is negative for (q), (r)

d. **r**

The graph of $y = |\sin x|$



From the graph, dy/dx is negative for (r).

5. (3) **a.** If $xy - \log y = 1$ then $xy_1 + y = y_1/y$
so $y^2 + (xy - 1)y_1 = 0$

b. $y\sqrt{1-x^2} = \sin^{-1}x$

$$\Rightarrow \quad -\frac{yx}{\sqrt{1-x^2}} + y_1\sqrt{1-x^2} = \frac{1}{\sqrt{1-x^2}}$$

so $-xy + y_1(1-x^2) = 1$

c. $y^2 = 2x - x^2 \Rightarrow yy_1 = 1 - x$

$$\Rightarrow \quad y_1^2 + yy_2 = -1$$

So, $\frac{(1-x)^2}{y^2} + yy_2 = -1$

$$\Rightarrow \quad y^3y_2 = -[1 + x^2 - 2x + y^2] = -1$$

d. $y = e^{\sqrt{x}} + e^{-\sqrt{x}}$

$$\Rightarrow \quad y_1 = \frac{e^{\sqrt{x}} - e^{-\sqrt{x}}}{2\sqrt{x}}$$

$$\Rightarrow \quad 4y_1^2x = (e^{\sqrt{x}} - e^{-\sqrt{x}})^2 = y^2 - 4$$

$\Rightarrow \quad 8y_1y_2x + 4y_1^2 = 2yy_1$

$\Rightarrow \quad 4y_2x + 2y_1 = y$

$\Rightarrow \quad xy_2 - \frac{1}{2}y_1 - \frac{1}{4}y = 0$

Numerical Value Type

1. (9) $\frac{d}{dx}\{[f(x)]^2 - [\phi(x)]^2\}$

$= 2[f(x)\cdot f'(x) - \phi(x)\cdot\phi'(x)]$

$= 2[f(x)\cdot\phi(x) - \phi(x)\cdot f(x)]$ [$\because f'(x) = \phi(x)$ and $\phi'(x) = f(x)$]

$= 0$

or $[f(x)]^2 - [\phi(x)]^2 =$ constant

$\therefore \quad [f(10)]^2 - [\phi(10)]^2 = [f(3)]^2 - [\phi(3)]^2 = [f(3)]^2 - [f'(3)]^2$

$= 25 - 16 = 9.$

2. (-2) Since $f(x)$ is odd, $f(-x) = -f(x)$ or $f'(-x)(-1) = -f'(x)$

or $f'(-x) = f'(x)$

$\therefore \quad f'(-3) = f'(3) = -2$

3. (5) Here $x = \alpha$ is a repeated root of the equation $f(x) = 0$.

Hence, $x = \alpha$ is also a root of the equation $f'(x) = 0$, i.e.,

$3x^2 + 6x - 9 = 0$

or $x^2 + 2x - 3 = 0$ or $(x + 3)(x - 1) = 0$

has the root α once which can be either -3 or 1.

If $\alpha = 1$, then $f(x) = 0$ gives $c - 5 = 0$ or $c = 5$.

If $\alpha = -3$, then $f(x) = 0$ gives $-27 + 27 + 27 + c = 0$

$\therefore \quad c = -27$

4. (3) We have $f(5 - x) = -f(5 + x)$ or $-f'(5 - x) = -f'(5 + x)$

or $f'(5 - 2) = f'(5 + 2)$ or $f'(3) = f'(7) = 3$

5. (-2) We have $g(x) = f(x)\sin x$ (1)

On differentiating equation (1) w.r.t. x, we get

$g'(x) = f(x)\cos x + f'(x)\sin x$ (2)

Again differentiating equation (2) w.r.t. x, we get

$g''(x) = f(x)(-\sin x) + f'(x)\cos x + f'(x)\cos x + f''(x)\sin x$ (3)

or $g''(-\pi) = 2f'(-\pi)\cos(-\pi) = 2 \times 1 \times (-1) = -2$

Hence, $g''(-\pi) = -2$

6. (8) $\ln(f(x)) = \ln(x - 1) + \ln(x - 2) + \cdots + \ln(x - n)$

or $f'(x) = f(x)\left[\frac{1}{x-1} + \frac{1}{x-2} + \cdots \frac{1}{x-n}\right]$

$= (x-2)(x-3)(x-n) + (x-1)(x-3)\cdots(x-n) + \cdots$

$+ (x-1)(x-2)\cdots(x-(n-1))$

or $f'(n) = (n-1)(n-2)(n-3)\cdot 3 \cdot 2 \cdot 1$

(all other factors except the last vanishes when $x = n$)

or $5040 = (n-1)!$

or $n = 8$

7. (9) $f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x+0)}{h}$

or $\lim_{h\to 0}\frac{2f(x) + xf(h) + h\sqrt{f(x)} - 2f(x) - xf(0) - 0\sqrt{f(x)}}{h}$

as $f(0) = 0$

or $\lim_{h\to 0} x\left(\frac{f(h) - f(0)}{h - 0}\right) + \sqrt{f(x)} = f'(0) + \sqrt{f(x)}$

or $f'(x) = \sqrt{f(x)}$ [$\because f'(0) = 0$]

or $\int \frac{f'(x)}{\sqrt{f(x)}}dx = \int dx$

or $2\sqrt{f(x)} = x + c$

or $f(x) = \frac{x^2}{4}$ [$\because f(0) = 0$]

8. (3) $f(x)\cdot f'(-x) = f(-x)\cdot f'(x)$

or $f'(x)\times f(-x) - f(x)\times f'(-x) = 0$

or $\frac{d}{dx}[f(x)f(-x)] = 0$

or $f(x)f(-x) = k$

Given $(f(0))^2 = k = 9$ or $k = 9$

Then $f(3)f(-3) = 9$ or $f(-3) = 3$

9. (5) $y = \frac{a + bx^{3/2}}{x^{5/4}}$

or $y' = \frac{\frac{3}{2}bx^{1/2}x^{5/4} - \frac{5}{4}x^{1/4}(a + bx^{3/2})}{x^{5/2}}$

According to the question,

$0 = \frac{\frac{3}{2}b5^{1/2}5^{5/4} - \frac{5}{4}5^{1/4}(a + b5^{3/2})}{5^{5/2}}$

or $\frac{3b}{2}5^{7/4} - a\frac{5^{5/4}}{4} - 5b\frac{5^{7/4}}{4} = 0$

or $b5^{7/4} = a5^{5/4}$

or $b\sqrt{5} = a$

or $a : b = \sqrt{5} : 1$

10. (3) $y = \frac{x^4 - (x^2 + 2x + 1)}{x^2 - x - 1} = x^2 + x + 1$

$\therefore \quad \frac{dy}{dx} = 2x + 1 = ax + b$

Hence, $a = 2$ and $b = 1$

11. (2) Limit is $f'(e)$ where $f(x) = x^{\ln x} = e^{\ln^2 x}$

$\therefore \quad f'(x) = e^{\ln^2 x}\cdot\frac{2\ln x}{x}$

or $f'(e) = e\cdot\frac{2}{e} = 2$

12. (5) We have $(gof)(x) = x$

or $g'(f(x))f'(x) = 1$

When $f(x) = -\frac{7}{6}$, $x = 1$

or $g'\left(-\frac{7}{6}\right)f'(1) = 1$

Hence, $g'\left(-\frac{7}{6}\right) = \frac{1}{f'(1)} = \frac{1}{5}$

13. (6) $g(x) = f\left(-x + f\left(f(x)\right)\right)$; $f(0) = 0$; $f'(0) = 2$

$g'(x) = f'\left(-x + f\left(f(x)\right)\right)\cdot\left[-1 + f'\left(f(x)\right)\cdot f'(x)\right]$

$g'(0) = f'\left(f(0)\right)\cdot[-1 + f'(0)\cdot f'(0)]$

$= f'(0)[-1 + (2)(2)] = (2)(3) = 6$

14. (0) $f(x) = 0$ has roots 1, 2, 3 only.

$\Rightarrow$ Any one of 1, 2 or 3 is a repeated root of $f(x) = 0$

$\Rightarrow$ $f'(1)$ or $f'(2)$ or $f'(3)$ is zero.

$\Rightarrow$ $f'(1) \cdot f'(2) \cdot f'(3) = 0$

15. (18) Let degree of $f(x)$ is n, degree of $f'(x)$ is $n-1$, and degree of $f''(x)$ is $(n-2)$. Hence,

$$n = (n-1) + (n-2) = 2n - 3$$

$\therefore \quad n = 3$

Hence, $f(x) = ax^3 + bx^2 + cx + d \qquad (a \neq 0)$

$$f'(x) = 3ax^2 + 2bx + c$$

$$f''(x) = 6ax + 2b$$

$\therefore \quad ax^3 + bx^2 + cx + d = (3ax^2 + 2bx + c)(6ax + 2b)$

$\therefore \quad 18a^2 = a \quad \text{or} \quad a = \frac{1}{18}$

16. (1) $\frac{dx}{dt} = -\frac{3}{t^4} - \frac{2}{t^3} = -\left(\frac{3+2t}{t^4}\right)$

$$\frac{dy}{dt} = -\left(\frac{3}{t^3} + \frac{2}{t^2}\right) = -\left(\frac{3+2t}{t^3}\right)$$

or $\frac{dy}{dx} = t \quad$ or $\quad \frac{dy}{dx} - x\left(\frac{dy}{dx}\right)^3 = t - \left(\frac{1+t}{t^3}\right) \cdot t^3 = -1$

17. (2.5) $z = (\cos x)^5$; $y = \sin x$

$$\frac{dz}{dx} = -5\cos^4 x \cdot \sin x; \quad \frac{dy}{dx} = \cos x$$

$\therefore \quad \frac{dz}{dy} = -5\cos^3 x \cdot \sin x$

Now, $\frac{d^2z}{dy^2} = \frac{d}{dx}\left(\frac{dz}{dy}\right) \cdot \frac{dx}{dy}$

$$= -5\frac{d}{dx}[\cos^3 x \cdot \sin x]\ \frac{1}{\cos x}$$

$$= -5[\cos^4 x - 3\sin^2 x \cdot \cos^2 x]\ \frac{1}{\cos x}$$

$$= -5(\cos^3 x - 3\sin^2 x \cdot \cos x)$$

$$= -5(\cos^3 x - 3\cos x(1 - \cos^2 x))$$

$$= -5(4\cos^3 x - 3\cos x)$$

$$= -5\cos 3x$$

$\therefore \quad \left.\frac{d^2z}{dy^2}\right|_{x=\frac{2\pi}{9}} = -5\cos 120^\circ = \frac{5}{2}$

18. (7) $g'(0) = b = \lim_{x\to 0} \frac{x^2 + x\tan x - x\tan 2x}{x(ax + \tan x - \tan 3x)}$

$$= \lim_{x\to 0} \frac{x + \tan x - \tan 2x}{ax + \tan x - \tan 3x}$$

$$= \lim_{x\to 0} \frac{x + \left(x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots\infty\right) - \left(2x + \frac{8x^3}{3} + \frac{2}{15}\cdot 32x^5 + \cdots\infty\right)}{ax + \left(x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots\infty\right) - \left(3x + \frac{27x^3}{3} + \frac{2}{15}243x^5 + \cdots\infty\right)}$$

$$= \lim_{x\to 0} \frac{x^3\left(-\frac{7}{3} + \frac{-62}{15}x^2 + \cdots\infty\right)}{(a+1-3)x + \left(\frac{1}{3} - 9\right)x^3 + \frac{2}{15}(-242)x^5 + \cdots\infty}$$

b can be finite if $a + 1 - 3 = 0$. Therefore,

$$a = 2 \text{ and } b = \frac{-\frac{7}{3}}{\frac{1}{3} - 9} = \left(\frac{-7}{3}\right)\left(\frac{3}{-26}\right) = \frac{7}{26} \Rightarrow 52\frac{b}{a} = 7$$

19. (50) $f(x) = x + \cfrac{1}{x + x + \cfrac{1}{2x + \cfrac{1}{2x + \cdots\infty}}} = x + \frac{1}{x + f(x)}$

or $\quad f(x) - x = \frac{1}{x + f(x)}$

or $\quad f^2(x) - x^2 = 1$

Differentiating w.r.t. x, we get

or $\quad 2f(x) \cdot f'(x) - 2x = 0$

or $\quad f(x) \cdot f'(x) = x$

or $\quad f(50) \cdot f'(50) = 50$

20. (24) $F(x) = f(x)\, g(x)\, h(x)\ \forall\ x \in R$

$f(x)$, $g(x)$, $h(x)$ are differentiable functions. Then

$$F'(x) = f'(x)\, g(x)\, h(x) + f(x)\, g'(x)\, h(x) + f(x)\, g(x)\, h'(x)$$

At $x = x_0$,

$$F'(x_0) = f'(x_0)\, g(x_0)\, h(x_0) + f(x_0)\, g'(x_0)\, h(x_0) + f(x_0)g(x_0)\, h'(x_0).$$

Using the given values of $F'(x_0)$, $f'(x_0)$, $g'(x_0)$, and $h'(x_0)$, we get

$$21F(x_0) = 4f(x_0)\, g(x_0)\, h(x_0) - 7f(x_0)\, g(x_0)\, h(x_0) + kh(x_0)\, f(x_0)\, g(x_0)$$

or $\quad 21 = 4 - 7 + k \qquad [\because F(x_0) = f(x_0)\, g(x_0)\, h(x_0)]$

or $\quad k = 24$

21. (1) $y = \frac{\sqrt[3]{1+3x}\ \sqrt[4]{1+4x}\ \sqrt[5]{1+5x}}{\sqrt[7]{1+7x}\ \sqrt[8]{1+8x}}$

$$\log_e y = \frac{1}{3}\log_e(1+3x) + \frac{1}{4}\log_e(1+4x) + \frac{1}{5}\log_e(1+5x)$$

$$-\frac{1}{7}\log_e(1+7x) - \frac{1}{8}\log_e(1+8x)$$

Differentiating w.r.t. x

$$\frac{1}{y} \times \frac{dy}{dx} = \frac{1}{1+3x} + \frac{1}{1+4x} + \frac{1}{1+5x} - \frac{1}{1+7x} - \frac{1}{1+8x}$$

At $x = 0$, $y = 1$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=0} = 1 + 1 + 1 - 1 - 1 = 1$$

Archives

JEE MAIN

Single Correct Answer Type

1. (1) $x^{2x} - 2x^x \cot y - 1 = 0$

Now at $x = 1$,

$$1 - 2\cot y - 1 = 0 \Rightarrow \cot y = 0 \Rightarrow y = \frac{\pi}{2}$$

Now differentiating (i) w.r.t. x, we get

$$2x^{2x}(1+\log x)-2\left[x^x(-\operatorname{cosec}^2 y)\frac{dy}{dx}+\cot y\, x^x(1+\log x)\right]=0$$

Now at $(1, \pi/2)$,

$$2(1+\log 1)-2\left[1\,(-1)\left(\frac{dy}{dx}\right)_{(1,\pi/2)}+0\right]=0$$

$$\Rightarrow \quad 2+2\left(\frac{dy}{dx}\right)_{(1,\pi/2)}=0$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(1,\pi/2)}=-1$$

2. (3) $g'(x)=2\,[f(2f(x)+2)]\left[\frac{d}{dx}(f(2f(x)+2))\right]$

$$=2[f(2f(x)+2)][f'(2+f(x)+2)\times 2f'(x)]$$

$$\Rightarrow \quad g'(0)=2f(2f(0)+2)\,f'(2f(0)+2)\,2f'(0)$$

$$=4f(0)\,[f'(0)]^2$$

$$=4(-1)(1)=-4$$

3. (1) We have

$$\frac{d^2x}{dy^2}=\frac{d}{dy}\left(\frac{dx}{dy}\right)=\frac{d}{dy}\left(\frac{1}{\frac{dy}{dx}}\right)=\frac{d}{dx}\left(\frac{1}{\frac{dy}{dx}}\right)\cdot\frac{dx}{dy}$$

$$=-\frac{1}{\left(\frac{dy}{dx}\right)^2}\cdot\frac{d^2y}{dx^2}\cdot\frac{1}{\left(\frac{dy}{dx}\right)}$$

$$=-\frac{1}{\left(\frac{dy}{dx}\right)^3}\cdot\frac{d^2y}{dx^2}$$

$$=-\left(\frac{dy}{dx}\right)^{-3}\left(\frac{d^2y}{dx^2}\right)$$

4. (4) $y=\sec(\tan^{-1}x)$

$$\Rightarrow \quad \frac{dy}{dx}=\sec(\tan^{-1}x)\cdot\tan(\tan^{-1}x)\cdot\frac{1}{1+x^2}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=1}=\sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right)\frac{1}{1+1}=\frac{1}{\sqrt{2}}$$

5. (4) Since g is inverse of f, $f(g(x))=x$

$$\Rightarrow \quad f'(g(x))\,g'(x)=1$$

$$\Rightarrow \quad g'(x)=1+(g(x))^5 \qquad \left(\because f'(x)=\frac{1}{1+x^5}\right)$$

6. (2) We have

$$y=\tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)$$

$$=\tan^{-1}\left(\frac{2\cdot(3x^{3/2})}{1-(3x^{3/2})^2}\right)=2\tan^{-1}(3x^{3/2})$$

$$\therefore \quad \frac{dy}{dx}=2\times\frac{1}{1+9x^3}\times 3\times\frac{3}{2}\times x^{1/2}=\frac{9\sqrt{x}}{1+9x^3}$$

$$\therefore \quad g(x)=\frac{9}{1+9x^3}$$

JEE ADVANCED

Multiple Correct Answers Type

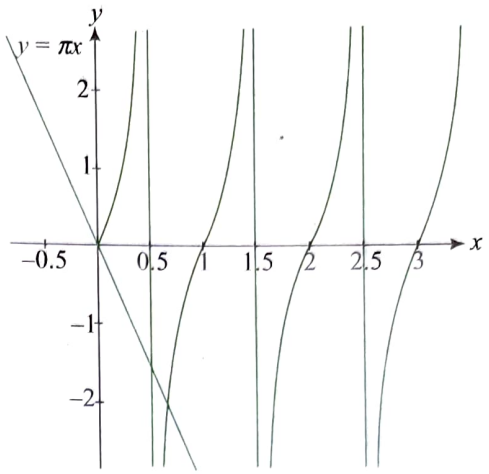
1. **(2), (3)**

We have $f'(x)=\sin\pi x+\pi x\cos\pi x=0$

or $\tan\pi x=-\pi x$

The graph of $y=\tan\pi x$ and $y=-\pi x$ is as shown in the following figure. Therefore,

$$\tan\pi x=-\pi x$$



From the graph

$$x\in\left(n+\frac{1}{2},\,n+1\right) \text{ or } (n,\,n+1)$$

2. **(1)** $f'(x)+\frac{f(x)}{x}=2$

$$\Rightarrow \quad xf'(x)+f(x)=2x$$

$$\Rightarrow \quad \int d(x.f(x))=\int 2x\,dx$$

$$\Rightarrow \quad xf(x)=x^2+c$$

$$\Rightarrow \quad f(x)=x+\frac{c}{x} \qquad (c\neq 0 \text{ as } f(1)\neq 1)$$

$$\lim_{x\to 0^+} f'\left(\frac{1}{x}\right)=\lim_{x\to 0^+}(1-cx^2)=1$$

$$\lim_{x\to 0^+} xf\left(\frac{1}{x}\right)=\lim_{x\to 0^+}(1+cx^2)=1$$

$$\lim_{x\to 0^+} x^2f'(x)=\lim_{x\to 0^+}(x^2-c)=-c\neq 0$$

$\lim_{x\to 0^+} f(x)=\infty$ or $-\infty$, so option (4) is incorrect

3. **(2), (3)**

$$f(x)=x^3+3x+2$$

$\therefore$ $f(1)=6$

$\therefore$ In $g(f(x))=x$, putting $x=1$

$g(6)=1$

Also $g(f(x))=x$

$\Rightarrow$ $g'(f(x))\times f'(x)=1$

Put $x=0$

$\therefore$ $g'(f(0)).\,f'(0)=1$

$\therefore$ $g'(2)=\frac{1}{f'(0)}=\frac{1}{3}$

$f(3)=38$

$\therefore$ $g(38)=3$

$\therefore$ $h(g(3))=h(g(g(38)))=38$

$f(2)=16\Rightarrow g(16)=2$

$\therefore$ $h(g(g(16))=h(g(2))=h(0)$

$\therefore$ $16=h(g(g(16))=h(0)$

$\therefore$ Option (3) is correct.

$f'(x)=3x^2+3$

$\therefore \quad f'(6) = 111, f(1) = 6 \Rightarrow g'(6) = \dfrac{1}{6}$

$h(g(g(x))) = x$

$\Rightarrow \quad h'(g(g(x))) \times g'(g(x)) \times g'(x) = 1$

Put $x = 236$,

$\therefore \quad h'(g(g(236))) \times g'(g(236)) \times g'(236) = 1$

$\Rightarrow \quad h'(g(6))\, g'(6) \times \dfrac{1}{f'(6)} = 1$

$\Rightarrow \quad h'(1) = 666$

Matrix Match Type

1. (1) $y = \cos(3\cos^{-1} x)$

$\Rightarrow \quad \dfrac{3\sin(3\cos^{-1} x)}{\sqrt{1-x^2}}$

$\Rightarrow \quad \sqrt{1-x^2}\; y' = 3\sin(3\cos^{-1} x)$

$\Rightarrow \quad \dfrac{-x}{\sqrt{1-x^2}}\, y' + \sqrt{1-x^2}\; y'' = 3\cos(3\cos^{-1} x) \cdot \dfrac{-3}{\sqrt{1-x^2}}$

$\Rightarrow \quad -xy' + (1-x^2)\, y'' = -9y$

$\Rightarrow \quad \dfrac{1}{y}\,[(x^2-1)\, y'' + xy'] = 9$

Numerical Value Type

1. (1) $f(\theta) = \sin\left(\tan^{-1}\left(\dfrac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $\theta \in \left(-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right)$

$= \sin\left(\tan^{-1}\left(\dfrac{\sin\theta}{\sqrt{2\cos^2\theta - 1}}\right)\right) = \sin(\sin^{-1}(\tan\theta)) = \tan\theta$

$\therefore \quad \dfrac{d(\tan\theta)}{d(\tan\theta)} = 1$

2. (8) $(y - x^5)^2 = x(1 + x^2)^2$

Differentiating both sides w.r.t. x, we get

$$2(y - x^5)\left(\frac{dy}{dx} - 5x^4\right) = 1(1 + x^2)^2 + (x)\,(2\,(1 + x^2)\,(2x))$$

On putting $x = 1, y = 3$ in above equation, we get

$$\frac{dy}{dx} = 8$$

3. (2) $f(x + y) = f(x) \cdot f'(y) + f'(x) \cdot f(y)$ ...(1)

Putting $y = 0$

$$f(x) = f(x)\, f'(0) + f'(x) \cdot f(0) = f(x)\, f'(0) + f'(x) \qquad \ldots(2)$$

To find $f'(0)$, in (1), put $x = y = 0$.

$\therefore \quad f(0) = 2f(0) \cdot f'(0)$

$\therefore \quad f'(0) = \dfrac{1}{2}$

So, from (2), we get

$$f'(x) = \frac{f(x)}{2}$$

$\Rightarrow \quad \displaystyle\int \frac{f'(x)}{f(x)}dx = \int \frac{1}{2}dx$

$\Rightarrow \quad \log_e f(x) = (x/2) + c$

$\Rightarrow \quad \log_e f(x) = x/2 \qquad (\text{as } f(0) = 1)$

$\Rightarrow \quad \log_e(f(4)) = 2$

Chapter 4

Concept Application Exercises

Exercise 4.1

1. Given relation $f(x\cdot y)=f(x)f(y)$.

Put $x=y=1$. Then $f(1)=f(1)f(1)$, i.e., $f(1)=0$ or 1.

If $f(1)=0$, then $f(x\cdot 1)=f(x)f(1)=0$ or $f(x)$ is identically zero which is continuous for all x.

For $f(1)=1$:

Consider some arbitrary point $x=a$.

$$\text{L.H.L.}=\lim_{h\to 0} f(a-h)$$

$$=\lim_{h\to 0} f\left(a\left(1-\frac{h}{a}\right)\right)$$

$$=\lim_{h\to 0} f(a)f\left(1-\frac{h}{a}\right)$$

$$=f(a)\lim_{h\to 0} f\left(1-\frac{h}{a}\right)$$

$$=f(a)f(1) \qquad [\text{as } f(x) \text{ is continuous at } x=1]$$

$$=f(a)$$

Similarly, we can prove that R.H.L. $=\lim_{h\to 0} f(a+h)=f(a)$.

Hence, $f(x)$ is continuous for all x.

2. We must have

$$f(0)=\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{(1+x)^{\frac{1}{2}}-(1+x)^{\frac{1}{3}}}{x}$$

$$=\lim_{x\to 0}\left[\frac{(1+x)^{\frac{1}{2}}-1}{x}-\frac{(1+x)^{\frac{1}{3}}-1}{x}\right]$$

$$=\frac{1}{2}-\frac{1}{3}$$

$$=\frac{1}{6}$$

3. For continuity at $x=0$,

$$f(0)=\lim_{x\to 0}\left(\frac{1}{x}-\frac{2}{e^{2x}-1}\right)$$

$$=\lim_{x\to 0}\frac{e^{2x}-1-2x}{x\left(e^{2x}-1\right)}$$

$$=\lim_{x\to 0}\frac{\left(1+(2x)+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\cdots\right)-1-2x}{x\left(1+(2x)+\frac{(2x)^2}{2!}+\cdots-1\right)}$$

$$=\lim_{x\to 0}\frac{\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\cdots}{x\left((2x)+\frac{(2x)^2}{2!}+\cdots\right)}$$

$$=\lim_{x\to 0}\frac{\frac{2^2}{2!}+\frac{2^3}{3!}x+\cdots}{2+\frac{(2)^2}{2!}x+\cdots}$$

$$=\frac{2+0+\cdots}{2+0+\cdots}$$

$$=\frac{2}{2}=1$$

4. f is continuous at $\frac{\pi}{4}$, if

$$f\left(\frac{\pi}{4}\right)=\lim_{x\to\frac{\pi}{4}}\frac{1-\tan x}{4x-\pi}$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{0-\sec^2 x}{4} \qquad \text{[L' Hopital's rule]}$$

$$=-\frac{1}{4}\sec^2\frac{\pi}{4}$$

$$=-\frac{1}{4}(2)$$

$$=-\frac{1}{2}$$

5. $f(x)$ is continuous at $x=1$ if

$$f(1)=\lim_{x\to 1}\left(\tan\left(\frac{\pi}{4}+\log_e x\right)\right)^{\log_x e}$$

$$=\lim_{h\to 0}\left(\tan\left(\frac{\pi}{4}+h\right)\right)^{\frac{1}{h}}$$

$$=e^{\lim_{h\to 0}\left(\tan\left(\frac{\pi}{4}+h\right)-1\right)\frac{1}{h}}$$

$$=e^{\lim_{h\to 0}\left(\frac{2\tan h}{1-\tan^2 h}\right)\frac{1}{h}}$$

$$=e^2$$

6. Since $f(x)$ is continuous at $x=2$,

$$f(2)=\lim_{x\to 2} f(x)$$

$$=\lim_{x\to 2}\frac{x^2-(A+2)x+A}{x-2}$$

$$=\lim_{x\to 2}\frac{x(x-2)-A(x-1)}{x-2}$$

Now, $f(2)$ is finite only when $A=0$.

7. Given that $f(x)$ is continuous in $[0,\pi]$.

Since each of the branch functions is continuous in its corresponding interval, $f(x)$ must be continuous at $x=\frac{\pi}{4}$ and $x=\pi/2$

$$\therefore \quad \lim_{x\to\pi/4^-} f(x)=f(\pi/4)=\lim_{x\to\pi/4^+} f(x)$$

$$\Rightarrow \quad \lim_{x\to\pi/4^-}(x+a\sqrt{2}\sin x)=f(\pi/4)=\lim_{x\to\pi/4^+}(2x\cot x+b)$$

$$\Rightarrow \quad \frac{\pi}{4}+a\sqrt{2}\sin\frac{\pi}{4}=\frac{2\pi}{4}\cot\frac{\pi}{4}+b$$

$$\Rightarrow \quad a-b=\frac{\pi}{4} \qquad \ldots(1)$$

Also, $\lim_{x\to\pi/2^-} f(x)=f(\pi/2)=\lim_{x\to\pi/2^+} f(x)$

$$\therefore \quad \lim_{x\to\pi/2^-}(2x\cot x+b)=f(\pi/2)=\lim_{x\to\pi/2^+}(a\cos 2x-b\sin x)$$

$$\Rightarrow \quad 2\cdot\frac{\pi}{2}\cot\frac{\pi}{2}+b=a\cos\pi-b\sin\frac{\pi}{2}$$

$$\Rightarrow \quad a+2b=0 \qquad \ldots(2)$$

Solving (1) and (2), we get $a = \pi/6$ and $b = -\pi/12$

8. L.H.L. at $x = 0$,

$$f(0^-)=\lim_{x\to0^-}\frac{1-\cos 4x}{x^2}$$

$$=\lim_{x\to0^-}\frac{2\sin^2 2x}{4x^2}\times 4=8$$

R.H.L at $x = 0$ is,

$$f(0^+)=\lim_{x\to0^+}\frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}$$

$$=\lim_{x\to0^+}\frac{\sqrt{x}(\sqrt{16+\sqrt{x}}+4)}{16+\sqrt{x}-16}$$

$$=\lim_{x\to0^+}(\sqrt{16+\sqrt{x}}+4)$$

$$=4+4=8$$

Since $f(x)$ is continuous at $x = 0$, $f(0) = f(0^-) = f(0^+)$

$\therefore \quad a = 8.$

9. $f(x)$ is continuous at $x = 0$.

$$\therefore \quad \lim_{x\to0^-} f(x)=f(0)=\lim_{x\to0^+} f(x)$$

$$\Rightarrow \quad \lim_{x\to0^-}(1+|\sin x|)^{\frac{a}{|\sin x|}}=b=\lim_{x\to0^+}e^{\frac{\tan 2x}{\tan 3x}}$$

$$\Rightarrow \quad \left(\lim_{x\to0^-}(1+|\sin x|)^{\frac{1}{|\sin x|}}\right)^a=b=e^{\lim_{x\to0^+}\frac{\frac{\tan 2x}{2x}\times 2x}{\frac{\tan 3x}{3x}\times 3x}}$$

$$\Rightarrow \quad e^a=b=e^{\frac{2}{3}}$$

$$\Rightarrow \quad a=\frac{2}{3},\ b=e^{\frac{2}{3}}$$

10. RH.L. at $x = 0$,

$$f(0^+)=\lim_{x\to0^+}\frac{\sin(e^{x^2}-1)}{x^2}$$

$$=\lim_{x\to0^+}\frac{\sin(e^{x^2}-1)}{e^{x^2}-1}\cdot\frac{e^{x^2}-1}{x^2}$$

$$=(1)\cdot(1)=1$$

Also, $f(0) = 1$

Now L.H.L. $=\lim_{x\to0^-}\frac{\log_e(px^2+qx+r)}{x^2}=1.$

As denominator $\to 0$ for $x \to 0^-$ and limit exist so numerator also $\to 0$ for $x \to 0^-$

$\Rightarrow \quad \log_e r = 0 \Rightarrow r = 1.$

$$\Rightarrow \quad \lim_{x\to0^-}\frac{\log_e((px^2+qx)+1)}{px^2+qx}\cdot\frac{px^2+qx}{x^2}=1$$

$$\Rightarrow \quad \lim_{x\to0^-}\frac{px+q}{x}=1$$

$\Rightarrow \quad p = 1$ and $q = 0$

11. **a → s, q; b → t, p; c → r, q; d → u, q**

a. $f(x)=\frac{1}{x-1}$

$$\Rightarrow \quad \lim_{x\to1^+}\frac{1}{x-1}=\infty \text{ and } \lim_{x\to1^-}\frac{1}{x-1}=-\infty$$

$\therefore$ $f(x)$ has discontinuity due to vertical asymptote which is nonremovable.

b. $f(x)=\frac{x^3-x}{x^2-1}=x,\ x\neq\pm1,$

$\therefore$ $f(x)$ has missing point discontinuity which is removable.

c. $f(x)=\frac{|x-1|}{x-1}=\begin{cases}1, & x>1\\ -1, & x<1\end{cases}$

$\therefore$ $f(x)$ has jump discontinuity which is nonremovable.

d. $f(x)=\sin\left(\frac{1}{x-1}\right)$

$$\therefore \quad \lim_{x\to1+}\sin\left(\frac{1}{x-1}\right)=\sin(\infty)=\text{any value between } -1 \text{ and } 1$$

So, $f(x)$ has oscillating discontinuity which is nonremovable.

12. (d) $f_1(x)=\sqrt{2\sin x+3}$

$$-1\le\sin x\le1$$

or $\quad -2\le 2\sin x\le 2$

or $\quad 1\le 2\sin x+3\le 5$

So, $\sqrt{2\sin x+3}$ is defined and, hence, continuous $\forall\ x\in R$.

$f_2(x)=\frac{e^x+1}{e^x+3}$. Here, $e^x + 3 > 3\ \forall\ x \in R$.

So, $f_2(x)$ is continuous, $\forall\ x \in R$

$$f_3(x)=\left(\frac{2^{2x}+1}{2^{3x}+5}\right)^{5/7}.$$

Here, $2^{3x} + 5 = 8^x + 5 > 5\ \forall\ x \in R$.

Hence, $f_3(x)$ is continuous $\forall\ x \in R$.

$$f_4(x)=\sqrt{\text{sgn}(x)+1}$$

$$=\begin{cases}\sqrt{\frac{|x|}{x}+1} & x\neq0\\ \sqrt{1}, & x=0\end{cases}$$

$$=\begin{cases}0, & x<0\\ \sqrt{2}, & x>0\\ 1, & x=0\end{cases}.$$

Clearly, $f_4(x)$ is discontinuous at $x = 0$.

13. $\lim_{x\to0} f(x)=\lim_{x\to0}\frac{x^2}{|x|}=\lim_{x\to0}|x|=0=f(0)$

So, $f(x)$ is continuous at $x = 0$.

Also, if $x \neq 0$, $f(x) = |x|$, which is continuous for nonzero x.

Therefore, $f(x)$ is continuous everywhere.

Clearly, continuous at $x = 1$.

To check continuity at $x = 0$, $f(0) = e^3$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}(1+3x)^{1/x} \qquad (1^\infty \text{ form})$$

$$= e^{\lim_{x\to 0} 3x\left(\frac{1}{x}\right)}$$

$$= e^{\lim_{x\to 0}(3)} = e^3$$

$$\therefore \lim_{x\to 0} f(x) = f(0)$$

Thus, continuous at $x = 0$.

$$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} \frac{h}{e^{\frac{1}{h}}+1} = 0$$

and, $$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} \frac{-h}{e^{-\frac{1}{h}}+1} = 0$$

Also, $f(1) = 0$

Therefore, $f(x)$ is continuous at $x = 0$.

16. For any $x \neq 1, 2$, we find that $f(x)$ is the quotient of two polynomials and a polynomial is everywhere continuous. Therefore, $f(x)$ is continuous for all $x \neq 1, 2$.

$$f(x) = \begin{cases} \dfrac{x^4 - 5x^2 + 4}{|(x-1)(x-2)|}, & x \neq 1, 2 \\ 6, & x = 1 \\ 12, & x = 2 \end{cases}$$

$$= \begin{cases} \dfrac{(x-1)(x-2)}{|(x-1)(x-2)|}(x^2+3x+2), & x \neq 1, 2 \\ 6, & x = 1 \\ 12, & x = 2 \end{cases}$$

$$= \begin{cases} (x^2+3x+2), & x < 1 \text{ or } x > 2 \\ -(x^2+3x+2), & 1 < x < 2 \\ 6, & x = 1 \\ 12, & x = 2 \end{cases}$$

$f(1^+) = -6, f(1^-) = 6, f(2^+) = 12$, and $f(2^-) = -12$

Hence, $f(x)$ is discontinuous at $x = 1$ and $x = 2$.

17. We have $f(x) = [x] + |1 - x|, -1 \le x \le 3$.

$[x]$ is discontinuous at $x = -1, 0, 1, 2$ and 3.

$|1 - x|$ is continuous $\forall\, x \in [-1, 3]$.

$\therefore$ $f(x)$ is discontinuous at $x = 0, 1, 2$ and 3.

Exercise 4.2

1. $f(x) = [x^2 + 1] = [x^2] + 1$

Now, x^2 is monotonic in the range of $[1, 3]$.

Hence, $[x^2]$ is discontinuous when x^2 is integer or

$x^2 = 2, 3, 4, \ldots, 9$

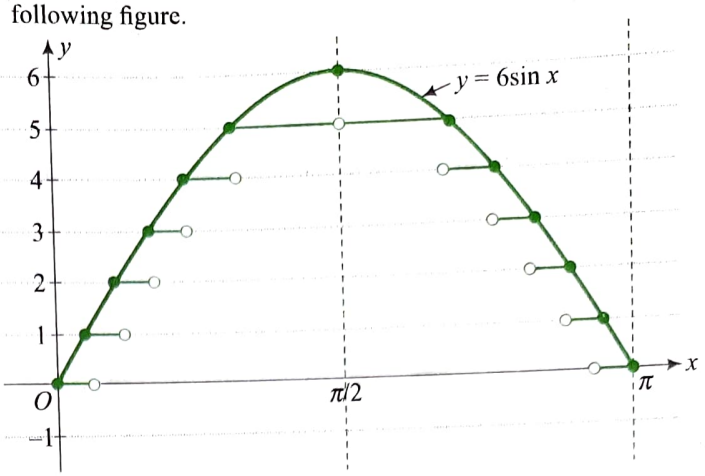
or $x = \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{9}$

Note that it is right continuous at $x = 1$ but not left continuous at $x = 3$. Therefore,

$$\lim_{x\to 1^+}[x^2+1] = 2 = f(1)$$

and $$\lim_{x\to 3^-}[x^2+1] = 9 \neq 10 [= f(3))]$$

2. Graph of $y = 6\sin x$ and $f(x) = [6\sin x]$ are as shown in the following figure.



From the figure, $f(x)$ has 11 points of discontinuity.

3. Since $g(x) = \tan^{-1} x$ is a strictly increasing function,

$f(x) = [\tan^{-1} x]$ is discontinuous when $\tan^{-1} x$ is an integer.

Now, integral values of $\tan^{-1} x$ are -1, 0, and 1.

Hence, $f(x)$ is discontinuous when

$\tan^{-1} x = -1, 0, 1$

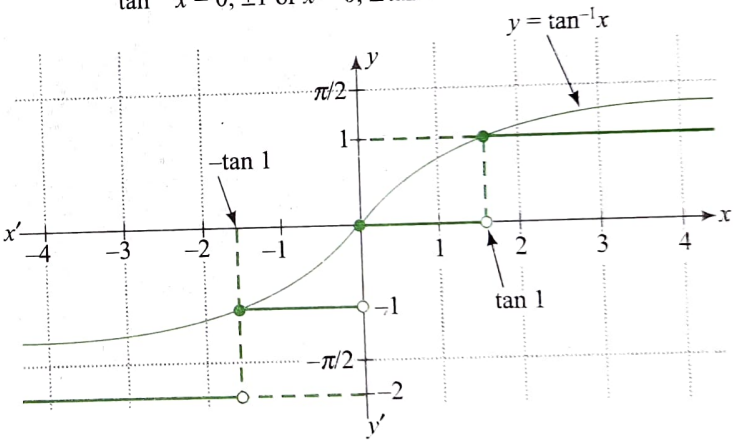
or $x = \tan(-1), \tan 0, \tan 1$

or $x = -\tan 1, 0, \tan 1$

Graphically, also this can be analyzed.

Clearly, from the graph given in the figure, $f(x)$ is discontinuous when

$\tan^{-1} x = 0, \pm 1$ or $x = 0, \pm \tan 1$



4. $f(x) = \{\cot^{-1} x\}$

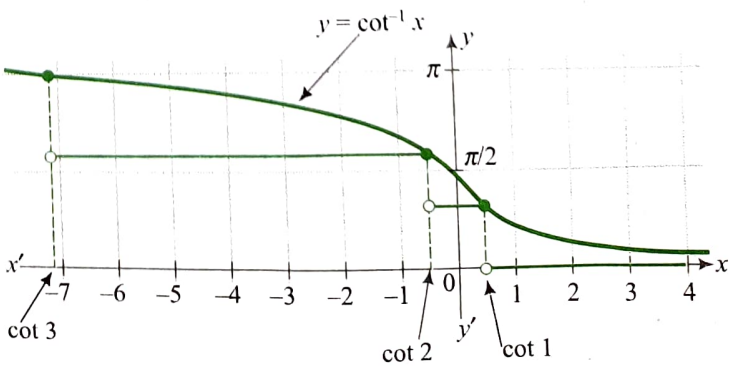
$= \cot^{-1} x - [\cot^{-1} x]$

$f(x)$ is discontinuous where $\cot^{-1} x$ is an integer.

Clearly, from graph shown in the figure, $f(x)$ is discontinuous when

$\cot^{-1} x = 1, 2, 3$

or $x = \cot 1, \cot 2, \cot 3$



5. $\text{sgn}(x^2-1)$ is discontinuous when $x^2-1=0$ or $x=\pm1$.
But $\log|x|=0$ when $x=\pm1$. Hence, $f(x)$ is continuous at $x=\pm1$.
Then $f(x)$ is continuous in its entire domain.

6. a. Continuity should be checked at the end-points of intervals of each definition, i.e., $x=0, 1, 2$.

b. For $[\sin \pi x]$, continuity should be checked at all values of x at which $\sin \pi x \in I$, i.e., $x=0, \frac{1}{2}$

c. For $\left(\text{sgn}\left(x-\frac{5}{4}\right)\right)\left\{x-\frac{2}{3}\right\}$, continuity should be checked when $x-\frac{5}{4}=0$ [as sgn $(g(x))$ is discontinuous at $g(x)=0$], i.e., $x=\frac{5}{4}$, and when $x-\frac{2}{3}\in I$,

Finally, $\left\{x-\frac{2}{3}\right\}$ is discontinuous when $\left(x-\frac{2}{3}\right)\in I$.

i.e., $x=\frac{5}{3}$ (as $\{x\}$ is discontinuous when $x\in I$).

Therefore, overall discontinuity should be checked at $x=0$, $\frac{1}{2}$, 1, $\frac{5}{4}$, $\frac{5}{3}$, and 2. Check the discontinuity yourself.

Hence, $f(x)$ is discontinuous at $x=\frac{1}{2}, 1\ \frac{5}{4}, \frac{5}{3}$.

At $x=0$ and $x=2$, $f(x)$ is continuous as
$\lim_{x\to0^-} f(x)=f(0)$ and $\lim_{x\to2^-} f(x)=f(2)$.

7. $f(x)=\lim_{n\to\infty}\dfrac{1-\dfrac{\sin x^n}{x^n}}{1+\dfrac{\sin x^n}{x^n}}=\begin{cases}1 \text{ for } x<1\\ 0 \text{ for } x=1\\ 0 \text{ for } x>1\end{cases}$

Hence, $f(x)$ is discontinuous at $x=1$.

8. Since $\lim_{n\to\infty} x^{2n}=\begin{cases}0, & |x|<1\\ 1, & |x|=1\end{cases}$

We have $f(x)=\lim_{n\to\infty}\left(\sin\left(\frac{\pi x}{2}\right)\right)^{2n}=\begin{cases}0; & \left|\sin\frac{\pi x}{2}\right|<1\\ 1; & \left|\sin\frac{\pi x}{2}\right|=1\end{cases}$

Thus, $f(x)$ is continuous for all x, except for those values of x for which $\left|\sin\frac{\pi x}{2}\right|=1$, i.e., x is an odd integer. Therefore,

$x=(2n+1)$ where $n\in I$

Check continuity at $x=(2n+1)$.

L.H.L $=\lim_{x\to2n+1} f(x)=0$ and $f(2n+1)=1$

Thus, L.H.L $\neq f(2n+1)$.

Therefore, $f(x)$ is discontinuous at $x=(2n+1)$ (i.e., at odd integers)

9. $f(x)=\begin{cases}x^2, & x \text{ is rational}\\ -x^2, & x \text{ is irrational}\end{cases}$

$f(x)$ is continuous when $x^2=-x^2$ or $x=0$.

10. $f(x)$ is continuous if $x^2=x+a$ or $x^2-x-a=0$.
For $f(x)$ to be discontinuous, for all real x, equation must have imaginary roots. Therefore,

$D<0$

$\therefore\quad 1+4a<0$

$\therefore\quad a<-\frac{1}{4}$

11. $t=\frac{1}{x-1}$ is discontinuous at $x=1$. Also, $y=\frac{1}{t^2+t-2}$ is discontinuous at $t=-2$ and $t=1$.

When $t=-2$, $\frac{1}{x-1}=-2$ or $x=\frac{1}{2}$. When $t=1$, $\frac{1}{x-1}=1$ or $x=2$.

So, $y=f(x)$ is discontinuous at three points: $x=1, \frac{1}{2}, 2$.

12. Here $h(x)=f(g(x))$ where $g(x)=\frac{x-1}{x-2}$ and $f(x)=\frac{1}{1-e^{-x}}$

Therefore, $f(g(x))$ is discontinuous when $1-e^{-x}=0$ or $x=0$ and when $x=2$, where $g(x)$ is discontinuous.

13. Clearly $f(x)$ is discontinuous at $x=2$.

$\therefore$ $f(f(x))$ will be discontinuous at $x=2$.

$f(f(x))$ is also discontinuous when $f(x)=2$.

If $1+x=2$, then $x=1$, which is possible as $0\le x\le2$.

If $3-x=2$, then $x=1$, which is not possible as $2<x\le3$.

So, $f(f(x))$ is discontinuous at $x=1$ and $x=2$.

Exercise 4.3

1. The function $f(x)=\frac{x^3}{4}-\sin\pi x+3$ is continuous within the interval $[-2, 2]$. Now, $f(-2)=1$ and $f(2)=5$. By intermediate value theorem, f takes all values between 1 and 5. Thus, f takes the value $\alpha=7/3$.

2. Consider a continuous function

$g(x)=f\left(x+\frac{1}{2}\right)-f(x)\quad\left(g \text{ is continuous } \forall\ x\in\left[0,\frac{1}{2}\right]\right)$

$g(0)=f\left(\frac{1}{2}\right)-f(0)=f\left(\frac{1}{2}\right)-f(1)\qquad[\text{As } f(0)=f(1)]$

and $g\left(\frac{1}{2}\right)=f(1)-f\left(\frac{1}{2}\right)=-\left[f\left(\frac{1}{2}\right)-f(1)\right]$

Since g is continuous and $g(0)$ and $g(1/2)$ have opposite signs, the equation $g(x)=0$ must have at least one root in $[0, 1/2]$.

Hence, for some $c\in\left[0,\frac{1}{2}\right]$, $g(c)=0$ implies $f\left(c+\frac{1}{2}\right)=f(c)$.

3. If $f(a)=a$, then $b=a$ solves the problem.
So, assume $f(a)>a$.
Then $g(x)=f(x)-x$ is positive at $x=a$ and is negative at $c=f(a)$ since $g(c)=f(f(a))-f(a)=a-f(a)<0$.
Since $g: R\to R$ is continuous, there must be some b, $a<b<c$, such that $g(x)=0$, i.e., $f(b)=b$.
The same argument works if $f(a)<a$.

Exercise 4.4

1. Let $f'(0^+)=\lim_{h\to0}\dfrac{f(h)-f(0)}{h}=k$ (say)

$\therefore\quad f'(0^-)=\lim_{h\to0}\dfrac{f(0-h)-f(0)}{-h}$

$=\lim_{h\to0}\dfrac{f(h)-f(0)}{-h}$

$=-k$

$\because\quad f'(0^+)\neq f'(0^-)$, but both are finite

So, $f(x)$ is continuous at $x=0$ but not differentiable at $x=0$.

2. $f(x+y)=f(x)+f(y)$ and $f(x)=xg(x)$ for all $x, y \in R$, where $g(x)$ is continuous.

We have $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$=\lim_{h\to 0}\frac{f(x)+f(h)-f(x)}{h}$

$=\lim_{h\to 0}\frac{f(h)}{h}$

$=\lim_{h\to 0}\frac{hg(h)}{h}=\lim_{h\to 0}g(h)=g(0).$

$[\because g$ is continuous at $x=0]$

3. $f'(k^+)=\lim_{h\to 0}\frac{f(k+h)-f(k)}{h}$

$=\lim_{h\to 0}\frac{[2k+2h]\sin(3\pi(k+h))-[2k]\sin(3k\pi)}{h}$

$=\lim_{h\to 0}\frac{[2k]\sin(3k\pi+3\pi h)}{h}$ (as $\sin(3k\pi)=0$)

$=2k\times\lim_{h\to 0}\frac{(-1)^k\sin(3\pi h)}{h}$ (as 2k is integer)

$=2k\times(-1)^k\times(3\pi)\lim_{h\to 0}\frac{\sin(3\pi h)}{(3\pi h)}$

$=6k\pi(-1)^k$

4. $g(x)$ is an even function.

$\therefore \quad g(x)=g(-x)$

$\Rightarrow \quad g'(x)=-g'(-x)$

$\Rightarrow \quad g'(0)=-g'(0)$ (as $g(x)$ is differentiable at $x=0$)

$\Rightarrow \quad g'(0)=0$

$f'(0)=\lim_{h\to 0}\frac{g(0+h)\cos(1/h)-0}{h}$

$=\lim_{h\to 0}\frac{g(h)\cos(1/h)}{h}$

$=\lim_{h\to 0}g'(0)\cos(1/h)$

$=0$

5. $f(x)=\begin{cases}(\sin^{-1}x)^2.\cos\frac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$

$\lim_{x\to 0}f(x)=\lim_{x\to 0}(\sin^{-1}x)^2.\cos\left(\frac{1}{x}\right)$

$=0\times$ (any value between -1 to 1)

$=0=f(0)$

Hence $f(x)$ is continuous at $x=0$.

$f'(0^+)=\lim_{h\to 0}\frac{(\sin^{-1}h)^2.\cos\frac{1}{h}-0}{h}$

$=\left(\lim_{h\to 0}\frac{\sin^{-1}h}{h}\right).\left(\lim_{h\to 0}\sin^{-1}h\right).\left(\lim_{h\to 0}\cos\left(\frac{1}{h}\right)\right)$

$=1\times(0)\times$ (any value b/w -1 to 1)

$=0$

Similarly $f'(0^-)=0$

Hence $f(x)$ is differentiable at $x=0$.

6. For continuity at $x=0$,

R.H.L. $f(0^+)=\lim_{h\to 0}\sqrt{h}.\left(1+h.\sin\frac{1}{h}\right)$

$=0\times[1+0\times$ (any value between -1 and 1)$]$

$=0$

L.H.L. $f(0^-)=\lim_{h\to 0}\left[-\sqrt{-(-h)}\left(1-h\sin\left(-\frac{1}{h}\right)\right)\right]$

$=\lim_{h\to 0}\left[-\sqrt{h}\left(1+h\sin\frac{1}{h}\right)\right]$

$=-0\times[1+0\times$ (any value between -1 and 1)$]$

$=0$

So, $f(x)$ is continuous at $x=0$.

For differentiability,

$f'(0^+)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}$

$=\lim_{h\to 0}\frac{\sqrt{h}\left[1+h.\sin\frac{1}{h}\right]-0}{h}$

$=\lim_{h\to 0}\frac{1+h.\sin\frac{1}{h}}{\sqrt{h}}$

$=\lim_{h\to 0}\left[\frac{1}{\sqrt{h}}+\sqrt{h}\sin\frac{1}{h}\right]$

$=\infty$

For differentiability derivative must be finite.

Hence $f(x)$ is non-differentiable at $x=0$.

Exercise 4.5

1. $f(x)=|x+1|+|x|+|x-1|$

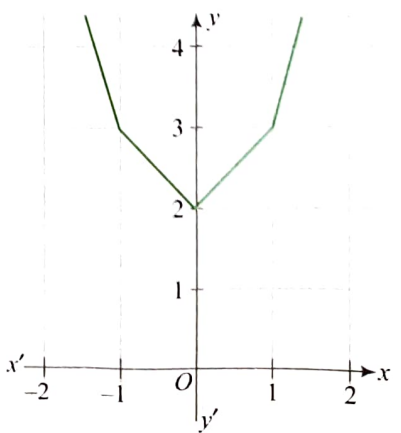
$|x+1|, |x|, |x-1|$ are continuous for all x, but non-differentiable at $x=-1, 0, 1$, respectively.

Hence, $f(x)$ is non-differentiable at $x=-1, 0, 1$.

$$f(x)=\begin{cases}(-x-1)+(-x)+(1-x), & x<-1\\ (x+1)+(-x)+(1-x), & -1\le x<0\\ (x+1)+(x)+(1-x), & 0\le x<1\\ (x+1)+(x)+(x-1) & x\ge 1\end{cases}$$

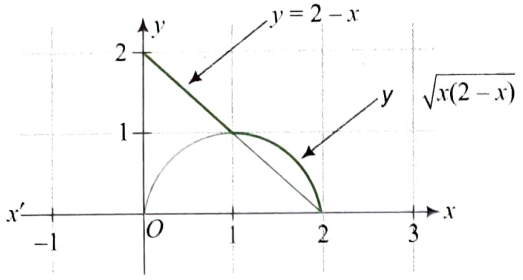
$$=\begin{cases}-3x, & x<-1\\ -x+2 & -1\le x<0\\ x+2, & 0\le x<1\\ 3x, & x\ge 1\end{cases}$$

Therefore, graph of $f(x)$ is given in the figure.



It is clear from the graph that $f(x)$ is continuous $\forall\, x \in R$ but not differentiable at $x = -1, 0, 1$.

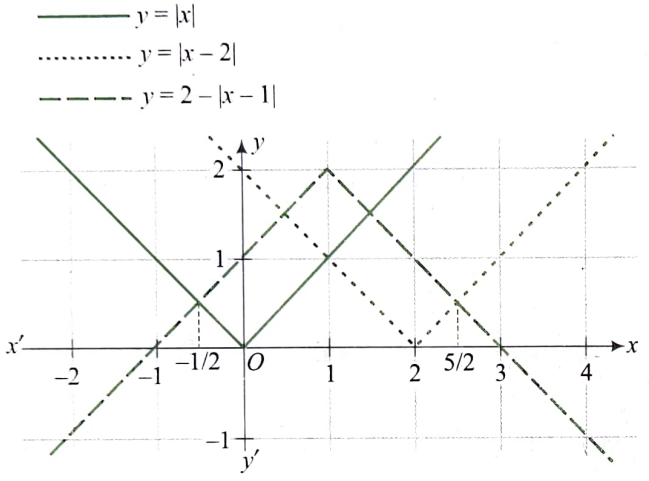
2. Domain of $f(x)$ is $[0, 2]$.



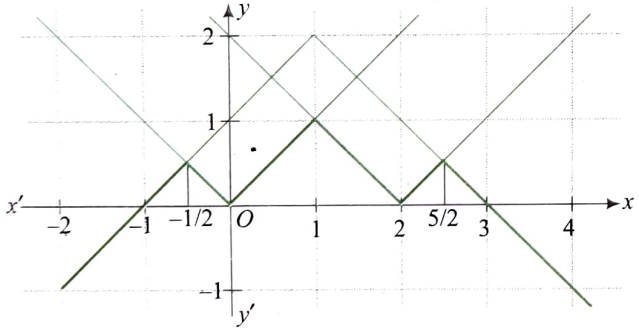
Clearly, from the graph given in the figure, $f(x)$ is non-differentiable at $x = 1$.

3. $f(x) = \min\{|x|, |x-2|, 2-|x-1|\}$

Draw the graphs of $y = |x|$, $y = |x-2|$, and $y = 2-|x-1|$.



Graph of $y = f(x)$



It is clear from the graph, $f(x) = \min\{|x|, |x-2|, 2-|x-1|\}$ is continuous $\forall\, x \in R$ and non-differentiable at $x = -\frac{1}{2}, 0, 1, 2, \frac{5}{2}$.

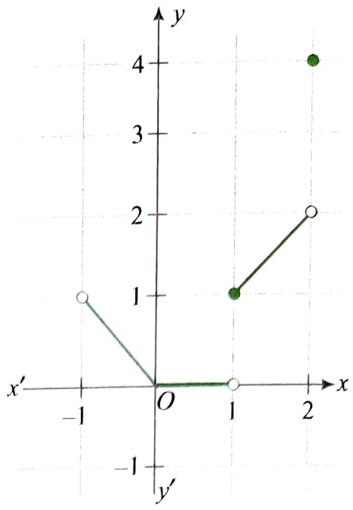
4. Since x and $|x - x^2|$ are continuous for all x, $f(x) = x - |x - x^2|$ is continuous for all x.

Also, x is differentiable but $|x - x^2|$ is non-differentiable at $x = 0$ and 1. Hence, $f(x)$ is non-differentiable at $x = 0$ and 1.

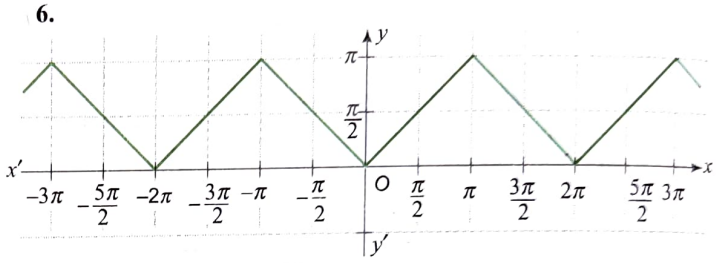
5. We have $f(x) = |[x]x|$ in $-1 < x \le 2$

$$\therefore \quad f(x) = \begin{cases} -x, & -1 < x < 0 \\ 0, & 0 \le x < 1 \\ x, & 1 \le x < 2 \\ 2x, & x = 2 \end{cases}$$

It is evident from the graph given in the figure for this function that it is continuous but not differentiable at $x = 0$. Also, it is discontinuous at $x = 1$ and $x = 2$ and, hence, non-differentiable at $x = 1$ and $x = 2$.

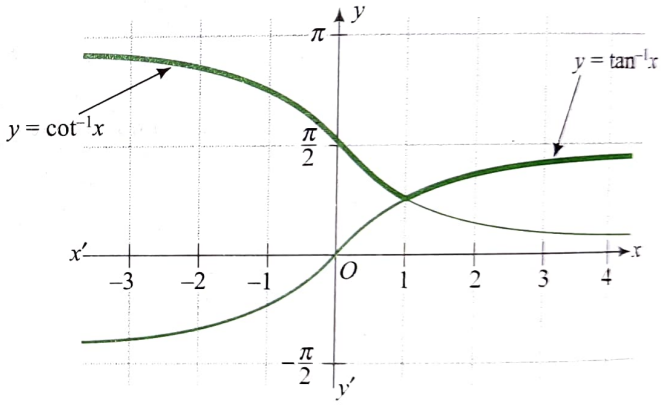


6.



Clearly, from the graph given in the figure, $f(x)$ is non-differentiable at $x = n\pi$, $n \in Z$.

7.



Clearly, from the graph given in the figure, $f(x)$ is non-differentiable at $x = 1$.

8. $f(x) = \begin{cases} ax^2 + 1, & x \le 1 \\ x^2 + ax + b, & x > 1 \end{cases}$

or $\quad f'(x) = \begin{cases} 2ax, & x \le 1 \\ 2x + a, & x > 1 \end{cases}$

$f(x)$ is differentiable at $x = 1$.

Then we must have

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) \text{ and } \lim_{x \to 1^+} f'(x) = \lim_{x \to 1^-} f'(x)$$

or $\quad a + 1 = 1 + a + b$ and $2a = 2 + a$

or $\quad a = 2$ and $b = 0$

9. $f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x, & \text{if } 0 \le x < \infty \\ -2\tan^{-1}x, & \text{if } -\infty < x \le 0 \end{cases}$

or $\quad f'(x) = \begin{cases} \frac{2}{1+x^2}, & \text{if } 0 < x < \infty \\ -\frac{2}{1+x^2}, & \text{if } -\infty < x < 0 \end{cases}$

or $f'(0^+) = 1$ and $f'(0^-) = -1$

Hence, $f(x)$ is continuous and non-differentiable at $x = 0$.

Using the shortcut method, differentiate $f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ w.r.t. x. Then

$$f'(x) = -\frac{1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}}\left(\frac{-2x(1+x^2)-2x(1-x^2)}{(1+x^2)^2}\right)$$

$$= \frac{4x}{\sqrt{(1+x^2)^2-(1-x^2)^2}}\frac{1}{1+x^2}$$

$$= \frac{4x}{\sqrt{4x^2}}\frac{1}{1+x^2}$$

$$= \frac{2x}{|x|}\frac{1}{1+x^2}$$

which is discontinuous at $x = 0$.

Hence, $f(x)$ is non-differentiable at $x = 0$.

10. $f(x) = ||x^2 - 4| - 12|$ is non-differentiable when

$x^2 - 4 = 0$ and $|x^2 - 4| - 12 = 0$

or $x = \pm 2$ and $x^2 - 4 = \pm 12$ or $x = \pm 2$ and $x^2 = 16$

or $x = \pm 2$ and $x = \pm 4$

Hence, there are four points of non-differentiability.

11. $f(x) = x[x]\{x\}$

$= x[x](x - [x])$

$$= \begin{cases} -x(x+1), & -1 \le x < 0 \\ 0, & 0 \le x \le 1 \\ x(x-1), & 1 < x < 2 \\ 0, & x = 2 \end{cases}$$

Clearly, $f(x)$ is discontinuous at $x = 2$ and hence, not differentiable at $x = 2$.

Now $f'(x) = \begin{cases} -2x-1, & -1 < x < 0 \\ 0, & 0 < x < 1 \\ 2x-1, & 1 < x < 2 \end{cases}$

$f'(0^-) = -1$ and $f'(0^+) = 0$.

So, $f(x)$ is not differentiable at $x = 0$.

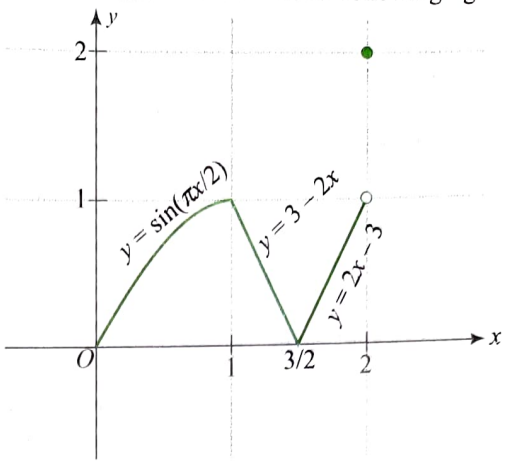
Also, $f'(1^-) = 0$ and $f'(1^+) = 1$

So, $f(x)$ is not differentiable at $x = 1$

12. We have $f(x) = \begin{cases} \sin\left(\frac{\pi x}{2}\right), & 0 \le x < 1 \\ (3-2x)[x], & 1 \le x < 3/2 \\ (2x-3)[x], & \frac{3}{2} \le x \le 2 \end{cases}$

$$= \begin{cases} \sin\left(\frac{\pi x}{2}\right), & 0 \le x < 1 \\ 3-2x, & 1 \le x < 3/2 \\ (2x-3), & \frac{3}{2} \le x < 2 \\ 2, & x = 2 \end{cases}$$

Graph of the function is as shown in the following figure.



Clearly $f(x)$ is discontinuous at $x = 2$ and non-differentiable at $x = 1, 3/2$ and 2.

13. $g(x) = |(x-1)(x-2)(x-3)|$ is non-differentiable at $x = 1, 2, 3$.

But for $h(x) = \sin \pi x$, $h(1) = h(2) = h(3) = 0$

So, $f(x)$ is differentiable at $x = 1, 2, 3$ and hence for all real x.

14. $f(x) = |x \operatorname{sgn}(1-x^2)|$

$= |x||\operatorname{sgn}(1-x^2)|$

Clearly $|\operatorname{sgn}(1-x^2)|$ is discontinuous when $1 - x^2 = 0$ or $x = \pm 1$, where $|x| = 1$

$\therefore$ $f(x)$ is discontinuous and hence non-differentiable at $x = \pm 1$

Also $|x|$ is continuous but non-differentiable at $x = 0$, where $\operatorname{sgn}(1-x^2) = 1$

Hence $f(x)$ is non-differentiable at $x = 0$

Thus $f(x)$ is non-differentiable at $x = 0, \pm 1$.

15. $\lim_{x\to 0}\left(\cos\frac{1}{x}\right)(\log(1+x))^2$

= (any value between -1 and 1) $\times 0$

$= 0$

$\therefore$ $f(x)$ is continuous at $x = 0$.

$$f'(0^+) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\cos\left(\frac{1}{h}\right)\cdot\frac{\log_e(1+h)}{h}\cdot\log_e(1+h)$$

= (any value between -1 and 1) $\times 1 \times 0$

$= 0$

Similarly, $f'(0^-) = 0$.

$\therefore$ $f(x)$ is differentiable at $x = 0$.

Now

$$f'(x) = \begin{cases} \frac{1}{x^2}\sin\left(\frac{1}{x}\right)(\log_e(1+x))^2 + \cos\left(\frac{1}{x}\right)\cdot\frac{2\log_e(1+x)}{1+x}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

$$\lim_{x\to 0}\left[\frac{1}{x^2}\sin\left(\frac{1}{x}\right)(\log_e(1+x))^2 + \cos\left(\frac{1}{x}\right)\cdot\frac{2\log_e(1+x)}{1+x}\right]$$

$$= \lim_{x\to 0}\left[\sin\left(\frac{1}{x}\right)\left(\frac{\log_e(1+x)}{x^2}\right)+0\right]$$

= (any value between -1 and 1) $\times 1$

$\therefore$ $\lim_{x\to 0} f'(x)$ does not exist.

Thus $f'(x)$ is not continuous at $x = 0$, i.e. derivative is not continuous at $x = 0$.

Exercises

Single Correct Answer Type

1. (3) $f(x) = \tan x$ is discontinuous when $x = (2n + 1)\pi/2$, $n \in Z$

$f(x) = x[x]$ is discontinuous when $x = k$, $k \in Z$

$f(x) = \sin[\pi x]$ is discontinuous when $\pi x = k$, $k \in Z$

Thus, all the above functions have infinite number of points of discontinuity.

But $f(x) = \dfrac{|x|}{x}$ is discontinuous when $x = 0$ only.

2. (3) We have $f(x) = \dfrac{4 - x^2}{x(4 - x^2)}$

Clearly, there are three points of discontinuity, viz., 0, 2, –2.

3. (2) $f(x) = \dfrac{\tan\left(\frac{\pi}{4} - x\right)}{\cot 2x}$, $(x \neq \pi/4)$, is continuous at $x = \pi/4$.

Therefore,

$$f\left(\frac{\pi}{4}\right) = \lim_{x \to \frac{\pi}{4}} f(x)$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\tan\left(\frac{\pi}{4} - x\right)}{\cot 2x}$$

Now, by applying L'Hopital's rule,

$$\lim_{x \to \frac{\pi}{4}} \frac{-\sec^2\left(\frac{\pi}{4} - x\right)}{-2\operatorname{cosec}^2(2x)} = \frac{1}{2}$$

4. (2) The function f is clearly continuous at each point in its domain except possibly at $x = 0$. Given that $f(x)$ is continuous at $x = 0$.

$$\therefore \quad f(0) = \lim_{x \to 0} f(x)$$

$$= \lim_{x \to 0} \frac{2x - \sin^{-1} x}{2x + \tan^{-1} x}$$

$$= \lim_{x \to 0} \frac{2 - (\sin^{-1} x)/x}{2 + (\tan^{-1} x)/x} = \frac{1}{3}$$

5. (2) Given $f(x)$ is continuous at $x = 0$. Therefore,

$$\lim_{x \to 0} f(x) = f(0)$$

or $$\lim_{x \to 0} \frac{(3^x - 1)^2}{\sin x \ln(1 + x)} = f(0)$$

or $$f(0) = \lim_{x \to 0} \frac{\left(\frac{3^x - 1}{x}\right)^2}{\left(\frac{\sin x}{x}\right)\left(\frac{\ln(1 + x)}{x}\right)} = (\ln 3)^2$$

6. (4) $$\lim_{x \to 0} \frac{x - e^x + 1 - (1 - \cos 2x)}{x^2}$$

$$= \lim_{x \to 0} \left[\frac{x - e^x + 1}{x^2} - \frac{(1 - \cos 2x)}{x^2}\right]$$

$$= \lim_{x \to 0} \left[\frac{x + 1 - \left(1 + x + \frac{x^2}{2}\right)}{x^2} - \frac{2\sin^2 x}{x^2}\right]$$

(Using expansion of e^x)

$$= -\frac{1}{2} - 2$$

$$= -\frac{5}{2}$$

Hence, for continuity, $f(0) = -\dfrac{5}{2}$.

Now, $[f(0)] = -3$; $\{f(0)\} = \left\{-\dfrac{5}{2}\right\} = \dfrac{1}{2}$.

Hence, $[f(0)]\{f(0)\} = -\dfrac{3}{2} = -1.5$.

7. (3) $f(0) = 0 + 0 + \lambda \ln 4 = \lambda \ln 4$ (1)

$$\text{R.H.L.} = \lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0 + h)$$

$$= \lim_{h \to 0} \frac{8^h - 4^h - 2^h + 1}{h^2}$$

$$= \lim_{h \to 0} \frac{(4^h - 1)(2^h - 1)}{h.h}$$

$$= \lim_{h \to 0}\left(\frac{4^h - 1}{h}\right)\lim_{h \to 0}\left(\frac{2^h - 1}{h}\right)$$

$$= \ln 4 \ln 2 \quad (2)$$

$\therefore$ $f(0) = \text{R.H.L.}$

$\therefore$ $\lambda = \ln 2$

8. (2) We must have $\lim_{x \to 0} \dfrac{a\cos x - \cos bx}{x^2} = 4$

or $$\lim_{x \to 0} \frac{a\left(1 - \frac{x^2}{2!}\right) - \left(1 - \frac{b^2 x^2}{2!}\right)}{x^2} = 4$$

or $$\lim_{x \to 0}\left[\frac{(a - 1)}{x^2} - \left(\frac{a}{2} - \frac{b^2}{2}\right)\right] = 4$$

or $a = 1$ and $\dfrac{a}{2} - \dfrac{b^2}{2} = -4$

or $a = 1$ and $b^2 = 9$

or $a = 1$ and $b = \pm 3$

9. (1) As f is continuous,

$$f(0) = \lim_{x \to 0} f(x)$$

$$= \lim_{n \to \infty} f(1/4n)$$

$$= \lim_{n \to \infty}\left((\sin e^n)\, e^{-n^2} + \frac{1}{1 + 1/n^2}\right)$$

$$= 0 + 1 = 1$$

10. (1) $f(x) = \dfrac{x^2 - bx + 25}{x^2 - 7x + 10}$, $x \neq 5$

$f(x)$ is continuous at $x = 5$ only if $\lim_{x \to 5} \dfrac{x^2 - bx + 25}{x^2 - 7x + 10}$ is finite.

Now, $x^2 - 7x + 10 \to 0$ when $x \to 5$.

Then we must have $x^2 - bx + 25 \to 0$ for which $b = 10$.

Hence, $\lim_{x \to 5} \dfrac{x^2 - 10x + 25}{x^2 - 7x + 10} = \lim_{x \to 5} \dfrac{x - 5}{x - 2} = 0$.

11. (4) $f(x)=\dfrac{\tan\left(\pi\left[x-\pi\right]\right)}{1+[x]^2}$

By definition, $[x-\pi]$ is an integer whatever be the value of x and so $\pi[x-\pi]$ is an integral multiple of π.

Consequently, $\tan(\pi[x-\pi])=0\ \forall\ x$.

Also, since $1+[x]^2\neq 0$ for any x, we conclude that $f(x)=0$.

Thus, $f(x)$ is constant function and, so, it is continuous and differentiable.

12. (2) We have $f(x)=\begin{cases}\dfrac{1-|x|}{1+x}, & x\neq -1\\ 1, & x=-1\end{cases}$

$$=\begin{cases}1, & x<0,\\ \dfrac{1-x}{1+x}, & x\geq 0\end{cases}\qquad [\because f(-1)=1 \text{ is given}]$$

$$\Rightarrow\quad f([2x])=\begin{cases}1, & [2x]<0\\ \dfrac{1-[2x]}{1+[2x]}, & [2x]\geq 0\end{cases}$$

$$=\begin{cases}1, & x<0\\ 1, & 0\leq x<1/2\\ 0, & 1/2\leq x<1\\ -1/3, & 1\leq x<\dfrac{3}{2}\end{cases}$$

Clearly, $f(x)$ is continuous for all $x<\dfrac{1}{2}$ and discontinuous at $x=\dfrac{1}{2}, 1$.

13. (4) We have

$$\text{L.H.L.}=\lim_{x\to 4^-}f(x)$$

$$=\lim_{h\to 0}f(4-h)$$

$$=\lim_{h\to 0}\frac{4-h-4}{|4-h-4|}+a$$

$$=\lim_{h\to 0}\left(-\frac{h}{h}+a\right)=a-1$$

$$\text{R.H.L.}=\lim_{x\to 4^+}f(x)$$

$$=\lim_{h\to 0}f(4+h)$$

$$=\lim_{h\to 0}\frac{4+h-4}{|4+h-4|}+b=b+1$$

Also, $f(4)=a+b$

Since $f(x)$ is continuous at $x=4$,

$$\lim_{x\to 4^-}f(x)=f(4)=\lim_{x\to 4^+}f(x)$$

or $\quad a-1=a+b=b+1$ or $b=-1$ and $a=1$

14. (3) $\displaystyle\lim_{x\to 2^+}\frac{(x-2)}{|x-2|}\left(\frac{x^2-1}{x^2+1}\right)=\lim_{x\to 2^+}\frac{(x-2)}{(x-2)}\left(\frac{x^2-1}{x^2+1}\right)$

$$=\lim_{x\to 2^+}\left(\frac{x^2-1}{x^2+1}\right)=\frac{3}{5}$$

$$=\lim_{x\to 2^-}\frac{(x-2)}{|x-2|}\left(\frac{x^2-1}{x^2+1}\right)$$

$$=\lim_{x\to 2^-}\frac{(x-2)}{(2-x)}\left(\frac{x^2-1}{x^2+1}\right)=-\frac{3}{5}$$

Thus, L.H.L. $\neq$ R.H.L.

Hence, the function has non-removable discontinuity at $x=2$.

15. (1) $f(x)=\begin{cases}x+2, & x<0\\ -x^2-2, & 0\leq x<1\\ x, & x\geq 1\end{cases}$

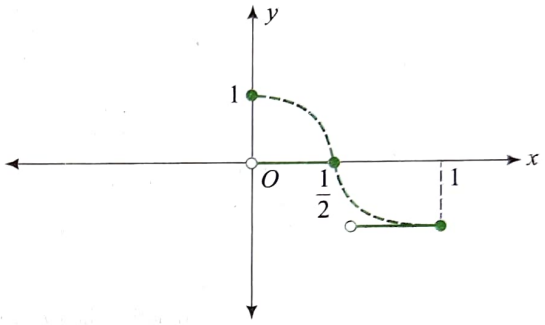
$$\therefore\quad |f(x)|=\begin{cases}-x-2, & x<-2\\ x+2, & -2\leq x<0\\ x^2+2, & 0\leq x<1\\ x, & x\geq 1\end{cases}$$

It is discontinuous at $x=1$. Therefore, number of discontinuity is 1.

16. (4) Refer theory.

17. (2) Consider $x\in[0,1]$.

From the graph given, it is clear that $[\cos\pi x]$ is discontinuous at $x=0,\ 1/2$ (1)



Now, consider $x\in(1,2]$.

$f(x)=[x-2]\,|2x-3|$

For $x\in(1,2)$, $[x-2]=-1$, and for $x=2$, $[x-2]=0$.

Also, $|2x-3|=0$ or $x=3/2$.

Therefore, $x=3/2$ and 2 may be the points at which $f(x)$ is discontinuous.

$$f(x)=\begin{cases}1, & x=0\\ 0, & 0<x\leq\dfrac{1}{2}\\ -1, & \dfrac{1}{2}<x\leq 1\\ -(3-2x), & 1<x\leq 3/2\\ -(2x-3), & 3/2<x\leq 2\\ 0, & x=2\end{cases}$$

Thus, $f(x)$ is continuous when $x\in[0,2]-\{0,1/2,2\}$.

18. (4) For $0\leq x<1$, $f(x)=[\sin 0]=0$,
$1\leq x<2$, $f(x)=[\sin 1]=0$,
$2\leq x<3$, $f(x)=[\sin 2]=0$,
$3\leq x<4$, $f(x)=[\sin 3]=0$,
$4\leq x<5$, $f(x)=[\sin 4]=-1$

Hence, there is discontinuity at point $(4,-1)$

19. (3) $f(x)=\{x\}\sin(\pi[x])$

$=\{x\}\sin(\text{integral multiple of }\pi)$

$=0$

Hence, $f(x)$ is continuous for all x.

20. (4) We have $\displaystyle\lim_{x\to 1}f(x)=\lim_{h\to 0}f(1-h)$

$$=\lim_{h\to 0}\frac{\log(4+h^2)}{\log(1-4h)}=-\infty$$

and $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} \frac{\log(4+h^2)}{\log(1+4h)} = \infty$

So, $f(1^-)$ and $f(1^+)$ do not exist.

21. (4) $f(x) = [x^2] - \{x\}^2$

$f(-1) = 1, f(-1^-) = 1 - 1 = 0$

$f(1) = 1, f(1^+) = 1 - 0 = 1$

$f(1^-) = 0 - 1 = -1$

Thus, $f(x)$ is discontinuous at $x = 1, -1$.

22. (2) $f(2) = 0$

$f(2^+) = \{4^+\} - \{2^+\}^2 = 0 - 0 = 0$

$f(2^-) = \{4^-\} - \{2^-\}^2 = 1 - 1 = 0$

Hence, $f(x)$ is continuous at $x = 2$.

$f(-2) = 0$

$f(-2^+) = \{4^-\} - \{-2^+\}^2 = 1 - 0 = 1$

Hence, $f(x)$ is discontinuous at $x = -2$.

23. (1) $\lim_{x\to 0+}\left(3-\left[\cot^{-1}\frac{2x^3-3}{x^2}\right]\right) = (3-[\cot^{-1}(-\infty)]) = (3-[\pi]) = 0$

$$\lim_{x\to 0-}\{x^2\}\cos(e^{1/x}) = \left(\lim_{x\to 0-}\{x^2\}\right)\left(\lim_{x\to 0-}\cos(e^{1/x})\right)$$
$$= (0)\,(\cos(e^{-\infty}))$$
$$= 0$$

Thus, $f(0) = 0$.

24. (2) $f(x)$ is discontinuous at $x = 1$ and $x = 2$.

Therefore, $f(f(x))$ may be discontinuous when $f(x) = 1$ or 2.

Now,

$1 - x = 1 \Rightarrow x = 0$, where $f(x)$ is continuous

$x + 2 = 1 \Rightarrow x = -1 \notin (1, 2)$

$4 - x = 1 \Rightarrow x = 3 \in [2, 4]$

Now, $1 - x = 2 \Rightarrow x = -1 \notin [0, 1]$

$x + 2 = 2 \Rightarrow x = 0 \notin (1, 2]$

$4 - x = 2 \Rightarrow x = 2 \in [2, 4]$

Hence, $f(f(x))$ is discontinuous at $x = 2, 3$.

25. (1) $f(x) = (-1)^{[x^3]}$ is discontinuous when $x^3 = n, n \in Z$, or $x = n^{1/3}$.

$$f\left(\frac{3}{2}\right) = (-1)^3 = -1$$

For $x \in (-1, 0)$, $f(x) = (-1)^{-1} = -1$

$\therefore \quad f'(x) = 0$

For $x \in [0, 1)$, $f(x) = (-1)^0 = 1$

$\therefore \quad f'(x) = 0$

26. (3) $f(x) = \lim_{n\to\infty}\frac{\left[(x-1)^2\right]^n - 1}{\left[(x-1)^2\right]^n + 1} = \lim_{n\to\infty}\frac{1-\frac{1}{\left[(x-1)^2\right]^n}}{1+\frac{1}{\left[(x-1)^2\right]^n}}$

$$= \begin{cases} -1, & 0 \le (x-1)^2 < 1 \\ 0, & (x-1)^2 = 1 \\ 1, & (x-1)^2 > 1 \end{cases}$$

$$= \begin{cases} 1, & x < 0 \\ 0, & x = 0 \\ -1, & 0 < x < 2 \\ 0, & x = 2 \\ 1, & x > 2 \end{cases}$$

Thus, $f(x)$ is discontinuous at $x = 0, 2$.

27. (1) $f(x)$ is continuous when $5x = x^2 + 6$ or $x = 2, 3$.

28. (4) Since $\lim_{n\to\infty} x^{2n} = \begin{cases} 0, & \text{if } |x| < 1 \\ 1, & \text{if } |x| = 1 \end{cases}$,

$$f(x) = \lim_{x\to\infty}(\sin x)^{2n} = \begin{cases} 0, & \text{if } |\sin x| < 1 \\ 1, & \text{if } |\sin x| = 1 \end{cases}$$

Thus, $f(x)$ is continuous at all x except for those values of x for which $|\sin x| = 1$, i.e., $x = (2k+1)\frac{\pi}{2}, k \in Z$.

29. (1) $f(x)$ is continuous at some x where $\sin x = \cos x$ or $\tan x = 1$ or $x = n\pi + \pi/4, n \in I$.

30. (3) For $|x| < 1$, $x^{2n} \to 0$ as $n \to \infty$, and for $|x| > 1$, $1/x^{2n} \to 0$ as $n \to \infty$. So,

$$f(x) = \begin{cases} \log(2+x), & |x|<1 \\ \lim_{n\to\infty}\dfrac{x^{-2n}\log(2+x) - \sin x}{x^{-2n}+1} = -\sin x, & |x|>1 \\ \dfrac{1}{2}[\log(2+x) - \sin x], & |x|=1 \end{cases}$$

Thus, $\lim_{x\to 1+} f(x) = \lim_{x\to 1}(-\sin x) = -\sin 1$

and $\lim_{x\to 1-} f(x) = \lim_{x\to 1}\log(2+x) = \log 3$.

31. (1) $f(x) = \lim_{n\to\infty}\left(\sin^2(\pi x)\right)^n + \left[x + \frac{1}{2}\right]$

Now, $g(x) = \lim_{n\to\infty}\left(\sin^2(\pi x)\right)^n$ is discontinuous when

$$\sin^2(\pi x) = 1 \text{ or } \pi x = (2n+1)\frac{\pi}{2} \text{ or } x = \frac{(2n+1)}{2}, n \in Z$$

Thus, $g(x)$ is discontinuous at $x = 3/2$.

Also, $h(x) = \left[x + \frac{1}{2}\right]$ is discontinuous at $x = 3/2$.

But $f(3/2) = \lim_{n\to\infty}\left(\sin^2(3\pi/2)\right)^n + \left[\frac{3}{2} + \frac{1}{2}\right] = 1 + 2 = 3$

$$f(3/2^+) = \lim_{n\to\infty}\left(\sin^2((3\pi/2)^+)\right)^n + \left[\left(\frac{3}{2}\right)^+ + \frac{1}{2}\right]$$
$$= 0 + 2 = 2$$

Hence, $f(x)$ is discontinuous at $x = 3/2$.

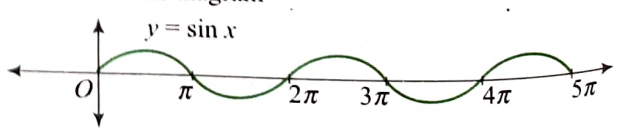
Both $g(x)$ and $h(x)$ are continuous at $x = 1$. Hence, $f(x)$ is continuous at $x = 1$.

32. (3) $f(x) = \text{sgn}(\sin^2 x - \sin x - 1)$ is discontinuous when

$\sin^2 x - \sin x - 1 = 0$

or $\sin x = \frac{1 \pm \sqrt{5}}{2}$ or $\sin x = \frac{1-\sqrt{5}}{2}$.

For exactly four point of discontinuity, n can take value 4 or 5 as shown in the diagram



33. (3) $f(x) = \begin{cases} x^2 - ax + 3, & x \text{ is rational} \\ 2 - x, & x \text{ is irrational} \end{cases}$

It is continuous when

$x^2 - ax + 3 = 2 - x$ or $x^2 - (a-1)x + 1 = 0$

which must have two distinct roots for
$(a-1)^2 - 4 > 0$
or $(a-1-2)(a-1+2) > 0$
or $a \in (-\infty, -1) \cup (3, \infty)$

34. (1) Domain of the given function is $R \sim [-1, 0)$.
Possible points of discontinuity of the function are $x =$ Integer $\sim \{-1\}$.
$f(0) = 0, f(0+0) = 0$.
This means that $f(x)$ is continuous at $x = 0$.
Let $x = I_0$, where $I_0 \neq -1, 0$.

Then $f(I_0) = I_0 \cdot \sin\dfrac{\pi}{(I_0+1)}$,

$$f(I_0 - 0) = (I_0 - 1)\cdot \sin\frac{\pi}{I_0}$$

$$f(I_0 + 0) = I_0 \cdot \sin\frac{\pi}{(I_0+1)}$$

Thus, $f(x)$ is discontinuous at $x = I_0$.

35. (3) Given that $\cos\dfrac{x}{2}\cos\dfrac{x}{2^2}\cos\dfrac{x}{2^3}\cdots\cos\dfrac{x}{2^n} = \dfrac{\sin x}{2^n \sin\left(\dfrac{x}{2^n}\right)}$ (1)

Taking logarithm to the base e on both sides of equation (1) and then differentiating w.r.t. x, we get

$$\sum_{n=1}^{n}\frac{1}{2^n}\tan\frac{x}{2^n} = \left(\frac{1}{2^n}\cot\frac{x}{2^n} - \cot x\right)$$

$$\therefore \lim_{n\to\infty}\sum_{n=1}^{n}\frac{1}{2^n}\tan\frac{x}{2^n} = \lim_{n\to\infty}\left(\frac{1}{x}\times\frac{\dfrac{x}{2^n}}{\tan\dfrac{x}{2^n}} - \cot x\right) = \left(\frac{1}{x} - \cot x\right)$$

Thus, we have $f(x) = \begin{cases} \dfrac{1}{x} - \cot x, & x \in (0,\pi) - \left\{\dfrac{\pi}{2}\right\} \\ \dfrac{2}{\pi}, & x = \dfrac{\pi}{2} \end{cases}$

Clearly, $\lim\limits_{x\to\frac{\pi}{2}} f(x) = \lim\limits_{x\to\frac{\pi}{2}}\left(\dfrac{1}{x} - \cot x\right) = \dfrac{2}{\pi} = f\left(\dfrac{\pi}{2}\right)$

Hence, $f(x)$ is continuous at $x = \dfrac{\pi}{2}$.

36. (3) $f(0+0) = \lim\limits_{h\to 0} f(h) = \lim\limits_{h\to 0}\dfrac{h}{2h^2+h} = \lim\limits_{h\to 0}\dfrac{1}{2h+1} = 1$

and $f(0-0) = \lim\limits_{h\to 0} f(-h) = \lim\limits_{h\to 0}\dfrac{-h}{2h^2+|-h|}$

$= \lim\limits_{h\to 0}\dfrac{-h}{2h^2+h}$

$= \lim\limits_{h\to 0}\dfrac{-1}{2h+1} = -1$

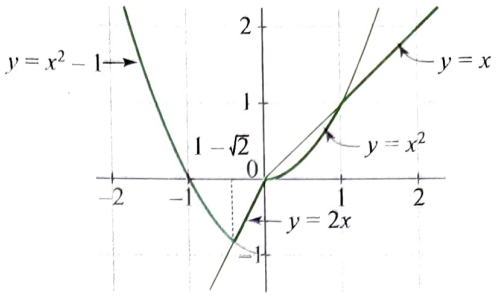
37. (2) We have

$$f(x) = \frac{x - |x-1|}{x} = \begin{cases} \dfrac{x+x-1}{x}, & x < 1, x \neq 0 \\ \dfrac{x-(x-1)}{x}, & x \geq 1 \end{cases}$$

$$= \begin{cases} \dfrac{2x-1}{x}, & x < 1, x \neq 0 \\ \dfrac{1}{x}, & x \geq 1 \end{cases}$$

Clearly, $f(x)$ is discontinuous at $x = 0$ as it is not defined at $x = 0$. Since $f(x)$ is not defined at $x = 0$, $f(x)$ cannot be differentiable at $x = 0$. Clearly, $f(x)$ is continuous at $x = 1$, but it is not differentiable at $x = 1$, because $Lf'(1) = 1$ and $Rf'(1) = -1$.

38. (4)



From the graph, it is clear that $f(x)$ is continuous everywhere but not differentiable at $x = 1 - \sqrt{2}, 0, 1$.

39. (2) Since both $\cos x$ and $\sin^{-1}x$ are continuous functions,
$f(x) = \sin^{-1}(\cos x)$ is also a continuous function.
Now,

$$f'(x) = \frac{-\sin x}{\sqrt{1-\cos^2 x}} = \frac{-\sin x}{|\sin x|}$$

Hence, $f(x)$ is non-differentiable at $x = n\pi$, $n \in Z$.

40. (2) $0 \leq \tan^2 x < 1$ when $-\dfrac{\pi}{4} < x < \dfrac{\pi}{4}$

or $f(x) = 0$ $-\dfrac{\pi}{4} < x < \dfrac{\pi}{4}$

Hence, $f(x)$ is continuous and differentiable at $x = 0$. Also, $f'(0) = 0$.

41. (4) $f(2^+) = 2 + 2\sin(0) = 2$
$f(2^-) = 3 + 2\sin 1$
Hence $f(x)$ is discontinuous at $x = 2$
Also $f(0^+) = 2(0) - 0 - 0\sin(0-0) = 0$
and $f(0^-) = 2(0) - (-1) - 0\sin(0-(-1)) = 1$
Hence $f(x)$ is discontinuous at $x = 0$

42. (4) $f(x) = (e^x - 1)|e^{2x} - 1|$
$= (e^x - 1)|e^x - 1||e^x + 1|$
$= (e^x + 1)(e^x - 1)|e^x - 1|$
Now, both $e^x + 1$ and $(e^x - 1)|e^x - 1|$ are differentiable, as $g(x)|g(x)|$ is differentiable when $g(x) = 0$.
Hence, $f(x)$ is differentiable.

$f(x) = \dfrac{x-1}{x^2+1}$ is rational function in which denominator never becomes zero.
Hence, $f(x)$ is differentiable.

$$f(x) = \begin{cases} ||x-3|-1|, & x < 3 \\ \dfrac{x}{3}[x] - 2, & x \geq 3 \end{cases}$$

$$= \begin{cases} |3-x-1|, & x < 3 \\ \dfrac{x}{3}3 - 2, & 3 \leq x < 4 \end{cases}$$

$$= \begin{cases} |x-2|, & x < 3 \\ x-2, & 3 \leq x < 4 \end{cases}$$

$= x - 2, x \in [2, 4)$

Hence, $f(x)$ is differentiable

$$f(x) = 3(x-2)^{3/4} + 3 \text{ or } f'(x) = \frac{9}{4}(x-2)^{-1/4}$$

which is non-differentiable at $x = 2$.

Here, $f(x)$ is continuous and the graph has vertical tangent at $x = 2$; however, the graph is smooth in the neighborhood of $x = 2$.

43. (3) $\left|x - \frac{1}{2}\right|$ is continuous everywhere but not differentiable at $x = \frac{1}{2}$, $|x-1|$ is continuous everywhere but not differentiable at $x = 1$, and $\tan x$ is continuous in $[0, 2]$ except at $x = \frac{\pi}{2}$.

Hence, $f(x)$ is not differentiable at $x = \frac{1}{2}, 1, \frac{\pi}{2}$.

44. (3) $f(x) = (x^2-1)\,|(x-1)(x-2)|$

$= (x^2-1)\,|(x-1)(x-2)|$

$= (x+1)\,[(x-1)\,|x-1|]\,|x-2|$

which is differentiable at $x = 1$.

For $f(x) = \sin(|x-1|) - |x-1|$,

$$f'(1^+) = \lim_{h\to 0}\frac{\sin h - h - 0}{h} = 0$$

$$f'(1^-) = \lim_{h\to 0}\frac{\sin|-h| - |-h|}{-h}$$

$$= \lim_{h\to 0}\frac{\sin h - h}{-h}$$

$$= 0$$

Hence, $f(x)$ is differentiable at $x = 1$.

For $f(x) = \tan(|x-1|) + |x-1|$,

$$f'(1^+) = \lim_{h\to 0}\frac{\tan h + h - 0}{h} = 2$$

$$f'(1^-) = \lim_{h\to 0}\frac{\tan|-h| + |-h|}{-h} = \lim_{h\to 0}\frac{\tan h + h}{-h} = -2$$

Hence, $f(x)$ is non-differentiable at $x = 1$.

45. (4) Clearly, $f(x)$ is continuous at $x = 0$ if $a = 0$. Now,

$$f'(0+0) = \lim_{h\to 0}\frac{he^{-\left(\frac{1}{h}+\frac{1}{h}\right)} - 0}{h}$$

$$= \lim_{h\to 0}\frac{he^{-2/h} - 0}{h} = 0$$

$$f'(0-0) = \lim_{h\to 0}\frac{-he^{-\left(-\frac{1}{h}+\frac{1}{h}\right)} - 0}{-h} = 1$$

Thus, no values of a exist.

46. (2) $|\sin x|$ and $e^{|x|}$ are not differentiable at $x = 0$ and $|x|^3$ is differentiable at $x = 0$.

Therefore, for $f(x)$ to be differentiable at $x = 0$, we must have $a = 0$, $b = 0$ and c can be any real number.

47. (3) Clearly, $f(x)$ is continuous for all x except possibly at $x = \pi/6$.

For $f(x)$ to be continuous at $x = \pi/6$, we must have

$$\lim_{x\to\pi/6^-} f(x) = \lim_{x\to\pi/6^+} f(x)$$

or $\lim_{x\to\pi/6} \sin 2x = \lim_{x\to\pi/6} ax + b$

or $\sin(\pi/3) = (\pi/6)\,a + b$

or $\frac{\sqrt{3}}{2} = \frac{a\pi}{6} + b$ (1)

For $f(x)$ to be differentiable at $x = \pi/6$, we must have

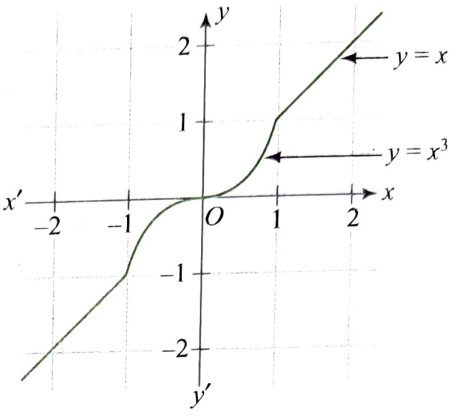
(L.H.D. at $x = \pi/6$) = (R.H.D. at $x = \pi/6$)

or $\lim_{x\to\pi/6} 2\cos 2x = \lim_{x\to\pi/6} a$

or $2\cos\pi/3 = a$ or $a = 1$

Putting $a = 1$ in equation (1), we get $b = (\sqrt{3}/2) - \pi/6$.

48. (2) $f(x)$ is clearly continuous for $x \in R$.



$$f'(x) = \begin{cases} 3x^2, & x^2 < 1 \\ 1, & x^2 > 1 \end{cases}$$

Thus, $f(x)$ is non-differentiable at $x = 1, -1$.

49. (4) $\frac{x}{1+|x|}$ is always differentiable (also at $x = 0$).

Also, $(x-2)(x+2)\,|(x-1)(x-2)(x-3)|$ is not differentiable at $x = 1, 3$.

So, $f(x)$ is not differentiable at $x = 1, 3$.

50. (2) $f(x) = \cos\pi(|x| + [x])$

$$= \begin{cases} \cos\pi(-x + (-1)), & -1 \le x < 0 \\ \cos\pi(x + 0), & 0 \le x < 1 \end{cases}$$

$$= \begin{cases} -\cos\pi x, & -1 \le x < 0 \\ \cos\pi x, & 0 \le x < 1 \end{cases}$$

Obviously, $f(x)$ is discontinuous at $x = 0$; otherwise $f(x)$ is continuous and differentiable in $(-1, 0)$ and $(0, 1)$.

51. (3) For $f(x)$ to be continuous at $x = 0$, we have

$f(0^-) = f(0^+)$ or $a(0) + b = 1$ or $b = 1$

$$f'(0^+) = \lim_{h\to 0}\frac{f(h) - f(0)}{h} = \lim_{h\to 0}\frac{e^{h^2+h} - b}{h}$$

$$= \lim_{h\to 0}\frac{e^{h^2+h} - 1}{h} = \lim_{h\to 0}\frac{e^{h^2+h} - 1}{h(h+1)}(h+1) = 1$$

$\therefore \quad f'(0^-) = a$

Hence, $a = 1$.

52. (3) $f(x) = \begin{cases} |x| - 1, & |x| < 0 \\ |x|^2 - 2|x|, & |x| \ge 0 \end{cases}$

where $|x| < 0$ is not possible. Thus, neglecting, we get

$f(|x|) = |x|^2 - 2|x|, |x| \ge 0$

$$f(|x|) = \begin{cases} x^2 + 2x, & x < 0 \\ x^2 - 2x, & x \ge 0 \end{cases} \quad (1)$$

$$\therefore \quad f'(|x|) = \begin{cases} 2x + 2, & x < 0 \\ 2x - 2, & x > 0 \end{cases}$$

Clearly, $f(|x|)$ is continuous at $x = 0$, but non-differentiable at $x = 0$.

$$\text{Also, } g(x) = |f(x)| = \begin{cases} 1-x, & x<0 \\ -x^2+2x, & 0 \le x < 2 \\ x^2-2x, & x \ge 2 \end{cases} \quad (2)$$

Clearly, $|f(x)|$ is discontinuous at $x = 0$, but continuous at $x = 2$.

$$\text{Also, } g'(x) = \begin{cases} -1, & x<0 \\ -2x+2, & 0<x<2 \\ 2x-2, & x>2 \end{cases}.$$

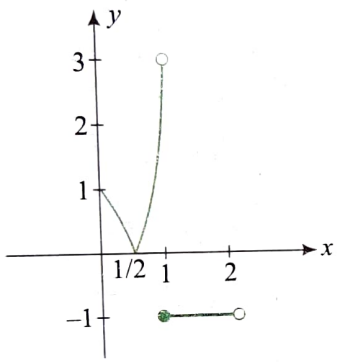
$|f(x)|$ is non-differentiable at $x = 0$ and $x = 2$.

53. (3) Since $1 \le x < 2$ or $0 \le x - 1 < 1$,

$$[x^2 - 2x] = [(x-1)^2 - 1] = [(x-1)^2] - 1 = 0 - 1 = -1$$

$$\therefore \quad f(x) = \begin{cases} 1-4x^2, & 0 \le x < \dfrac{1}{2} \\ 4x^2-1, & \dfrac{1}{2} \le x < 1 \\ -1 & 1 \le x < 2 \end{cases}$$

Therefore, graph of $f(x)$ is as follows.



It is clear from graph that $f(x)$ is discontinuous at $x = 1$ and not differentiable at $x = \dfrac{1}{2}$ and $x = 1$.

54. (3) $f'(0) = \lim\limits_{h\to 0} \dfrac{f(h) - f(0)}{h}$

$$= \lim_{h\to 0} \frac{h^a \sin\left(\frac{1}{h}\right)}{h} = \lim_{h\to 0} h^{a-1} \sin\left(\frac{1}{h}\right)$$

This limit will not exist if $a - 1 \le 0$ or $a \le 1$.

Now, $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} x^a \sin\left(\dfrac{1}{x}\right) = 0$ if $a > 0$.

Thus, $a \in (0, 1]$.

55. (1) We have $x + 4|y| = 6y$

or $\begin{cases} x - 4y = 6y, & \text{if } y < 0 \\ x + 4y = 6y, & \text{if } y \ge 0 \end{cases}$

or $y = \begin{cases} \dfrac{1}{2}x, & \text{if } x \ge 0 \\ \dfrac{1}{10}x, & \text{if } x < 0 \end{cases}$

or $f'(x) = \begin{cases} \dfrac{1}{2}, & x > 0 \\ \dfrac{1}{10}, & x < 0 \end{cases}$

Clearly, $f(x)$ is continuous at $x = 0$ but non-differentiable at $x = 0$.

56. (2) $f(0^+) = \lim\limits_{x\to 0+} |x|^{\sin x} = e^{\lim\limits_{x\to 0} \sin x \log |x|}$

$$= e^{\lim\limits_{x\to 0} \frac{\log x}{\operatorname{cosec} x}} = e^0 = 1 \qquad \text{(Using L' Hopital's rule)}$$

$f(0^-) = g(0) = 1$

Let $g(x) = ax + b$.

Then $b = 1$ or $g(x) = ax + 1$.

For $x > 0$,

$$f'(x) = e^{\sin x \ln(|x|)} \left[\cos x \ln(|x|) + \frac{\sin x}{x}\right]$$

$f'(1) = 1[0 + \sin 1] = \sin 1$

$f(-1) = -a + 1$ or $a = 1 - \sin 1$

or $\quad g(x) = (1 - \sin 1)x + 1$

57. (3) Given that $f(x) = |1 - x|$

$$\therefore \quad f(|x|) = \begin{cases} x-1, & x>1 \\ 1-x, & 0<x\le 1 \\ 1+x, & -1\le x\le 0 \\ -x-1, & x<-1 \end{cases}$$

Clearly, the domain of $\sin^{-1}(f|x|)$ is $[-2, 2]$.

Therefore, it is non-differentiable at the points $\{-1, 0, 1\}$.

58. (4) $f(x)$ is continuous at $x = 0$. Therefore,

$$\lim_{x\to 0} f(x) = f(0)$$

or $\quad f(0) = \lim\limits_{x\to 0} f(0+h)$

$$= \lim_{h\to 0} \frac{hg(h)}{|h|}$$

$$= \lim_{h\to 0} g(h) = g(0) = 0$$

Now, $f'(0^+) = \lim\limits_{h\to 0} \dfrac{f(0+h) - f(0)}{h}$

$$= \lim_{h\to 0} \frac{\frac{hg(h)}{|h|}}{h} = \lim_{h\to 0} \frac{g(h)}{h}$$

$$= \lim_{h\to 0} \frac{g(h) - g(0)}{h}$$

$= g'(0)$ [as $g(0) = 0$]

$= 0$

$f'(0^-) = \lim\limits_{h\to 0} \dfrac{f(0-h) - f(0)}{-h}$

$$= \lim_{h\to 0} \frac{\frac{-hg(-h)}{|-h|}}{-h} = \lim_{h\to 0} \frac{g(-h)}{h}$$

$$= -\lim_{h\to 0} \frac{g(-h) - g(0)}{-h}$$

$= -g'(0) = 0$

Hence, $f'(0)$ exists and $f'(0) = 0$.

59. (3) $f(|x|) = \begin{cases} \sin|x|, & |x| < 0 \\ \cos(x) - ||x| - 1|, & |x| \ge 0 \end{cases}$

$= \cos(x) - ||x| - 1|, x \in R$

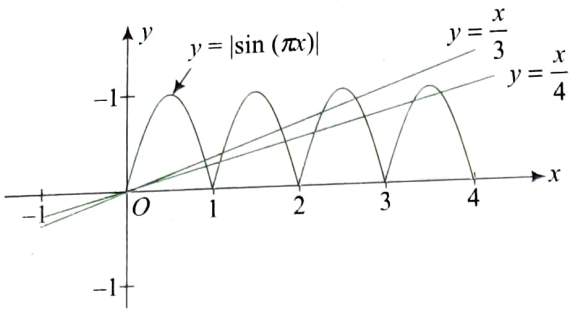
[as $|x| < 0$ is not possible and $|x| \ge 0$ is true $\forall\, x \in R$]

which is non-differentiable at $x = 0$ and when $|x| - 1 = 0$ or $x = \pm 1$.

Hence, $f(|x|)$ has exactly three points of non-differentiability.

60. (2) $f(x) = \max.\left(\frac{x}{n}, |\sin \pi x|\right), n \in N$

For maximum points of intersection, line $y = x/n$ must intersect the graph of $y = |\sin \pi x|$ maximum number of times.



Clearly, from the figure, for maximum points of intersection, least value of n is 4.

61. (1) $f(e) = [\log_e e] + \sqrt{\{\log_e e\}}$

$= [1] + \sqrt{\{1\}} = 1 + 0 = 1$

$f(e^+) = [\log_e e^+] + \sqrt{\{\log_e e^+\}}$

$= \lim_{h\to 0}[1+h] + \sqrt{\{1+h\}}$

$= 1 + 0 = 1$

$f(e^-) = [\log_e e^-] + \sqrt{\{\log_e e^-\}}$

$= \lim_{h\to 0}[1-h] + \sqrt{\{1-h\}}$

$= 0 + 1 = 1$

Hence, $f(x)$ is continuous at $x = e$. Now,

$f'(e^+) = \lim_{h\to 0}\frac{f(e+h)-f(e)}{h}$

$= \lim_{h\to 0}\frac{[1+h]+\sqrt{\{1+h\}}-1}{h}$

$= \lim_{h\to 0}\frac{1+\sqrt{h}-1}{h} = \lim_{h\to 0}\frac{1}{\sqrt{h}} \to \infty$

Hence, $f(x)$ is non-differentiable at $x = 0$.

62. (2) $f(1) = 1 - \sqrt{1-1^2} = 1$

$f(1^-) = \lim_{x\to 1^-}\left(1 - \sqrt{1-x^2}\right) = 1$

$f(1^+) = \lim_{x\to 1^+}\left(1 + \log\frac{1}{x}\right) = 1 + \log\frac{1}{1} = 1$

Hence, $f(x)$ is continuous at $x = 1$.

$f'(1^+) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$

$= \lim_{h\to 0}\frac{1 + \log\frac{1}{1+h} - 1}{h}$

$= -\lim_{h\to 0}\frac{\log(1+h)}{h} = -1$

$f'(1^-) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$

$= \lim_{h\to 0}\frac{1-\sqrt{1-(1-h)^2}-1}{-h}$

$= \lim_{h\to 0}\frac{\sqrt{2-h}}{\sqrt{h}} = \infty$

Hence, $f(x)$ is non-differentiable at $x = 1$.

63. (4) $f(x) = \sqrt[3]{|x|} - |x| - 1$

$= |x| - |x| - 1 = -1$

Hence, differentiable for all x.

64. (2) $g'(0^+) = \lim_{h\to 0}\frac{f(|h|) - |\sin h| - f(0)}{h}$

$= \lim_{h\to 0}\frac{f(h)-f(0)}{h} - \lim_{h\to 0}\frac{\sin h}{h}$

$= 1 - 1$

$= 0$

$g'(0^-) = \lim_{h\to 0}\frac{f(|-h|) - |\sin(-h)| - f(0)}{-h}$

$= \lim_{h\to 0}\frac{f(h)-f(0)}{-h} + \lim_{h\to 0}\frac{\sin h}{h}$

$= -1 + 1$

$= 0$

Thus, $g(x)$ is differentiable and $g'(0) = 0$.

65. (3) At $x = 0$,

$\text{L.H.L.} = \lim_{x\to 0-} f(x) = \lim_{h\to 0} f(0-h)$

$= \lim_{h\to 0} h^2\left(\frac{e^{-1/h} - e^{1/h}}{e^{-1/h} + e^{1/h}}\right)$

$= \lim_{h\to 0} h^2\left(\frac{e^{-2/h} - 1}{e^{-2/h} + 1}\right)$

$= 0\left(\frac{0-1}{0+1}\right) = 0$

$\text{R.H.L.} = \lim_{x\to 0+} f(x) = \lim_{h\to 0} f(0+h)$

$= \lim_{h\to 0} h^2\left(\frac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}}\right)$

$= \lim_{h\to 0} h^2\left(\frac{1 - e^{-2/h}}{1 + e^{-2/h}}\right)$

$= 0\left(\frac{1-0}{1+0}\right) = 0$

and $f(0) = 0$

$\therefore$ L.H.L. = R.H.L. = $f(0)$

Hence, $f(x)$ is continuous at $x = 0$.

$\text{Also, L.H.D.} = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$

$= \lim_{h\to 0}\frac{h^2\frac{e^{-1/h} - e^{1/h}}{e^{-1/h} + e^{1/h}} - 0}{-h}$

$= -\lim_{h\to 0} h\frac{e^{-2/h} - 1}{e^{-2/h} + 1}$

$= 0$

and $\text{R.H.D.} = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$

$= \lim_{h\to 0}\frac{h^2\frac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}} - 0}{-h}$

$= -\lim_{h\to 0} h\frac{1 - e^{-2/h}}{1 + e^{-2/h}}$

$= 0$

Hence, $f(x)$ is differentiable at $x = 0$ and $f'(0) = 0$

(3) Obviously, $\lim_{x\to 0+} e^{-1/x^2} = \lim_{x\to 0-} e^{-1/x^2} = 0$.

Hence, $f(x)$ is continuous at $x = 0$.

$$f'(0) = \lim_{h\to 0} \frac{e^{-1/h^2}}{h} = \lim_{h\to 0} \frac{1/h}{e^{1/h^2}}$$

$$= \lim_{h\to 0} \frac{-1/h^2}{-e^{1/h^2}\cdot\frac{2}{h^3}} = \lim_{h\to 0} \frac{2h^3}{h^2 e^{1/h^2}} = 0$$

Hence, f is differentiable at $x = 0$. Also, $\lim_{x\to\pm\infty} e^{-\frac{1}{x^2}} \to 1$.

(3) $f(2 + x) = f(-x)$ (1)

Replacing x by $x - 1$, we get

$f(2 + x - 1) = f(-x + 1)$ or $f(1 + x) = f(1 - x)$

Hence, $f(x)$ is symmetrical about line $x = 1$.

Now, putting $x = 2$ in (1), we get $f(4) = f(-2)$. Hence, differentiability at $x = 4$ implies differentiability at $x = 2$.

(3) $2 + 10 \sin x \in [2, 12]$

Clearly, $f(x)$ is discontinuous where $2 + 10 \sin x = 3, 4, 5, \ldots, 12$, i.e., at 10 points.

Number of points of non-differentiability of $g(x) = (x-1)(x-2)|(x-1)(x-2)\cdots(x-2m)|$ is $(2m - 2)$ as $g(x)$ is differentiable at $x = 1$ and $x = 2$.

$\therefore \quad 2m - 2 = 10$

$\therefore \quad m = 6$

(2) $f(x) = \begin{cases} 1+\left[\cos\frac{\pi x}{2}\right], & 1<x\le 2 \\ 1-\{x\}, & 0\le x<1 \\ |\sin \pi x|, & -1\le x<0 \end{cases} = \begin{cases} 1-1, & 1<x\le 2 \\ 1-x, & 0\le x<1 \\ -\sin \pi x, & -1\le x<0 \end{cases}$

$f(x)$ is continuous at $x = 1$ but not differentiable.

(1) $x^2 + 2x + 3 + \sin \pi x = (x + 1)^2 + 2 + \sin \pi x > 1$

$\therefore \quad f(x) = 1 \quad \forall\ x \in R$

Multiple Correct Answers Type

1. (1), (2), (3), (4)

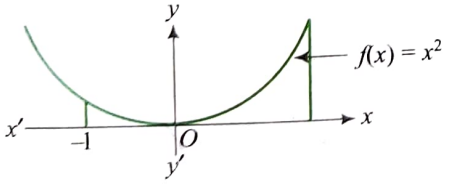
(1), (2), and (3) are false. Refer theory.

For (4), f must be continuous so, it is also false.

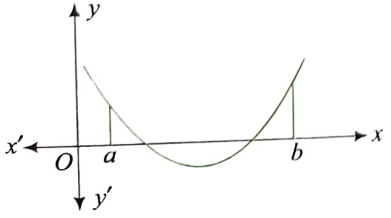
2. (1), (3), (4)

(1) is wrong as continuity is a must for $f(x)$.

(2) is the correct form of intermediate value theorem.



(3) as per the graph in the figure, it is incorrect.



(4) is wrong if f is discontinuous.

3. (1), (3), (4)

$$f(x) = \frac{x^2 - 2x - 8}{x + 2} = \frac{(x+2)(x-4)}{x+2} = x - 4, \quad x \ne -2$$

Hence, $f(x)$ has removable discontinuity at $x = -2$.

Similarly, $f(x)$ in options (3) and (4) has also removable discontinuity.

$$f(x) = \frac{x-7}{|x-7|} = \begin{cases} -1, & x < 7 \\ 1, & x > 7 \end{cases}$$

Hence, $f(x)$ has non-removable discontinuity at $x = 7$.

4. (2), (4)

(1) $\lim_{x\to 1^+} \frac{1}{\ln|x|} = \infty$ and $\lim_{x\to 1^-} \frac{1}{\ln|x|} = -\infty$.,

Hence, $f(x)$ has non-removable discontinuity.

(2) $\lim_{x\to 1} \frac{x^2-1}{x^3-1} = \frac{2}{3}$

Hence, $f(x)$ has removable discontinuity at $x = 1$.

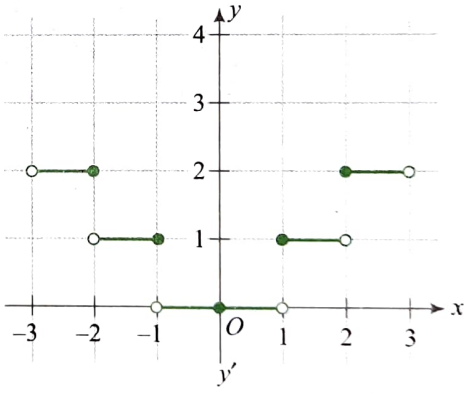
(3) $\lim_{x\to 1^+}\left(2^{-2^{\frac{1}{1-x}}}\right) = 1$ and $\lim_{x\to 1^-}\left(2^{-2^{\frac{1}{1-x}}}\right) = 0$.

Hence, the limit does not exist.

(4) $\lim_{x\to 1} \frac{\sqrt{x+1} - \sqrt{2x}}{x^2 - x} = \frac{-1}{2\sqrt{2}}$ (Rationalizing)

Hence, $f(x)$ has removable discontinuity at $x = 1$.

5. (2), (4)



6. (1), (2), (3), (4)

Given function is discontinuous when $a + \sin \pi x = \pm 1$.

Now, if $a = 1$, then $\sin \pi x = 0$ or $x = 1, 2, 3, 4, 5$.

If $a = 3$, then $\sin \pi x = -2$, not possible.

If $a = 0.5$, then $\sin \pi x = 0.5$.

Therefore, x has 6 values, 2 each for one cycle of period 2.

If $a = 0$, then $\sin \pi x = \pm 1$ or $x = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}$.

Hence, all the options are correct.

7. (1), (2)

For maximum points of discontinuity of

$f(x) = \text{sgn}(x^2 - ax + 1)$,

$x^2 - ax + 1 = 0$ must have two distinct roots, for which

$D = a^2 - 4 > 0$

or $\quad a \in (-\infty, -2) \cup (2, \infty)$

. (1), (2), (3)

$$f(0^-) = \lim_{n\to\infty}\left[\lim_{x\to 0^-}\left(\cos^2 x\right)^n\right]$$
$$= (\text{a value lesser than } 1)^\infty = 0$$
$$f(0^+) = \lim_{n\to\infty}\left[\lim_{x\to 0^+}\left(1+x^n\right)^{1/n}\right] = 1$$

Also, $f(0) = 1$. So, the function is discontinuous at $x = 0$.

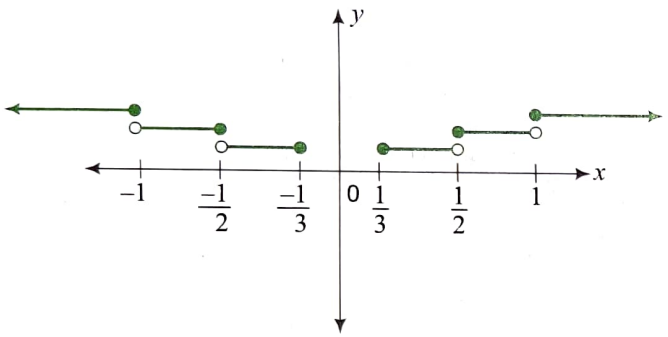
Further, $f(1^-) = 1; f(1^+) = 0; f(1) = 1$.

So, the function is discontinuous at $x = 1$.

. (1), (3)

$$f(x) = \begin{cases} 1, & |x| \geq 1 \\ \dfrac{1}{n^2}, & \dfrac{1}{n} \leq |x| < \dfrac{1}{n-1}, n = 2, 3, \ldots \\ 0, & x = 0 \end{cases}$$

$$= \begin{cases} 1, & x \leq -1 \text{ or } x \geq 1 \\ \dfrac{1}{4}, & x \in \left(-1, -\dfrac{1}{2}\right] \cup \left[\dfrac{1}{2}, 1\right) \\ \dfrac{1}{9}, & x \in \left(\dfrac{-1}{2}, \dfrac{-1}{3}\right] \cup \left[\dfrac{1}{3}, \dfrac{1}{2}\right) \\ \vdots \end{cases}$$



The function f is clearly continuous for $|x| > 1$.

We observe that

$$\lim_{x\to -1^-} f(x) = 1, \quad \lim_{x\to -1^+} f(x) = \frac{1}{4}$$

Clearly, f is discontinuous for $x = \pm\dfrac{1}{n}$, $n = 1, 2, 3, \ldots$.

Also, f is continuous for $x = 0$.

Hence, (1) and (3) are the correct answers.

0. (1), (2), (3)

Since $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^+} g(x) = 1$ and $g(1) = 0$,

$g(x)$ is not continuous at $x = 1$ but $\lim_{x\to 1} g(x)$ exists.

We have $\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0}[1-h] = 0$

and $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0}[1+h] = 1$

So, $\lim_{x\to 1} f(x)$ does not exist and, hence, $f(x)$ is not continuous at $x = 1$.

We have $gof(x) = g(f(x)) = g([x]) = 0 \ \forall\ x \in R$.

So, gof is continuous for all x.

We have $fog(x) = f(g(x)) = \begin{cases} f(0), & x \in Z \\ f(x^2), & x \in R - Z \end{cases}$

$$= \begin{cases} 0, & x \in Z \\ [x^2], & x \in R - Z \end{cases}$$

which is clearly not continuous.

11. (1), (2)

We know that $|x|^r$, $r = 0, 1, 2, \ldots$, is continuous everywhere.

$\therefore \quad f(x) = \sum_{r=1}^{n} a_r |x|^r$ is everywhere continuous

Also, $|x|, |x|^3, |x|^5, \ldots$ are not differentiable at $x = 0$.

Whereas, $|x|^2, |x|^4, \ldots$ are differentiable everywhere.

So, $f(x) = \sum_{r=1}^{n} a_r |x|^r$ is not differentiable at $x = 0$, if any one of $a_1, a_3, a_5, \ldots$ is non-zero

Thus, for $f(x)$ to be differentiable at $x = 0$, we must have

$a_1 = a_3 = a_5 \ldots = 0$

i.e., $a_{2k+1} = 0$.

12. (1), (2)

$f(x)$ is continuous for all x if it is continuous at $x = 1$,

for which $|1| - 3 = |1 - 2| + a$ or $a = -3$.

$g(x)$ is continuous for all x if it is continuous at $x = 2$,

for which $2 - |2| = \text{sgn}(2) - b = 1 - b$ or $b = 1$.

Thus, $f(x) + g(x)$ is continuous for all x if $a = -3$, $b = 1$.

Hence, $f(x)$ is discontinuous at exactly one point for options (1) and (2).

13. (1), (3)

$$f\left(\frac{\pi^-}{2}\right) = \lim_{h\to 0}\left(\frac{3}{2}\right)^{\cot\left(3\left(\frac{\pi}{2}-h\right)\right)\Big/\cot\left(2\left(\frac{\pi}{2}-h\right)\right)}$$
$$= \lim_{h\to 0}\left(\frac{3}{2}\right)^{\frac{\tan 3h}{-\cot 2h}}$$
$$= \lim_{h\to 0}\left(\frac{3}{2}\right)^{-(\tan 3h)(\tan 2h)}$$
$$= 1$$
$$f\left(\frac{\pi^+}{2}\right) = \lim_{h\to 0}\left[1+\left|\cot\left(\frac{\pi}{2}+h\right)\right|\right]^{\left[a\left|\tan\left(\frac{\pi}{2}+h\right)\right|\right]/b}$$
$$= \lim_{h\to 0}(1+\tan h)^{\frac{a\cot h}{b}}$$
$$= e^{\lim_{h\to 0}(1+\tan h-1)\frac{a\cot h}{b}}$$
$$= e^{a/b}$$

Also, $f\left(\dfrac{\pi}{2}\right) = b + 3$

$f(x)$ is continuous at $x = \pi/2$. Therefore,

$1 = b + 3 = e^{a/b}$ or $b = -2$ and $a = 0$

14. (3), (4)

$f(x) = \text{sgn}(\cos 2x - 2\sin x + 3)$

$= \text{sgn}(1 - 2\sin^2 x - 2\sin x + 3)$

$= \text{sgn}(-2\sin^2 x - 2\sin x + 4)$

$f(x)$ is discontinuous when

$-2\sin^2 x - 2\sin x + 4 = 0$ or $\sin^2 x + \sin x - 2 = 0$

i.e., $(\sin x - 1)(\sin x + 2) = 0$ or $\sin x = 1$

Hence, $f(x)$ is discontinuous.

15. (1), (2), (3), (4)

(1) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \dfrac{e^x + a}{2x} = \dfrac{1}{2}$ or $a = -1$

If $a=-1$, then $\lim_{x\to 0^+} f(x) = \frac{1}{2}$, $\lim_{x\to 0^-} f(x) = \frac{1}{2}$

Therefore, $f(x)$ is continuous at $x=0$ if $b=\frac{1}{2}$.

(3) If $a\neq -1$, then $\lim_{x\to 0}\frac{e^x+a}{2x}$ does not exist.

Therefore, $x = 0$ is a point of irremovable type of discontinuity.

(4) If $a=-1$, then $\lim_{x\to 0} f(x)=\frac{1}{2}$.

Therefore, $b \neq \frac{1}{2}$. Thus, there is removable type of discontinuity at $x=0$.

16. (1), (2), (3)

At $x=0$,

$$\text{R.H.L.} = \lim_{h\to 0}\frac{e^{1/h}+e^{2/h}+e^{3/h}}{ae^{2/h}+be^{3/h}}$$

$$= \lim_{h\to 0}\frac{e^{-2/h}+e^{-1/h}+1}{ae^{-1/h}+b} = \frac{1}{b}$$

and $$\text{L.H.L.} = \lim_{h\to 0}\left(\frac{e^h+h^2-a}{h}\right)^{1/h}$$

$$= \lim_{h\to 0}\left(h+\frac{e^h-a}{h}\right)^{1/h}$$

$\Rightarrow$ $a=1$ as it must be one power infinity form

$\therefore$ $$\text{L.H.L.} = e^{\lim_{h\to 0}\frac{1}{h}\left(h+\frac{e^h-1}{h}-1\right)}$$

$$= e^{\lim_{h\to 0}\frac{e^h+h^2-h-1}{h^2}}$$

$$= e^{\lim_{h\to 0}\frac{e^h+2h-1}{2h}} \quad \text{(Using L'Hopspital Rule)}$$

$$= e^{\lim_{h\to 0}\frac{e^h+2}{2}} \quad \text{(Using L'Hopspital Rule)}$$

$$= e^{3/2}$$

$\Rightarrow$ $f(0) = e^{3/2} = \frac{1}{b}$

17. (1), (2)

$f(-2) = b$ ...(1)

$$\text{R.H.L.} = \lim_{x\to -2^+} f(x)$$

$$= \lim_{x\to -2^+}\sin\left(\frac{x^4-16}{x^5+32}\right)$$

$$= \lim_{x\to -2^+}\sin\left(\frac{\frac{x^4-(-2)^4}{x-(-2)}}{\frac{x^5-(-2)^5}{x-(-2)}}\right)$$

$$= \sin\left(\frac{4(-2)^3}{5(-2)^4}\right)$$

$$= -\sin\frac{2}{5} \quad \ldots(2)$$

$$\text{L.H.L.} = \lim_{x\to -2^-} f(x)$$

$$= \lim_{h\to 0}\frac{ae^{1/(|-2-h+2|)}-1}{2-e^{1/(|-2-h+2|)}}$$

$$= \lim_{h\to 0}\frac{ae^{1/h}-1}{2-e^{1/h}}$$

$$= \lim_{h\to 0}\frac{a-e^{-1/h}}{2e^{-1/h}-1} = \frac{a-0}{0-1} = -a \quad \ldots(3)$$

From (1), (2), and (3)

$$a = \sin\frac{2}{5} \text{ and } b = -\sin\frac{2}{5}$$

18. (1), (2), (4)

We have $0 < x < 1$

$\Rightarrow$ $0 < 2^n x < 2^n$

$\Rightarrow$ $[2^n x]$ is discontinuous when $2^n x = 1, 2, 3, \ldots, 2^n - 1$

$\{2^m x\}$ is discontinuous when $2^m x = 1, 2, 3, \ldots, 2^m - 1$

But $(2^m - 1)$ number of points are common where both are discontinuous but their sum is continuous,

Hence total number of points of discontinuity

$(2^n - 1) + (2^m - 1) - 2(2^m - 1) = 2^n - 2^m$ $(n > m)$

Now $2^n - 2^m = 24$, for $n = 5$, $m = 3$

$2^n - 2^m = 28$, for $n = 5$, $m = 2$

$2^n - 2^m = 496$, for $n = 9$, $m = 4$

19. (1), (2)

$f(1^-) = 1$; $f(1^+) = 1$; $f(1) = 1$

$f'(1^-) = 5$; $f'(1^+) = 5$

$f(2^+) = 10$; $f(2^-) = 10$

$f'(2^+) = 3$; $f'(2^-) = 13$.

20. (1), (2)

$f(x) = \text{sgn}(x)\sin x$

$f(0^+) = \text{sgn}(0^+)\sin(0^+) = 1 \times (0) = 0$

$f(0^-) = \text{sgn}(0^-)\sin(0^-) = (-1) \times (0) = 0$

Also, $f(0) = 0$.

Hence, $f(x)$ is continuous everywhere.

Both sgn(x) and sin(x) are odd functions.

Hence, $f(x)$ is an even function.

Obviously, $f(x)$ is non-periodic. Now,

$$f'(0^+) = \lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{\text{sgn}(h)\sin h - 0}{h}$$

$$= \lim_{h\to 0}\frac{\sin h}{h} = 1$$

and $$f'(0^-) = \lim_{h\to 0}\frac{\text{sgn}(-h)\sin(-h)-0}{-h}$$

$$= \lim_{h\to 0}\frac{-1\times(-\sin h)}{-h} = -1$$

Hence, $f(x)$ is non-differentiable at $x = 0$.

21. (2), (3)

Option (1) is wrong as $f(x) = \sin x$ and $g(x) = |x|$.

$g(x)$ is non-differentiable at $x = 0$, but $f(x)\,g(x)$ is differentiable at $x = 0$.

22. (1), (3)

$$f(x)=\begin{cases} xe^x, & x\le 0 \\ x+x^2-x^3, & x>0 \end{cases}$$

Clearly, $f(x)$ is continuous at $x=0$.

$$\Rightarrow \quad f'(x)=\begin{cases} e^x(x+1), & x<0 \\ 1+2x-3x^2, & x>0 \end{cases}$$

$$f'(0^-)=\lim_{x\to 0^-} e^x(x+1)=1$$

$$f'(0^+)=\lim_{x\to 0^+}(1+2x-3x^2)=1$$

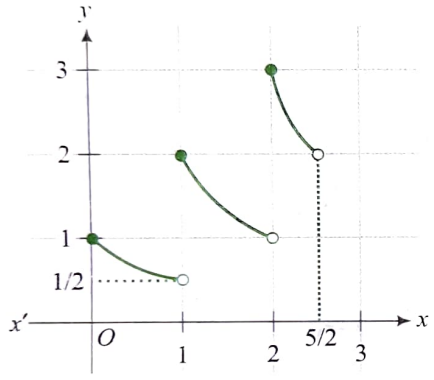
Hence, $f(x)$ is differentiable at $x=0$

$$\text{Again } f''(x)=\begin{cases} e^x(x+2), & x<0 \\ 2-6x, & x>0 \end{cases}$$

$$\lim_{x\to 0^+} f''(x)=\lim_{x\to 0^-} f''(x)=2$$

Therefore, $f'(x)$ is also continuous and differentiable.

23. (1), (2), (4)



$$f(x)\begin{cases} \dfrac{1}{x+1}, & 0\le x<1 \\ \dfrac{2}{x}, & 1\le x<2 \\ \dfrac{3}{x-1}, & 2\le x<\dfrac{5}{2} \end{cases}$$

Clearly, $f(x)$ is discontinuous and bijective function:

$$\lim_{x\to 1^-} f(x)=\frac{1}{2},\quad \lim_{x\to 1^+} f(x)=2$$

$$\min\left(\lim_{x\to 1^-} f(x), \lim_{x\to 1^+} f(x)\right)=\frac{1}{2}\ne f(1)$$

$$\max(1,2)=2=f(1)$$

24. (2), (4)

$$\text{We have } \lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{\log\cos x}{\log(1+x^2)}$$

$$=\lim_{x\to 0}\frac{\log(1-1+\cos x)}{\log(1+x^2)}\cdot\frac{1-\cos x}{1-\cos x}$$

$$=\lim_{x\to 0}\frac{\log\{1-(1-\cos x)\}}{1-\cos x}\cdot\frac{1-\cos x}{\log(1+x^2)}$$

$$=-\lim_{x\to 0}\frac{\log[1-(1-\cos x)]}{-(1-\cos x)}\cdot\frac{2\sin^2\frac{x}{2}}{4\left(\frac{x}{2}\right)^2}\cdot\frac{x^2}{\log(1+x^2)}=-\frac{1}{2}$$

Hence, $f(x)$ is differentiable at $x=0$.

Hence, (2) and (4) are the correct answers.

25. (1), (3)

$f(x)=x+|x|+\cos 9x$, $g(x)=\sin x$

Since both $f(x)$ and $g(x)$ are continuous everywhere,

$f(x)+g(x)$ is also continuous everywhere.

$f(x)$ is non-differentiable at $x=0$.

Hence, $f(x)+g(x)$ is non-differentiable at $x=0$. Now,

$$h(x)=f(x)\cdot g(x)$$

$$=\begin{cases} (\cos 9x)(\sin x), & x<0 \\ (2x+\cos 9x)(\sin x), & x\ge 0 \end{cases}$$

Clearly, $h(x)$ is continuous at $x=0$. Also,

$$h'(x)=\begin{cases} \cos x\cos 9x-9\sin x\sin 9x, & x<0 \\ (2-9\sin 9x)\sin x+\cos x(2x+\cos 9x), & x>0 \end{cases}$$

$h'(0^-)=1$, $h'(0^+)=1$

So, $f(x)\cdot g(x)$ is differentiable everywhere.

26. (1), (3), (4)

(1) is not correct as $f(x)=x$ from R to R is onto but its reciprocal function $g(x)=\dfrac{1}{x}$ is not onto on R.

Option (2) is obviously true.

Also, $g(x)$ is not continuous and, hence, not differentiable though $f(x)$ is continuous and differentiable in the above case.

27. (2), (3), (4)

For continuity at $x=1$,

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(x^2\operatorname{sgn}[x]+\{x\})=1+0=1$$

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(x^2\operatorname{sgn}[x]+\{x\})=1\operatorname{sgn}(0)+1=1$$

Also, $f(1)=1$

$\therefore$ L.H.L. = R.H.L $=f(1)$

Hence, $f(x)$ is continuous at $x=1$.

Now, for differentiability,

$$f'(1^+)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$=\lim_{h\to 0}\frac{(1+h)^2\operatorname{sgn}[1+h]+\{1+h\}-1}{h}$$

$$=\lim_{h\to 0}\frac{(1+h)^2+h-1}{h}$$

$$=\lim_{h\to 0}\frac{h^2+3h}{h}$$

$$=3$$

$$\text{and } f'(1^-)=\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$

$$=\lim_{h\to 0}\frac{(1-h)^2\operatorname{sgn}[1-h]+\{1-h\}-1}{-h}$$

$$=\lim_{h\to 0}\frac{0+1-h-1}{-h}$$

$$=\lim_{h\to 0}\frac{-h}{-h}=1$$

$f'(1^+)\ne f'(1^-)$

Hence, $f(x)$ is non-differentiable at $x=1$.

Now, at $x = 2$,

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2 \operatorname{sgn}[x] + \{x\}) = 4 \times 0 + 1$$

$$\lim_{h \to 2^+} f(x) = \lim_{h \to 2^+} (\sin x + |x - 3|) = 1 + \sin 2$$

Hence, L.H.L $\neq$ R.H.L.

Hence, $f(x)$ is discontinuous at $x = 2$ and then $f(x)$ is also non-differentiable at $x = 2$.

28. (2), (3), (4)

$$f(x) = |x^3| = \begin{cases} -x^3, & x < 0 \\ x^3, & x \geq 0 \end{cases} \text{ or } f''(x) \begin{cases} -6, & x < 0 \\ 6, & x > 0 \end{cases}$$

Hence, $f''(0)$ does not exist.

$$f(x) = x^3|x| = \begin{cases} -x^4, & x < 0 \\ x^4, & x \geq 0 \end{cases} \text{ or } f''''(x) = \begin{cases} -24x, & x < 0 \\ 24x, & x > 0 \end{cases}$$

Hence, $f''''(0) = 0$ and it exists.

Similarly, for $f(x) = |x|\sin^3 x$ and $f(x) = x|\tan^3 x|$ also, $f''''(0) = 0$ and it exists.

29. (1), (2)

$\sin^4 x \in (0, 1)$ for $x \in (-\pi/2, \pi/2)$

or $f(x) = 0$ for $x \in (-\pi/2, \pi/2)$

Hence, $f(x)$ is continuous and differentiable at $x = 0$.

30. (2), (3), (4)

$f(0) = 0$ and $f(1) = 0$

Also $\lim_{x \to 0} x^3(1 - x)\sin\frac{1}{x^2} = 0$

Further $x^3(1 - x)\sin\left(\frac{1}{x^2}\right)$ is continuous in $(0, 1)$

So $f(x)$ is bounded in $[0, 1]$.

$$f'(0^+) = \lim_{h \to 0} \frac{h^3(1 - h)\sin\frac{1}{h^2} - 0}{h} = 0$$

$$f'(1^-) = \lim_{h \to 0} \frac{(1 - h)^3(h)\sin\frac{1}{(1 - h)^2} - 0}{-h}$$

$$= -\lim_{h \to 0} (1 - h)^3 \sin\frac{1}{(1 - h)^2}$$

$$= -\sin 1$$

Hence f is derivable in $[0, 1]$.

$$f'(x) = (x^3 - x^4)\cos\left(\frac{1}{x^2}\right)\left(-\frac{2}{x^3}\right) + (3x^2 - 4x^3)\sin\frac{1}{x^2}, \quad x \neq 0$$

Clearly $f'(x)$ is continuous in $(0, 1]$.

And $\lim_{x \to 1^-} f'(x) = 0 - \sin 1 = -\sin 1$

Hence f' is also bounded.

31. (1), (2), (3)

$\frac{df}{dg}$ exists at $x = 1$ if $\frac{df}{dx}$ and $\frac{dg}{dx}$ exists at $x = 1$

$\therefore$ $f(x)$ and $g(x)$ must be continuous and differentiable at $x = 1$

$f(x)$ is continuous at $x = 1$ if $f(1^-) = f(1^+)$,

or $1 + a = 2 + b$ or $a - b = 1$ ...(i)

$f(x)$ is differentiable at $x = 1$ if $f'(1^-) = f'(1^+)$, which is true as both have value '2'

Now $g(x)$ is continuous at $x = 1$ if $g(1^-) = g(1^+)$,

or $3 + b = 1$ or $b = -2$

So, from eq. (i), we get $a = -1$

Further $g(x)$ is already differentiable at

$x = 1$ as $g'(1^-) = g'(1^+) = 3$

Also $\left(\frac{df}{dg}\right)_{x=1} = 2/3$

32. (2), (3)

$f(x)$ is continuous at $x = 0$ if

$$\lim_{x \to 0} \frac{a\cos x + bx\sin x + ce^x - 2x}{x^2} = 0$$

$$\Rightarrow \lim_{x \to 0} \frac{a\left(1 - \frac{x^2}{2!} + \ldots\right) + bx\left(x - \frac{x^3}{3!} + \ldots\right) + c\left(1 + \frac{x}{1!} + \frac{x^2}{2!} \ldots\right) - 2x}{x^2} = 0$$

$$\Rightarrow \lim_{x \to 0} \frac{a + c + (c - 2)x + \left(-\frac{a}{2} + b + \frac{c}{2}\right)x^2 + \ldots}{x^2} = 0$$

$$\Rightarrow a + c = 0,\ c - 2 = 0 \text{ and } \frac{-a}{2} + b + \frac{c}{2} = 0$$

$$\Rightarrow a = -2,\ b = -2 \text{ and } c = 2$$

$$\text{Now, } f'(0) = \lim_{h \to 0} \frac{\left(\frac{-2\cosh - 2h\sin h + 2e^h - 2h}{h^2}\right) - 0}{h}$$

$$= \lim_{h \to 0} \frac{-2\cosh - 2h\sin h + 2e^h - 2h}{h^3}$$

$$= \lim_{h \to 0} \frac{\begin{array}{r} -2\left(1 - \frac{h^2}{2!} + -\ldots\right) - 2h\left(h - \frac{h^3}{3!} + \ldots\right) \\ +2\left(1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \ldots\right) - 2h \end{array}}{h^3}$$

$$= \frac{1}{3}$$

Linked Comprehension Type

For Problems 1–3

1. (2), 2. (1), 3. (2)

$$f(x) = \begin{cases} \frac{a(1 - x\sin x) + b\cos x + 5}{x^2}, & x < 0 \\ 3, & x = 0 \\ \left\{1 + \left(\frac{P(x)}{x}\right)\right\}^{1/x}, & x > 0 \end{cases}$$

where $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

$f(0) = 3$

$$\text{R.H.L.} = \lim_{x \to 0^+} f(x)$$

$$= \lim_{h \to 0} f(0 + h)$$

$$= \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} \left\{1 + \left(\frac{P(h)}{h}\right)\right\}^{1/h}$$

Since f is continuous at $x = 0$, R.H.L. exists.

For the existence of R.H.L., $a_0, a_1 = 0$. Thus,

$$\text{R.H.L.} = \lim_{h\to 0}(1+a_2h+a_3h^2)^{1/h} \qquad (1^\infty \text{ form})$$

$$= e^{\lim_{h\to 0}(1+a_2h+a_3h^2-1)(1/h)} = e^{a_2}$$

$$\text{L.H.L.} = \lim_{x\to 0^-} f(x)$$

$$= \lim_{h\to 0} f(0-h)$$

$$= \lim_{h\to 0}\frac{a(1-(-h)\sin(-h))+b\cos(-h)+5}{(-h)^2}$$

$$= \lim_{h\to 0}\frac{a(1-h(h))+b\left(1-\frac{h^2}{2!}\right)+5}{h^2}$$

For finite value of L.H.L., $a + b + 5 = 0$ and $-a-\frac{b}{2} = 3$.

Solving, we get $a = -1$, $b = -4$.

Now, $g(x) = 3a \sin x - b \cos x = -3 \sin x + 4 \cos x$ which has range $[-5, 5]$.

Also, $P(x) = a_3x^3 + (\log_e 3)x^2$

$P''(x) = 6a_3x + 2\log_e 3$

$\therefore \quad P''(0) = 2\log_e 3$

Further, $P(x) = b$ or $a_3x^3 + (\log_e 3)x^2 = -4$ has only one real root, as the graph of $P(x) = a_3x^3 + (\log_e 3)x^2$ meets $y = -4$ only once for negative value of x.

For Problems 4–6

4. (1), 5. (3), 6. (4)

$$F(x) = \lim_{n\to\infty}\frac{f(x)+x^{2n}g(x)}{1+x^{2n}}$$

$$= \begin{cases} f(x), & 0 \le x^2 < 1 \\ \dfrac{f(x)+g(x)}{2}, & x^2 = 1 \\ g(x), & x^2 > 1 \end{cases}$$

$$= \begin{cases} g(x), & x < -1 \\ \dfrac{f(-1)+g(-1)}{2}, & x = -1 \\ f(x), & -1 < x < 1 \\ \dfrac{f(1)+g(1)}{2}, & x = 1 \\ g(x) & x > 1 \end{cases}$$

For continuity at $x = -1$,

$f(-1) = g(-1)$ or $1 - a + 3 = b - 1$ or $a + b = 5$ (1)

For continuity at $x = 1$,

$f(1) = g(1)$ or $1 + a + 3 = 1 + b$ or $a - b = -3$ (2)

Solving equations (1) and (2), we get $a = 1$ and $b = 4$.

$f(x) = 2g(x) \Rightarrow x^2 + x + 3 = 2x + 8$ or $x^2 - x - 5 = 0$

For Problems 7–9

7. (3), 8. (2), 9. (3)

For $0 \le x < \frac{\pi}{4}$, $g(x) = 1 + \tan x$

$$x \in \left[0, \frac{\pi}{4}\right) \Rightarrow 1 + \tan x \in [1, 2)$$

So, $f(g(x)) = f(1 + \tan x) = 1 + \tan x + 2$

and for $x \in \left[\frac{\pi}{4}, \pi\right)$, $g(x) = 3 - \cot x$

$$x \in \left[\frac{\pi}{4}, \pi\right) \Rightarrow 3 - \cot x \in [2, \infty)$$

So, $f(g(x)) = f(3 - \cot x) = 6 - (3 - \cot x)$

Let $h(x) = f(g(x)) = \begin{cases} 3 + \tan x, & 0 \le x < \frac{\pi}{4} \\ 3 + \cot x, & \frac{\pi}{4} \le x < \pi \end{cases}$

Clearly, $f(g(x))$ is continuous in $[0, \pi)$. Now,

$$h'\left(\frac{\pi}{4}^+\right) = \lim_{x\to\frac{\pi}{4}^+}(-\text{cosec}^2 x) = -2$$

$$h'\left(\frac{\pi}{4}^-\right) = \lim_{x\to\frac{\pi}{4}^-}(\sec^2 x) = 2$$

So, $f(g(x))$ is differentiable everywhere in $[0, \pi)$ other than at $x = \frac{\pi}{4}$.

$$|f(g(x))| = \begin{cases} |3 + \tan x|, & 0 \le x < \frac{\pi}{4} \\ |3 + \cot x|, & \frac{\pi}{4} \le x < \pi \end{cases}$$

which is non-differentiable at $x = \pi/4$ and where $3 + \cot x = 0$ or $x = \cot^{-1}(-3)$.

For $x \in \left[0, \frac{\pi}{4}\right)$, $3 + \tan x \in [3, 4)$.

For $x \in \left[\frac{\pi}{4}, \pi\right)$, $3 + \cot x \in (-\infty, 4]$.

Hence, the range is $(-\infty, 4]$.

For Problems 10–12

10. (1), 11. (4), 12. (2)

$$f(x) = \begin{cases} [x], & -2 \le x \le -\frac{1}{2} \\ 2x^2 - 1, & -\frac{1}{2} < x \le 2 \end{cases} = \begin{cases} -2, & -2 \le x < -1 \\ -1, & -1 \le x \le -\frac{1}{2} \\ 2x^2 - 1, & -\frac{1}{2} < x \le 2 \end{cases}$$

$$|f(x)| = \begin{cases} 2, & -2 \le x < -1 \\ 1, & -1 \le x \le -\frac{1}{2} \\ |2x^2 - 1|, & -\frac{1}{2} < x \le 2 \end{cases}$$

$$= \begin{cases} 2, & -2 \le x < -1 \\ 1, & -1 \le x \le -\frac{1}{2} \\ 1 - 2x^2, & -\frac{1}{2} < x \le \frac{1}{\sqrt{2}} \\ 2x^2 - 1, & \frac{1}{\sqrt{2}} < x \le 2 \end{cases}$$

$$f(|x|) = \begin{cases} -2, & -2 \le |x| < -1 \\ -1, & -1 \le |x| \le -\frac{1}{2} \\ 2|x|^2 - 1, & -\frac{1}{2} < |x| \le 2 \end{cases}$$

$$= 2x^2 - 1, \; -2 \le x \le 2$$

$$\therefore\quad g(x)=f(|x|)+|f(x)|=\begin{cases}2x^2+1, & -2\le x<-1\\ 2x^2, & -1\le x\le -\frac{1}{2}\\ 0, & -\frac{1}{2}<x<\frac{1}{\sqrt{2}}\\ 4x^2-2, & \frac{1}{\sqrt{2}}\le x\le 2\end{cases}$$

$g(-1^-)=\lim_{x\to -1}(2x^2+1)=3$, $g(-1^+)=\lim_{x\to -1}2x^2=2$

$g\left(-\frac{1}{2}^-\right)=\lim_{x\to -\frac{1}{2}}2x^2=\frac{1}{2}$, $g\left(-\frac{1}{2}^+\right)=\lim_{x\to -\frac{1}{2}}0=0$

$g\left(\frac{1}{\sqrt{2}}^-\right)=\lim_{x\to \frac{1}{\sqrt{2}}}0=0$, $g\left(\frac{1}{\sqrt{2}}^+\right)=\lim_{x\to \frac{1}{\sqrt{2}}}(4x^2-2)=0.$

Hence, $g(x)$ is discontinuous at $x=-1,-\frac{1}{2}$.

$g(x)$ is continuous at $x=\frac{1}{\sqrt{2}}$.

Now, $g'\left(\frac{1}{\sqrt{2}}^-\right)=0, g'\left(\frac{1}{\sqrt{2}}^+\right)=8\left(\frac{1}{\sqrt{2}}\right)=\frac{8}{\sqrt{2}}$

Hence, $g(x)$ is non-differentiable at $x=\frac{1}{\sqrt{2}}$.

For Problems 13–15

13. (3), 14. (4), 15. (2)

$$f(x)=\begin{cases}x^2+10x+8, & x\le -2\\ ax^2+bx+c, & -2<x<0, a\ne 0\\ x^2+2x, & x\ge 0\end{cases}$$

Continuous at $x=0 \Rightarrow c=0$

Continuous at $x=-2 \Rightarrow 4-20+8=4a-2b$

or $2a-b=-4$ (1)

Now, let the line $y=mx+p$ be tangent to all the three curves.

Solving $y=mx+p$ and $y=x^2+2x$, we get

$x^2+2x=mx+p$

$x^2+(2-m)x-p=0$

$D=0$

$\Rightarrow (2-m)^2+4p=0$ (2)

Again, solving $y=mx+p$ and $y=x^2+10x+8$, we get

$x^2+10x+8=mx+p$

or $x^2+(10-m)x+8-p=0$

$D=0$

$\Rightarrow (10-m)^2-4(8-p)=0$

or $(10-m)^2-(2-m)^2=32$

or $(100-20m)-(4-4m)=32$

or $m=4$ and $p=-1$

Hence, equation of the tangent to first and last curves is (3)

$y=4x-1$

Now, solving this with $y=ax^2+bx$ (as $c=0$),

$ax^2+bx=4x-1$ or $ax^2+(b-4)x+1=0$

$D=0$

$\Rightarrow (b-4)^2=4a$ [from (1)]

Also, $b=2a+4$

$\therefore\quad 4a^2=4a$ or $a=1$ and $b=6$ (as $a\ne 0$)

$f'(0^-)=\lim_{x\to 0}(2ax+b)=b$

$f'(0^+)=\lim_{x\to 0}(2x+2)=2$

$\therefore\quad b=2$

For Problems 16–17

16. (1), 17. (3)

Given that,

$$f(x)=\begin{cases}b\sin^{-1}\left(\frac{c+x}{2}\right), & -\frac{1}{2}<x<0\\ \frac{1}{2}, & x=0 \quad \text{where } |c|\le 1/2\\ \frac{e^{ax/2}-1}{x}, & 0<x<1/2\end{cases}$$

$f(x)$ is differentiable at $x=0$, then $f(x)$ will also be continuous at $x=0$

$\Rightarrow \lim_{h\to 0} f(0+h)=f(0)$

$\Rightarrow \lim_{h\to 0}\frac{e^{ah/2}-1}{h}=\frac{1}{2}$

$\Rightarrow \lim_{h\to 0}\frac{e^{ah/2}-1}{\frac{ah}{2}}\times\frac{a}{2}=\frac{1}{2}\Rightarrow a=1$

Also $Lf'(0)=Rf'(0)$

$\Rightarrow \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$

$\Rightarrow \lim_{h\to 0}\frac{b\sin^{-1}\left(\frac{c-h}{2}\right)-\frac{1}{2}}{-h}=\lim_{h\to 0}\frac{\frac{e^{\frac{ah}{2}}-1}{h}-\frac{1}{2}}{h}$

$=\lim_{h\to 0}\frac{2e^{ah/2}-2-h}{2h^2}$

For these limits to exist we must get the form 0/0 and hence using L Hospital's rule, we get

$$\lim_{h\to 0}\frac{\left(-\frac{1}{2}\right)\frac{b}{\sqrt{1-\left(\frac{c-h}{2}\right)^2}}}{-1}$$

$=\lim_{h\to 0}\frac{2.e^{ah/2}(a/2)-1}{4h}$

$=\lim_{h\to 0}\frac{e^h/2-1}{8(h/2)}$ [Putting $a=1$]

$\Rightarrow \frac{b}{2\sqrt{1-\frac{c^2}{4}}}=\frac{1}{8}$

$\Rightarrow 4b=\sqrt{1-\frac{c^2}{4}}$

$\Rightarrow 16b^2=\frac{4-c^2}{4}$

$\Rightarrow 64b^2=4-c^2$

For Problems 18–20

18. (1), 19. (4), 20. (2)

$$f(x)=\frac{(x^2-1)(x^2-4)}{g(x)}, x\in R-\{a,b\}$$

Given $\lim_{x\to\pm\infty} f(x) = \infty$

$\therefore$ degree of $g(x)$ must be less than 4.

Also $\lim_{x\to -1} f(x) = 6$

$\therefore \quad \lim_{x\to -1} \frac{(x^2-1)(x^2-4)}{g(x)} = 6$

$\therefore$ $g(x)$ must have $(x + 1)$ as a factor.

Also $\lim_{x\to -2} f(x) \neq 0$ but finite

$\Rightarrow$ $g(x)$ must have $(x + 2)$ another factor.

$\Rightarrow$ $g(x) = k(x + 1)(x + 2)$

$\Rightarrow \quad \lim_{x\to -1} \frac{(x^2-1)(x^2-4)}{k(x+1)(x+2)} = 6$

$\Rightarrow \quad \lim_{x\to -1} \frac{(x-1)(x-2)}{k} = 6$

$\Rightarrow \quad k = 1$

Thus $f(x) = \frac{(x^2-1)(x^2-4)}{g(x)}$

$= \frac{(x^2-1)(x^2-4)}{(x+1)(x+2)}, x \neq 1, 2$

$= (x - 1)(x - 2), x \neq 1, 2$

$|f(x)| = |(x - 1)(x - 2)|$ is non-differentiable at $x = 1$

and $x = 2$

$f(|x|) = (|x| - 1)(|x| - 2)$ is non-differentiable at $|x| = 1, 2$

and $x = 0$

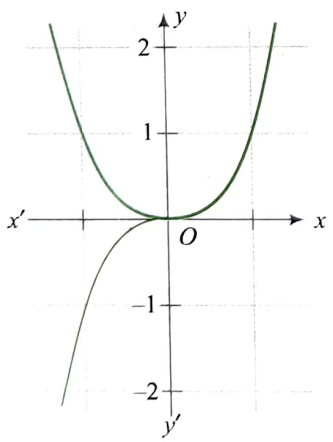
or $x = \pm 1, \pm 2, 0$

$f(-|x|) = (-|x| - 1)(-|x| - 2) = (|x| + 1)(|x| + 2)$ is non-differentiable at $x = 0$ only.

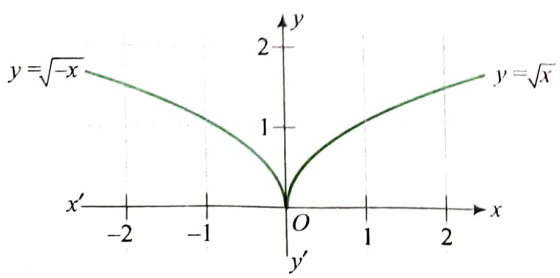
Matrix Match Type

1. a → p, q, r; b → p, r, s; c → p, r, s; d → p, r, s.

a. $f(x) = |x^3| = x(x|x|)$ is continuous and differentiable.

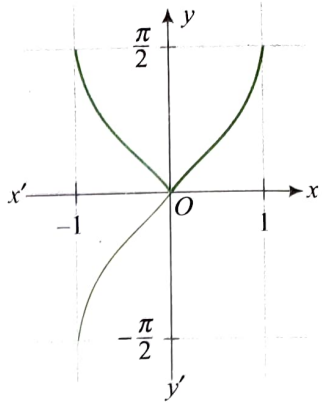


b. $f(x) = \sqrt{|x|}$ is continuous.



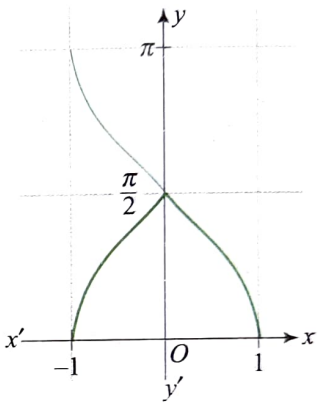
Clearly, from the graph, $f(x)$ is non-differentiable at $x = 0$.

c. $f(x) = |\sin^{-1} x|$ is continuous.



Clearly, from the graph, $f(x)$ is non-differentiable at $x = 0$.

d. $f(x) = \cos^{-1}|x|$ is continuous.

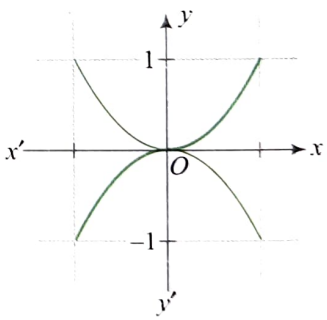


Clearly, from the graph, $f(x)$ is non-differentiable at $x = 0$.

2. a → p, q, r; b → p, s; c → r, s; d → p,q.

a→p, q, r.

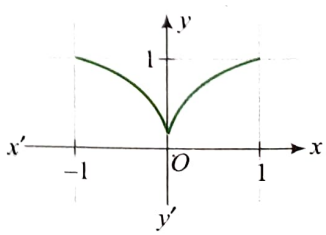
$y = x|x|$



From the graph, $f(x)$ is continuous and differentiable in $(-1, 1)$. Also, $f(x)$ is strictly increasing.

b→p, s.

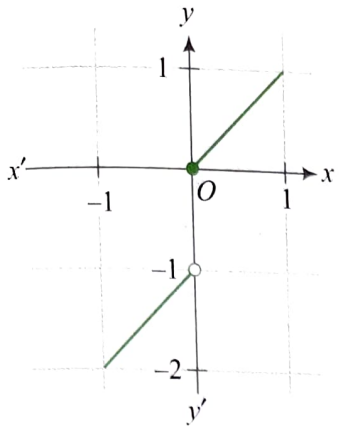
$y = \sqrt{|x|}$



From the graph, $f(x)$ is continuous in $(-1, 1)$, but non differentiable at $x = 0$.

c→r, s.

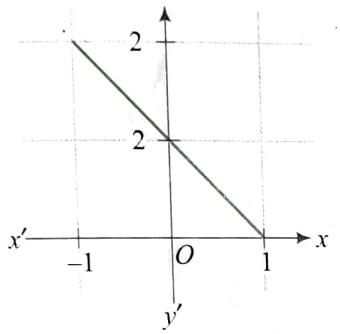
$y = x + [x]$



From the graph, $f(x)$ is discontinuous at $x = 0$. Also, $f(x)$ is increasing.

d→p, q.

$y = |x - 1|$



From the graph, $f(x)$ is continuous and differentiable in $(-1, 1)$.

3. **a → q, s; b → p, s; c → p, r; d → q, s.**

a. $f(x) = \begin{cases} \dfrac{5e^{1/x}+2}{3-e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$$f(0^+) = \lim_{h\to 0} \frac{5e^{1/h}+2}{3-e^{1/h}}$$

$$= \lim_{h\to 0} \frac{5+2e^{-1/h}}{3e^{-1/h}-1} = -5$$

Hence, $f(x)$ is discontinuous and non-differentiable at $x = 0$.

b. $g(x) = xf(x) = \begin{cases} x\dfrac{5e^{1/x}+2}{3-e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$$f(0^+) = \lim_{h\to 0} h\frac{5e^{1/h}+2}{3-e^{1/h}}$$

$$= \lim_{h\to 0} h\frac{5+2e^{-1/h}}{3e^{-1/h}-1} = 0 \times (-5) = 0.$$

$$f(0^-) = \lim_{h\to 0} h\frac{5e^{-1/h}+2}{3-e^{-1/h}} = 0 \times (2/3) = 0$$

Hence, $f(x)$ is continuous at $x = 0$.

$$Lg'(0) = \lim_{h\to 0} \frac{g(0-h)-g(0)}{-h}$$

$$= \lim_{h\to 0} \frac{-hf(-h)-0}{-h} = \lim_{h\to 0} f(-h)$$

$$= \lim_{h\to 0} \frac{5e^{-1/h}+2}{3-e^{-1/h}} = \frac{0+2}{3-0} = \frac{2}{3}$$

$$Rg'(0) = \lim_{h\to 0} \frac{g(0+h)-g(0)}{h}$$

$$= \lim_{h\to 0} \frac{g(h)-0}{h}$$

$$= \lim_{h\to 0} f(h) = \lim_{h\to 0} \frac{5e^{1/h}+2}{3-e^{1/h}}$$

$$= \lim_{h\to 0} \frac{5+2e^{-1/h}}{3e^{-1/h}-1}$$

$$= \frac{5+0}{0-1} = -5$$

$\therefore \quad LF'(0) \neq RF'(0)$

Hence, $F(x)$ is not differentiable, but continuous at $x = 0$.

c. For $x^2 f(x)$, Let $F(x) = x^2 f(x)$

Clearly, $F(x) = x(xf(x))$ is continuous at $x = 0$.

$$\therefore \quad LF'(0) = \lim_{h\to 0} \frac{F(0-h)-F(0)}{-h}$$

$$= \lim_{h\to 0} \frac{h^2 f(-h)-0}{-h} = 0$$

$$RF'(0) = \lim_{h\to 0} \frac{F(0+h)-F(0)}{h}$$

$$= \lim_{h\to 0} \frac{h^2 f(h)-0}{h} = 0$$

$\therefore \quad LF'(0) = RF'(0)$

Hence, $F(x)$ is differentiable at $x = 0$.

d. Clearly, from the above discussion, $y = x^{-1} f(x)$ is discontinuous and, hence, non-differentiable at $x = 0$.

4. **a → r, s; b → p, q; c → p, q; d → p, r.**

a. The given function is clearly continuous at all points except possibly at $x = \pm 1$.

As $f(x)$ is an even function, we need to check its continuity only at $x = 1$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$$\text{or} \quad \lim_{x\to 1^-} (ax^2 + b) = \lim_{x\to 1^+} \frac{1}{|x|} \text{ or } a + b = 1 \qquad (1)$$

Clearly, $f(x)$ is differentiable for all x, except possibly at $x = \pm 1$. As $f(x)$ is an even function, we need to check its differentiability at $x = 1$ only.

$$\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1}$$

$$\text{or} \quad \lim_{x\to 1} \frac{ax^2+b-1}{x-1} = \lim_{x\to 1^+} \frac{\frac{1}{|x|}-1}{x-1}$$

$$\text{or} \quad \lim_{x\to 1} \frac{ax^2-a}{x-1} = \lim_{x\to 1} \frac{-1}{x}$$

$$\text{or} \quad 2a = -1 \text{ or } a = -\frac{1}{2}$$

Putting $a = -1/2$ in (1), we get $b = 3/2$ or $|k| = 1$ or $k = \pm 1$.

b. If $f(x) = \text{sgn}(x^2 - ax + 1)$ is discontinuous at exactly one point, then $x^2 - ax + 1 = 0$ must have only one real root. Hence, $a = \pm 2$.

c. $f(x) = [2 + 3|n| \sin x]$, $n \in N$, has exactly 11 points of discontinuity in $x \in (0, \pi)$.

The required number of points are

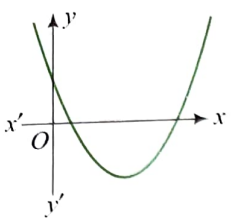
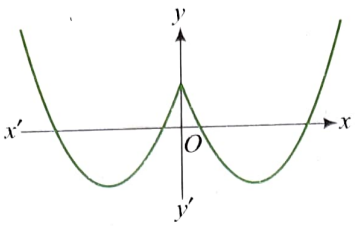
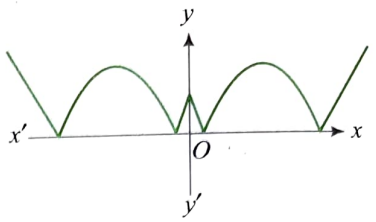
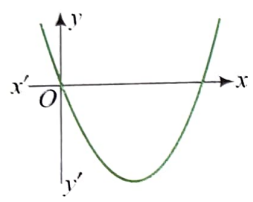
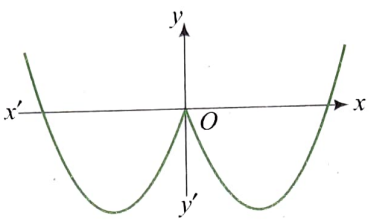
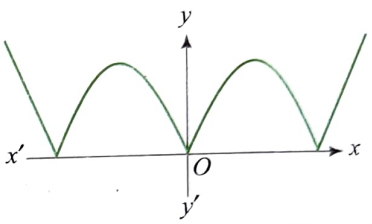
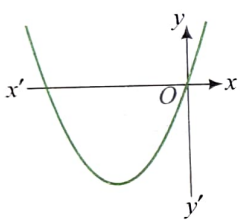
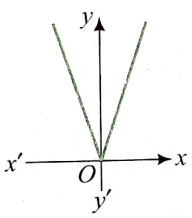
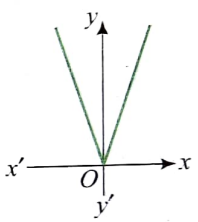
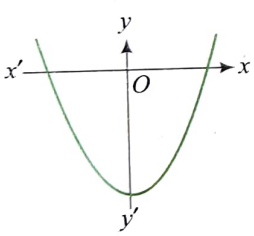
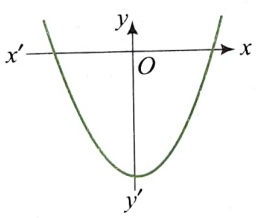
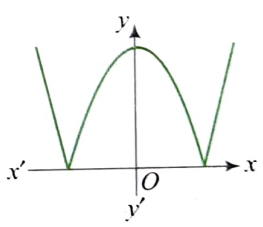
$$1 + 2(3|n| - 1) = 6|n| - 1 = 11 \text{ or } n = \pm 2$$

d. $f(x) = |||x| - 2| + a|$ has exactly three points of non-differentiability.

$f(x)$ is non-differentiable at $x = 0$, $|x| - 2 = 0$, or $x = 0, \pm 2$.

Hence, the value of a must be positive, as negative value of a allows $||x| - 2| + a = 0$ to have real roots, which gives more points of non-differentiability.

5. **a → s; b → r; c → p; d → q.**

$f(x) = x^2 + bx + c$	$f(\|x\|) = x^2 + b\|x\| + c$	$g(x) = \|f(\|x\|)\| = \|x^2 + b\|x\| + c\|$
$b < 0, c > 0$ 		
$c = 0, b < 0$ 		
$c = 0, b > 0$ 		
$b = 0, c < 0$ 		

6. **a → q, s; b → p, r; c → p, r; d → p, s.**

a. $f(x) = \lim_{n\to\infty}\left[\cos^2(2\pi x)\right]^n + \left\{x + \frac{1}{2}\right\}$

Obviously, $\lim_{x\to\frac{1}{2}^+} f(x) = 0 + 0 = 0$

and $\lim_{x\to\frac{1}{2}^-} f(x) = 0 + 1$

Therefore, $f(x)$ is discontinuous at $x = \frac{1}{2}$.

b. $f(x) = (\log x)(x-1)^{1/5}$

Obviously, $f(x)$ is continuous at $x = 1$.

$$f'(1^+) = \lim_{h\to 0}\frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0}\frac{\log(1+h)h^{1/5}}{h} = 0$$

$$f'(1^-) = \lim_{h\to 0}\frac{f(1-h) - f(1)}{h}$$

$$= \lim_{h\to 0}\frac{\log(1-h)(-h)^{1/5}}{-h} = 0$$

Hence, $f(x)$ is differentiable at $x = 1$.

c. $f(x) = [\cos 2\pi x] + \sqrt{\left\{\sin\left(\frac{\pi x}{2}\right)\right\}}$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1^-}[\cos 2\pi x] + \lim_{x\to 1^-}\sqrt{\left\{\sin\left(\frac{\pi x}{2}\right)\right\}}$$

$$= 0 + 1 = 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}[\cos(2\pi x)] + \lim_{x\to 1^+}\sqrt{\left\{\sin\left(\frac{\pi x}{2}\right)\right\}}$$

$$= 0 + 1 = 1$$

Also, $f(1) = 1 + 0 = 1$.

$f(x)$ is continuous at $x = 1$.

$$f'(1^+) = \lim_{h\to 0} \frac{[\cos 2\pi(1+h)] + \sqrt{\left\{\sin\left(\frac{\pi(1+h)}{2}\right)\right\}} - 1}{h}$$

$$= \lim_{h\to 0} \frac{[\cos 2\pi h] + \sqrt{\left\{\cos\left(\frac{\pi h}{2}\right)\right\}} - 1}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{\cos\left(\frac{\pi h}{2}\right)} - 1}{h} = \lim_{h\to 0} \frac{-\frac{\pi}{2}\sin\left(\frac{\pi h}{2}\right)}{2\sqrt{\cos\left(\frac{\pi h}{2}\right)}} = 0$$

Similarly, $f'(1^-) = 0$.

d. $f(x) = \begin{cases} \cos 2x, & x \in Q \\ \sin x, & x \notin Q \end{cases}$ at $\frac{\pi}{6}$.

$f(x)$ is continuous when $\cos 2x = \sin x$ which has $x = \frac{\pi}{6}$ as one of the solutions. Hence, it is continuous.

Also, in the neighborhood of $x = \frac{\pi}{6}$,

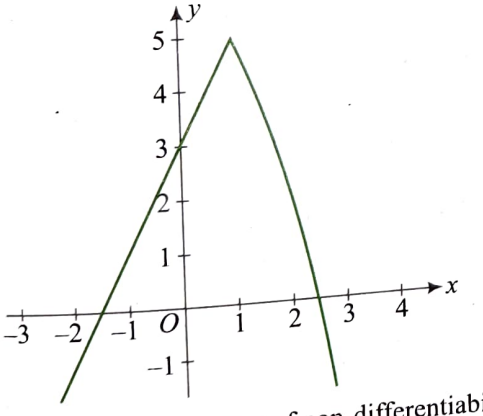
$$f'(x) = \begin{cases} -2\sin 2x, & \frac{\pi}{6} - \delta < x < \frac{\pi}{6} \\ \cos x, & \frac{\pi}{6} < x < \frac{\pi}{6} + \delta \end{cases}$$

Here, $f'\left(\frac{\pi^-}{6}\right) \neq f'\left(\frac{\pi^+}{6}\right)$.

Therefore, $f(x)$ is not differentiable at $x = \frac{\pi}{6}$.

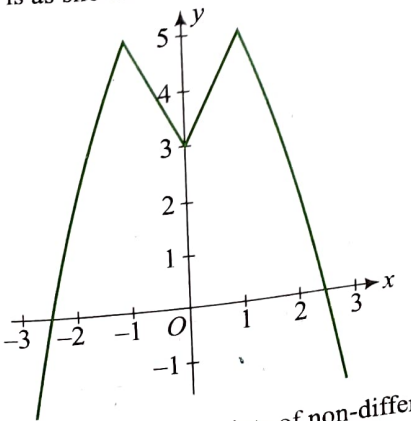
7(3) $f(x) = \begin{cases} 2x+3, & x \le 1 \\ -x^2+6, & x > 1 \end{cases}$

Graph of $y = f(x)$ is as shown in the following figure.



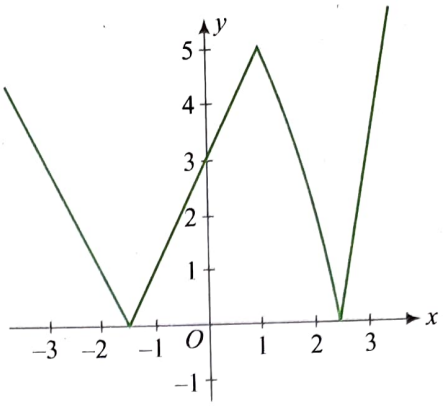
From the graph $f(x)$ has one point of non-differentiability.

Graph of $f(|x|)$ is as shown in the following figure.



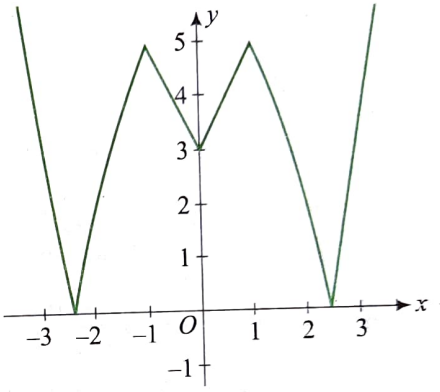
oints of non-differentiability.

Graph of $|f(x)|$ is as shown in the following figure.



From the graph $|f(x)|$ has three points of non-differentiability.

Graph of $|f(|x|)|$ is as shown in the following figure.



From the graph $|f(|x|)|$ has five points of non-differentiability.

Numerical Value Type

1. (5) $f(x) = \text{sgn}(\sin x)$ is discontinuous when $\sin x = 0$ or $x = 0, \pi, 2\pi, 3\pi, 4\pi$.

2. (4) $\text{sgn}(x^2 - 3x + 2)$ is discontinuous when $x^2 - 3x + 2 = 0$ or $x = 1, 2$.

$[x - 3] = [x] - 3$ is discontinuous at $x = 1, 2, 3, 4$.

Thus, $f(x)$ is discontinuous at $x = 3, 4$.

Now, both $\text{sgn}(x^2 - 3x + 2)$ and $[x - 3]$ are discontinuous at $x = 1$ and 2.

Then $f(x)$ may be continuous at $x = 1$ and 2.

But $f(1) = -2$ and $f(1^-) = -1 + 0 - 3 = -4$.

Thus, $f(x)$ is discontinuous at $x = 1$.

Also, $f(2) = -1$ and $f(2^+) = 1 - 1 = 0$.

Hence, $f(x)$ is discontinuous at $x = 2$ also.

3. (4) $f(x) = \sqrt{x^2} + [x]^2 = |x| + [x]^2$

$|x|$ is continuous function but $[x]$ is discontinuous at $x = -1, 0, 1, 2$

Hence $f(x)$ is discontinuous at $x = -1, 0, 1, 2$

4. (12) $f(x) = (x^3 + bx^2 + cx + d)|x - 1||x - 2|(x - 3)^2|x - 4|(x - 4)^2$

$f(x)$ is differentiable for all real x, if $(x - 1)$ and $(x - 2)$ are factors of $g(x)$.

Clearly $f'(3) = 0$.

$\therefore$ $f'''(4) = 0$, for which $(x - 4)$ must be factor of $g(x)$.

$\therefore$ $g(x) = x^3 + bx^2 + cx + d = (x - 1)(x - 2)(x - 4)$

So, $g(5) = 4 \times 3 \times 1 = 12$

5. (8) $f(x)=\begin{cases} ax^2+bx, & \text{for } -1<x<1 \\ \dfrac{a-b-1}{2}, & x=-1 \\ \dfrac{a+b+1}{2}, & x=1 \\ \dfrac{1}{x}, & \text{for } x>1 \text{ or } x<-1 \end{cases}$

For continuity at $x=1$, we have $a+b=\dfrac{a+b+1}{2}$.

Hence, $a+b=1$. (1)

For continuity at $x=-1$,

$a-b=-1$ (2)

Hence, $a=0$ and $b=1$.

6. (6) $g(f(x))=\begin{cases} g\left(\dfrac{x}{2}-1\right), 0\le x<1 \\ g\left(\dfrac{1}{2}\right), 1\le x\le 2 \end{cases}$

$$=\begin{cases} \dfrac{(x-1)(x-2-2k)}{2}+3, 0\le x<1 \\ 4-2k, 1\le x<2 \end{cases}$$

$\lim\limits_{x\to 1^-} g(f(x))=3, g(f(1))=4-2k$

and $\lim\limits_{x\to 1^+} g(f(x))=4-2k$

for $g(f(x))$ to be continuous at $x=1$, $4-2k=3$ or $k=\dfrac{1}{2}$.

7. (8) $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{f(x)+f(h)+2xh(x+h)-\frac{1}{3}-\left(f(x)+f(0)-\frac{1}{3}\right)}{h}$$

$$=\lim_{h\to 0}\frac{f(h)-f(0)}{h}+2x^2$$

$$=f'(0)+2x^2$$

$$\lim_{h\to 0}\frac{3f(h)-1}{6h}=\lim_{h\to 0}\frac{f(h)-\frac{1}{3}}{2h}=\lim_{h\to 0}\frac{f(h)-f(0)}{2h}$$

$$=\frac{f'(0)}{2}=\frac{2}{3}$$

$\therefore \quad f'(0)=\dfrac{4}{3}$

$\therefore \quad f'(x)=\dfrac{4}{3}+2x^2$

$f(x)=\lambda+\dfrac{4}{3}x+\dfrac{2x^3}{3}$ or $f(0)=\lambda=\dfrac{1}{3}$

$\therefore \quad f(x)=\dfrac{2x^3}{3}+\dfrac{4}{3}x+\dfrac{1}{3}$ or $f(2)=\dfrac{25}{3}$

8. (5) $\because \quad f(x)=\begin{cases} x^p\sin\left(\dfrac{1}{x}\right)+x^2, & x>0 \\ x^p\sin\left(\dfrac{1}{x}\right)-x^2, & x<0 \\ 0, & x=0 \end{cases}$

$$f''(x)=\begin{cases} -x^{p-4}\sin\left(\dfrac{1}{x}\right)-(p-2)x^{p-3}\cos\left(\dfrac{1}{x}\right) \\ -px^{p-3}\cos\left(\dfrac{1}{x}\right) \\ +p(p-1)x^{p-2}\sin\left(\dfrac{1}{x}\right)+2, & x>0 \\ -x^{p-4}\sin\left(\dfrac{1}{x}\right)-(p-2)x^{p-3}\cos\left(\dfrac{1}{x}\right) \\ px^{p-3}\cos\left(\dfrac{1}{x}\right) \\ +p(p-1))x^{p-2}\sin\left(\dfrac{1}{x}\right)-2, & x<0 \\ 0, & x=0 \end{cases}$$

RHL = LHL = $f(0)=0$

since sin ∞ and cos ∞ lie between –1 to 1, for $p\ge 5$, RHL = 2

LHL = –2.

$f(0)=0$

For $p\in[5,\infty)$, $f''(x)$ is not continuous.

9. (8) We have $f(x)=[x]+[x+1/3]+[x+2/3]=[3x]$,

which is discontinuous when $3x=k$ or $x=k/3$, $k\in I$.

Hence, points of discontinuity are 1/3, 2/3, 3/3, 4/3, 5/3, 6/3, 7/3, 8/3.

10. (1) $\lim\limits_{x\to 1^-} h(x)=\lim\limits_{x\to 1^-}\lim\limits_{n\to\infty}\dfrac{x^{2n}\cdot f(x)+x^{2m}\cdot g(x)}{(1+x^{2n})}=g(1)$

$$\lim_{x\to 1^-} h(x)=\lim_{x\to 1^+}\lim_{n\to\infty}\frac{x^{2n}\cdot f(x)+x^{2m}\cdot g(x)}{(1+x^{2n})}=f(1)$$

$\because \quad \lim\limits_{x\to 1} h(x)$ exists

$f(1)=g(1)$

Thus, $f(x)-g(x)=0$ has a root at $x=1$.

11. (11) $g(x)=x^3+3x^2+7x+2$

$g'(x)=3x^2+6x+7>0$ for all real x as its discriminant is negative.

Thus, $g(x)$ is an increasing function.

$g(0)=2$ and $g(1)=1+3+7+2=13$

Thus, $f(x)=[g(x)]$ is discontinuous when $g(x)=3, 4, \ldots, 13$.

So, there are 11 points of continuity

12. (7) Let $g(x)=(\ln x)(\ln x)\cdots\infty$.

$$g(x)=\begin{cases} 0, & 1<x<e \\ 1, & x=e \\ \infty, & x>e \end{cases}$$

$$\therefore \quad f(x)=\begin{cases} x, & 1<x<e \\ x/2, & x=e \\ 0, & e<x<3 \end{cases}$$

Hence, $f(x)$ is non-differentiable at $x=e$.

13. (2) $f(0)=\lim\limits_{x\to 0}\dfrac{\tan(\tan x)-\sin(\sin x)}{\tan x-\sin x}$

$$=\lim_{x\to 0}\frac{\tan(\tan x)-\sin(\sin x)}{\frac{\tan x}{x}\left(\frac{1-\cos x}{x^2}\right)x^3}$$

$$=2\lim_{x\to 0}\frac{\tan(\tan x)-\sin(\sin x)}{x^3}$$

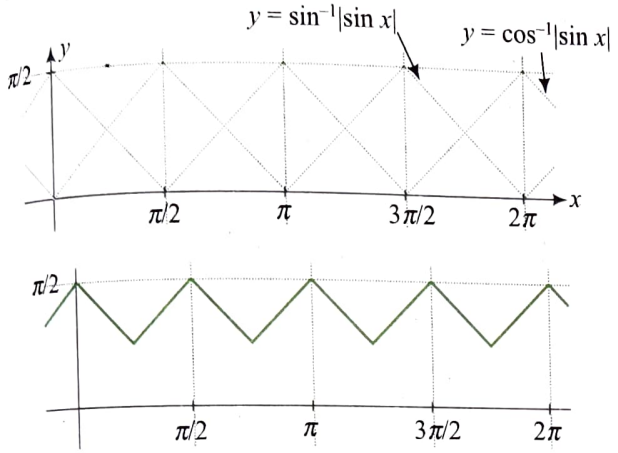
$$= 2\lim_{x\to 0}\frac{\left(\tan x+\frac{\tan^3 x}{3}+\frac{2}{15}\tan^5 x+\cdots\right)-\left(\sin x-\frac{\sin^3 x}{3!}+\frac{\sin^5 x}{5!}\cdots\right)}{x^3}$$

$$= 2\lim_{x\to 0}\left(\left(\frac{\tan x-\sin x}{x^3}\right)+\frac{\left(\frac{\tan^3 x}{3}+\frac{\sin^3 x}{3!}\right)}{x^3}+\cdots\right)$$

$$= 2\lim_{x\to 0}\left(\left(\frac{\tan x}{x}\right)\left(\frac{1-\cos x}{x^2}\right)+\frac{1}{3}+\frac{1}{6}\right)$$

$$= \left[\frac{1}{2}+\frac{1}{2}\right]2$$

14. (7) $\sin^{-1}|\sin x|$ is periodic with period π.



15. (6) $f(x) = \begin{cases} 1, & x \le 0 \\ -1, & x > 0 \end{cases}$

$f(x) = \begin{cases} 1, & x < 0 \\ -1, & x \ge 0 \end{cases}$

$f(x) = \begin{cases} -1, & x \le 0 \\ 1, & x > 0 \end{cases}$

$f(x) = \begin{cases} -1, & x < 0 \\ 1, & x \ge 0 \end{cases}$

$f(x) = \begin{cases} 1, & x > 0 \\ 1, & x < 0 \\ -1 & x = 0 \end{cases}$

$f(x) = \begin{cases} -1, & x > 0 \\ -1, & x < 0 \\ 1 & x = 0 \end{cases}$

Hence, there are six functions.

Archives

JEE Main

Single Correct Answer Type

1. (3) $f(x) = x|x|$ and $g(x) = \sin x$

$$gof(x) = \sin(x|x|) = \begin{cases} -\sin x^2, & x < 0 \\ \sin x^2, & x \ge 0 \end{cases}$$

$$\therefore \quad (gof)'(x) = \begin{cases} -2x\cos x^2, & x < 0 \\ 2x\cos x^2, & x > 0 \end{cases}$$

Clearly, $L(gof)'(0) = 0 = R(gof)'(0)$

Therefore, gof is differentiable at $x = 0$ and also its derivative is continuous at $x = 0$.

Now,

$$(gof)''(x) = \begin{cases} -2\cos x^2 + 4x^2 \sin x^2, & x < 0 \\ 2\cos x^2 + 4x^2 \sin x^2, & x > 0 \end{cases}$$

$\therefore$ $L(gof)''(0) = -2$ and $R(gof)''(0) = 2$

Hence $gof''(2)$ does not exist. (1)

2. (4) At $x = 0$, $f(0) = q$

For $x < 0$,

$$\text{L.H.L.} = \lim_{x\to 0-} f(x) = \lim_{h\to 0} f(0-h)$$

$$= \lim_{h\to 0}\frac{\sin(p+1)(0-h)+\sin(0-h)}{(0-h)}$$

$$= \lim_{h\to 0}\frac{-\sin(p+1)h-\sin h}{-h}$$

$$= \lim_{h\to 0}\left\{(p+1)\frac{\sin(p+1)h}{(p+1)h}+\frac{\sinh}{h}\right\}$$

$$= (p+1)\lim_{h\to 0}\frac{\sin(p+1)h}{(p+1)h}+\lim_{h\to 0}\frac{\sinh}{h}$$

$$= (p+1)\cdot 1 + 1 = p+2 \quad (2)$$

For $x > 0$,

$$\text{R.H.L.} = \lim_{x\to 0+} f(x) = \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0}\frac{(h+h^2)^{1/2}-h^{1/2}}{h^{3/2}}$$

$$= \lim_{h\to 0}\frac{(1+h)^{1/2}-1}{h}$$

$$= 1/2 \quad (3)$$

$f(x)$ is continuous at $x = 0$

$\Rightarrow$ L.H.L. = R.H.L. = $f(0)$ {from (1), (2) and (3)}

$\Rightarrow$ $p + 2 = 1/2 = q$

$\Rightarrow$ $p = -\frac{3}{2}, q = \frac{1}{2}$

3. (1) Function possibly discontinuous at $x = n$, $n \in I$

$$\text{L.H.L.} = \lim_{x\to n^-}[x]\cos\left(\left(\frac{2x-1}{2}\right)\pi\right)$$

$$= (n-1)\cos\left(\left(\frac{2n-1}{2}\right)\pi\right) = 0$$

$$\text{R.H.L.} = \lim_{x\to n^+}[x]\cos\left(\left(\frac{2x-1}{2}\right)\pi\right)$$

$$= n\cos\left(\frac{2n-1}{2}\pi\right) = 0$$

Also $f(n) = 0$

Therefore f is continuous at $x = n$, $n \in I$ and hence continuous everywhere.

4. (1) $g(x) = \begin{cases} k\sqrt{x+1}, & 0 \le x \le 3 \\ mx+2, & 3 < x \le 5 \end{cases}$

$$L(g'(3)) = \lim_{x\to 3^-}\frac{k\sqrt{x+1}-2k}{x-3} = \lim_{x\to 3^-} k\left\{\frac{(x+1-4)}{(x-3)(\sqrt{x+1}+2)}\right\} = \frac{k}{4}$$

$$R(g'(3)) = \lim_{x\to 3^+}\frac{mx+2-2k}{x-3}$$

Since this limit exists, $3m + 2 - 2k = 0$

$\Rightarrow$ $2k = 3m + 2$ (i)

So $R(g'(3)) = m$ by L'Hospital's rule.

Since $g(x)$ is differentiable, $k = 4m$ (ii)

Solving (i) and (ii), we get

$$m = \frac{2}{5}, \text{ and } n = \frac{8}{5}$$

$$\Rightarrow \quad k + m = 2$$

5. (1) $f(x) = |\log_e 2 - \sin x|$

$\therefore \quad f(f(x)) = |\log_e 2 - \sin |\log_e 2 - \sin x||$

In the neighborhood of $x = 0$

$$g(x) = \log_e 2 - \sin (\log_e 2 - \sin x)$$

Hence $g(x)$ is differentiable at $x = 0$ as it is sum and composite of differentiable function.

$$g'(x) = \cos (\log_e 2 - \sin x). \cos x$$

$\therefore \quad g'(0) = \cos (\log_e 2)$

6. (2) If $f(x) = h(x) \cdot g(x)$ and $h(x)$ is non-differentiable at $x = a$, but $g(a) = 0$, then $f(x)$ will be differentiable at $x = a$.

Now, $f(x) = |x - \pi| \times (e^{|x|} - 1) \sin |x|$

$|x - \pi|$ is non-differentiable at $x = \pi$.

But $\sin|\pi| = 0$.

So, $f(x)$ is differentiable at $x = \pi$.

$e^{|x|} - 1$ is non-differentiable at $x = 0$.

But $\sin|0| = 0$.

So, $f(x)$ is differentiable at $x = 0$.

Similarly, $\sin |x|$ is non-differentiable at $x = 0$.

But $e^{|0|} - 1 = 0$.

Thus, $f(x)$ is differentiable for all real x.

JEE ADVANCED

Single Correct Answer Type

1. (2) Clearly, $f(x)$ is continuous at $x = 0$.

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \left|\cos \frac{\pi}{h}\right| - 0}{h}$$

$$= \lim_{h \to 0} h \cos\left(\frac{\pi}{h}\right) = 0$$

So, $f(x)$ is differentiable at $x = 0$.

$$f'(2^+) = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{(2 + h)^2 \left|\cos \frac{\pi}{2 + h}\right| - 0}{h}$$

$$= \lim_{h \to 0} \frac{(2 + h)^2 \cos\left(\frac{\pi}{2 + h}\right)}{h}$$

$$= \lim_{h \to 0} \frac{(2 + h)^2}{h} \sin\left(\frac{\pi}{2} - \frac{\pi}{2 + h}\right)$$

$$= \lim_{h \to 0} \frac{(2 + h)^2}{\frac{\pi h}{2(2 + h)}} \sin \frac{\pi h}{2(2 + h)} \times \frac{\pi}{2(2 + h)} = \pi$$

Again, $f'(2^-) = \lim_{h \to 0} \frac{f(2 - h) - f(2)}{-h}$

$$= \lim_{h \to 0} \frac{(2 - h)^2 \left|\cos\left(\frac{\pi}{2 - h}\right)\right|}{-h}$$

$$= \lim_{h \to 0} \frac{-(2 - h)^2 \sin\left[\frac{\pi}{2} - \frac{\pi}{2 - h}\right]}{-h}$$

$$= \lim_{h \to 0} \frac{(2 - h)^2}{h} \cdot \sin\left[\frac{-\pi h}{2(2 - h)}\right]$$

$$= -\lim_{h \to 0} \frac{(2 - h)^2}{\frac{\pi h}{2(2 - h)}} \cdot \sin \frac{\pi h}{2(2 - h)} \times \frac{\pi}{2(2 - h)} = -\pi$$

Hence, $f(x)$ is not differentiable at $x = 2$.

Multiple Correct Answers Type

1. (1), (2), (3), (4)

Clearly, $f(x)$ is continuous at $x = -\frac{\pi}{2}$ 0, 1.

$$f'(x) = \begin{cases} -1, & x < -\pi/2 \\ \sin x, & -\pi/2 < x < 0 \\ 1, & 0 < x < 1 \\ 1/x, & x > 1 \end{cases}$$

Clearly, $f(x)$ is not differentiable at $x = 0$ as $f'(0^-) = 0$ and $f'(0^+) = 1$.

$f(x)$ is differentiable at $x = 1$ as $f'(1^-) = f'(1^+) = 1$.

2. (2), (3)

$\because \quad f(0) = 0$

and $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{f(h)}{h}$$

$$= f'(0) = k \text{ (say)}$$

We have $f(x) = kx + c$ or $f(x) = kx$ $[\because f(0) = 0]$

3. (1), (4)

$g(0) = 0$, $g'(0) = 0$ and $g'(1) \neq 0$

$$f(x) = \begin{cases} g(x); & x > 0 \\ -g(x); & x < 0 \\ 0; & 0 \end{cases}$$

$$\therefore \quad f'(x) = \begin{cases} g'(x), & x > 0 \\ -g'(x), & x < 0 \end{cases}$$

$f'(0^+) = g'(0^+) = 0$

$f'(0^-) = -g'(0^-) = 0$

Hence, $f(x)$ is differentiable at $x = 0$.

$$h(x) = e^{|x|} = \begin{cases} e^{-x}, & x < 0 \\ e^{x}, & x \geq 0 \end{cases}$$

$$\therefore \quad h'(x) = \begin{cases} -e^{-x}, & x < 0 \\ e^{x}, & x > 0 \end{cases}$$

$h'(0^+) = e^0 = 1$

$h'(0^-) = -e^0 = -1$

Hence, $h(x)$ is non-differentiable at $x = 0$.

Now, $f(h(x)) = g(e^x), \forall x \in R.$

$$\begin{cases} g(e^x), & x \geq 0 \\ g(e^{-x}), & x < 0 \end{cases}$$

$$(f(h(x)))' = \begin{cases} e^x g'(e^x), & x > 0 \\ -e^{-x} g'(e^{-x}), & x < 0 \end{cases}$$

$(f(h(0^+)))' = e^0 g'(e^0) = g'(1)$

$(f(h(0^-)))' = -e^0 g'(e^0) = -g'(1)$

Since $g'(1) \neq 0$, $f(h(x))$ is non-differentiable at $x = 0$.

$$h(f(x)) = \begin{cases} e^{|f(x)|}, & x \neq 0 \\ 1, & x = 0 \end{cases} = \begin{cases} e^{|g(x)|}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Given $g(0) = 0$, $g'(0) = 0$.

Hence, $x = 0$ is repeated root of $g(x) = 0$.

Therefore, $h(f(x))$ is differentiable at $x = 0$.

4. (1), (2)

$f(x) = a \cos(|x^3 - x|) + b|x| \sin(|x^3 + x|)$

If $a = 0, b = 1, f(x) = |x| \sin(|x^3 + x|)$

$\Rightarrow \quad f(x) = x \sin(x^3 + x) \; \forall \; x \in R$

Hence $f(x)$ is differentiable.

If $a = 1, b = 0, f(x) = \cos(|x^3 - x|)$

$\Rightarrow \quad f(x) = \cos(x^3 - x)$

Which is differentiable at $x = 1$ and $x = 0$.

If $a = 1, b = 1$

$f(x) = \cos(x^3 - x) + |x| \sin(|x^3 + x|)$

$= \cos(x^3 - x) + x \sin(x^3 + x)$

which is differentiable at $x = 1$

5. (2), (3)

$f(x) = [x^2 - 3] = [x^2] - 3$

$f(x)$ is discontinuous at $x = 1, \sqrt{2}, \sqrt{3}, 2$

$g(x) = (|x| + |4x - 7|)([x^2] - 3)$

$(|x| + |4x - 7|)$ is non-differentiable at $x = 0, 7/4$

So $g(x)$ is non-differentiable at $x = 0, 1, \sqrt{2}, \sqrt{3}$ (as at $x = 7/4$, $[x^2] - 3 = 0$)

6. (2), (3), (4)

$f(x) = x \cos(\pi x + [x]\pi)$

$\Rightarrow \quad f(x) = (-1)^{[x]} x \cos \pi x \qquad [\text{as } \cos(n\pi + x) = (-1)^n \cos x]$

Clearly, $f(x)$ is discontinuous at all integers except zero.

Alternative method:

$f(x) = x \cos(\pi(x + [x]))$

Let's check continuity at $x = n \; (\in Z)$.

$f(n^+) = n \cos 2n\pi = n$

$f(n^-) = n \cos(2n - 1)\pi = -n$

It is discontinuous at all integer points except 0.

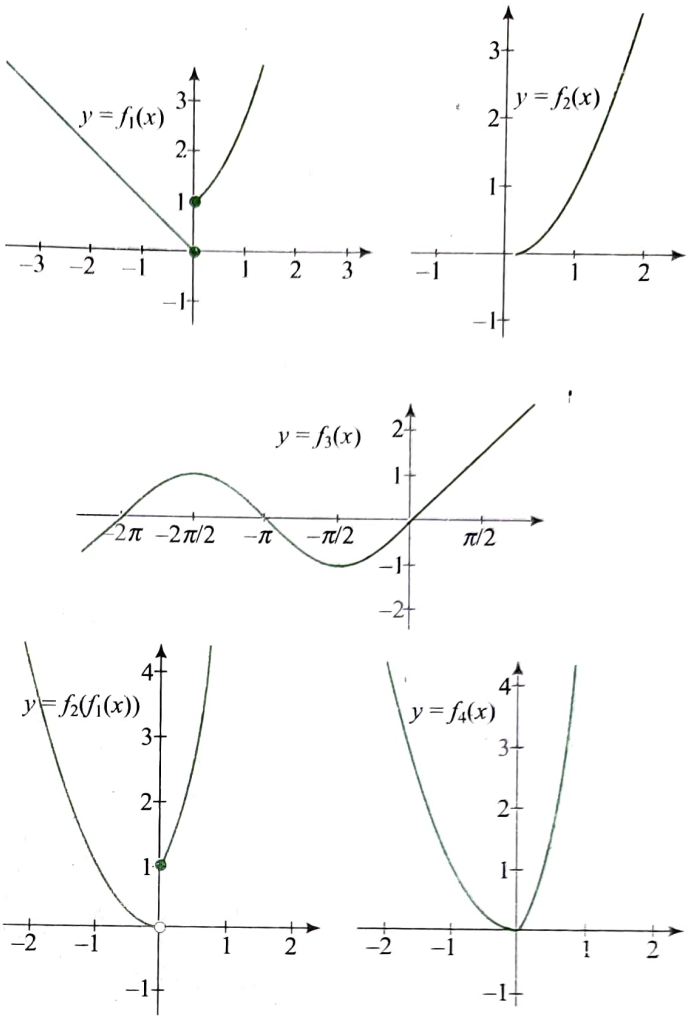
Matrix Match Type

1. (4) $f_2(f_1) = \begin{cases} x^2, & x < 0 \\ e^{2x}, & x \geq 0 \end{cases}$

$f_4 : R \to [0, \infty)$

$f_4(x) = \begin{cases} f_2(f_1(x)), & x < 0 \\ f_2(f_1(x)) - 1, & x \geq 0 \end{cases}$

$= \begin{cases} x^2, & x < 0 \\ e^{2x} - 1, & x \geq 0 \end{cases}$



2. **b – p, q**

$$f(x) = \begin{cases} -3ax^2 - 2, & x < 1 \\ bx + a^2, & x \geq 1 \end{cases}$$

$f(x)$ is continuous at $x = 1 \Rightarrow -3a - 2 = b + a^2$ ...(i)

$$f'(x) = \begin{cases} -6ax, & x < 1 \\ b, & x > 1 \end{cases}$$

$f(x)$ is differentiable at $x = 1$

$\Rightarrow \quad -6a = b$

$\Rightarrow \quad 6a = a^2 + 3a + 2$ (using (i))

$\Rightarrow \quad a^2 - 3a + 2 = 0$

$\Rightarrow \quad a = 1, 2$

Note: Solutions of the remaining parts are given in their respective chapters.

3. (4) **(i)** Clearly, $f_1(x)$ is continuous in R.

$$\text{Now, } f_1'(0) = \lim_{h \to 0} \frac{\sin\sqrt{1 - e^{-h^2}} - 0}{h}$$

$$= \lim_{h \to 0} \frac{\sin\sqrt{1 - e^{-h^2}}}{\sqrt{1 - e^{-h^2}}} \times \sqrt{\frac{e^{-h^2} - 1}{-h^2}} \times \frac{|h|}{h}$$

$$= 1 \times 1 \times \lim_{h \to 0} \frac{|h|}{h}$$

This limit does not exist.

Hence, $f_1(x)$ is not differentiable at $x = 0$.

$$= \lim_{x \to 0} \frac{|\sin x|}{|x|} \times \frac{x}{\tan^{-1} x} \times \frac{|x|}{x}$$

$$= 1 \times 1 \times \lim_{x \to 0} \frac{|x|}{x}$$

This limit does not exist.

So, $f_2(x)$ is not continuous at $x = 0$.

(iii) $f_3(x) = [\sin(\log_e(x + 2))]$

When $x \to 0$, $[\sin(\log_e(x + 2))] \to 0$.

Let us consider $[\sin(\log_e(x + 2))] = 0$.

$\Rightarrow \quad 0 \le \sin(\log_e(x + 2)) < 1$

$\Rightarrow \quad 0 \le \log_e(x + 2)) < \pi, \log_e(x + 2)) \ne \pi/2$

$\Rightarrow \quad 1 \le x + 2 < e^{\pi}, x + 2 \ne e^{\pi/2}$

$\Rightarrow \quad -1 \le x < e^{\pi} - 2, x \ne e^{\pi/2} - 2$

Thus, $f_3(x)$ is differentiable at $x = 0$ and derivative is continuous at $x = 0$.

(iv) $\lim_{x \to 0} f_4(x) = \lim_{x \to 0} \left(x^2 \sin \frac{1}{x}\right) = \lim_{x \to 0} x^2 \left(\sin \frac{1}{x}\right)$

$= 0 \times$ (value between -1 and 1) $= 0$

Thus, $f_4(x)$ is continuous at $x = 0$.

Also,

$$f_4'(0) = \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} = \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0$$

Thus, derivative of $f_4(x)$ exists at $x = 0$.

Also, $f_4'(x) = -\cos\frac{1}{x} + 2x \sin\frac{1}{x}, x \ne 0$

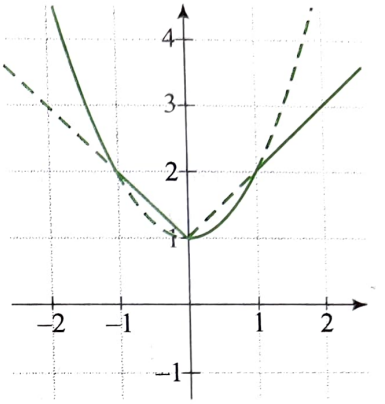
$$f_4'(0) = \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h}.$$

This limit does not exist.

Thus, $f_4(x)$ is continuous and differentiable at $x = 0$ but derivative is not continuous.

Numerical Value Type

1. (3) The graphs of $f(x) = |x| + 1$ and $g(x) = x^2 + 1$ are as shown in the following figure.



From the graph

$$h(x) = \begin{cases} \max\{f(x), g(x)\}, & \text{if } x \le 0 \\ \min\{f(x), g(x)\}, & \text{if } x > 0 \end{cases}$$

$$= \begin{cases} x^2 + 1, & x \in (-\infty, -1] \\ -x + 1, & x \in (-1, 0] \\ x^2 + 1, & x \in (0, 1] \\ x + 1, & x \in (1, \infty] \end{cases}$$

Hence, $h(x)$ is not differentiable at $x = -1, 0, 1$.

Chapter 5

Concept Application Exercises

Exercise 5.1

1. We have $(1+x^2)\,y = 2 - x$ or $y = \dfrac{2-x}{1+x^2}$

This meets x-axis at $(2, 0)$.

Also, $\dfrac{dy}{dx} = \dfrac{(1+x^2)(-1) - (2-x)\,2x}{(1+x^2)^2}$

At $(2, 0)$, $\dfrac{dy}{dx} = \dfrac{(1+4)(-1) - 0}{(1+4)^2} = -\dfrac{1}{5}$

Therefore, the required tangent is

$$y - 0 = -\frac{1}{5}(x-2) \text{ or } x + 5y = 2.$$

2. The curve is $3xy^2 - 2x^2y = 1$ (1)

Differentiating w.r.t. x, $\dfrac{dy}{dx} = \dfrac{4xy - 3y^2}{6xy - 2x^2}$

At the point $(1, 1)$, $dy/dx = 1/4$

Slope of line joining $P(1, 1)$ and $Q(-16/5, -1/20)$ is

$$\frac{1+\frac{1}{20}}{1+\frac{16}{5}} = \frac{21}{84} = \frac{1}{4}.$$

Also, point $Q(-16/5, -1/20)$ satisfies the curve.

Hence, proved.

3. Given that

$$x = 2at^2/(1+t^2),\ y = 2at^3/(1+t^2)$$

At $t = 1/2$, $x = 2a/5$, $y = a/5$.

Also, $\dfrac{dx}{dt} = \dfrac{4at}{(1+t^2)^2}$ and $\dfrac{dy}{dt} = \dfrac{2at^2(3+t^2)}{(1+t^2)^2}$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{2}t\,(3+t^2)$$

When $t = \dfrac{1}{2}$, $\dfrac{dy}{dx} = \dfrac{1}{2}\cdot\dfrac{1}{2}\left(3+\dfrac{1}{4}\right) = \dfrac{13}{16}$.

Therefore, the equation of the tangent when $t = 1/2$ is

$$y - a/5 = (13/16)(x - 2a/5) \quad \text{or } 13x - 16y = 2a.$$

And the equation of the normal is

$$(y - a/5)(13/16) + x - 2a/5 = 0$$

or $16x + 13y = 9a$

4. We have $x = a(1+\cos\theta),\ y = a\sin\theta$

$$\therefore \quad \frac{dx}{d\theta} = -a\sin\theta,\ \frac{dy}{d\theta} = a\cos\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\cos\theta}{-a\sin\theta} = -\cot\theta$$

or slope of normal $= \tan\theta$

Thus, the equation of normal is

$$y - a\sin\theta = \tan\theta\,(x - a(1+\cos\theta))$$

or $y\cos\theta - a\sin\theta\cos\theta = x\sin\theta - a\sin\theta - a\sin\theta\cos\theta$

or $x\sin\theta - y\cos\theta = a\sin\theta$

which clearly passes through $(a, 0)$.

5. The curves are $x^3 + y^3 = 8xy$ (1)

and $y^2 = 4x$ (2)

Solving equations (1) and (2), we get

$$x^3 + y\,(4x) = 8xy$$

or $x^3 = 4xy$

or $x^3 = 4x\,(2\sqrt{x})$

or $x^{3/2}(x^{3/2} - 8) = 0$

i.e., $x = 0$ or $x^{3/2} = 8 = 2^3$

i.e., $x = 0$ or $x = 2^2 = 4$

Now when $x = 0$, we get $y = 0$.

When $x = 4$, we get $y^2 = 16$ or $y = \pm 4$.

But $x = 4$ and $y = -4$ does not satisfy equation (1).

Thus, $(0, 0)$ and $(4, 4)$ are the points of intersection of equations (1) and (2).

Differentiating equation (1), we get $\dfrac{dy}{dx} = \dfrac{8y - 3x^2}{3y^2 - 8x}$.

At $(4, 4)$, $\dfrac{dy}{dx} = -1$.

Hence, the equation of the normal to (1) at $(4, 4)$ is

$$(y-4) = 1(x-4) \text{ or } y - x = 0$$

6. $y = ax^2 - 6x + b$ passes through $(0, 2)$, i.e.,

$$2 = 0 - 0 + b \text{ or } b = 2$$

Again, $\dfrac{dy}{dx} = 2ax - 6$

At $x = \dfrac{3}{2}$, $\dfrac{dy}{dx} = 3a - 6$

Since tangent is parallel to x-axis,

$$\frac{dy}{dx} = 0 \quad \text{or} \quad 3a - 6 = 0 \quad \text{or} \quad a = 2$$

Hence, $a = 2$, $b = 2$.

7. $\left(\dfrac{x}{a}\right)^n + \left(\dfrac{y}{b}\right)^n = 2$

or $\dfrac{n}{a}\left(\dfrac{x}{a}\right)^{n-1} + \dfrac{n}{b}\left(\dfrac{y}{b}\right)^{n-1}\dfrac{dy}{dx} = 0$

or $\dfrac{dy}{dx} = -\dfrac{b^n}{a^n} \times \dfrac{x^{n-1}}{y^{n-1}}$

At (a, b), $\dfrac{dy}{dx} = -\dfrac{b^n}{a^n}\dfrac{a^{n-1}}{b^{n-1}} = -\dfrac{b}{a}$

Therefore, tangent at (a, b) is $y - b = -\dfrac{b}{a}(x-a)$

or $\dfrac{y}{b} - 1 = -\dfrac{x}{a} + 1$ or $\dfrac{x}{a} + \dfrac{y}{b} = 2$

This shows that $\dfrac{x}{a} + \dfrac{y}{b} = 2$ touches the given curve for all n.

8. Given curve $xy + ax + by = 0$.

As $(1, 1)$ lies on curve,

$$1 + a + b = 0 \qquad (1)$$

Slope of tangent is $\tan(\tan^{-1} 2) = 2$ at $(1, 1)$

$$\therefore \quad \frac{dy}{dx} = 2 \text{ at } (1, 1)$$

Differentiating equation of curve w.r.t. x,

$$x\frac{dy}{dx} + y + a + b\frac{dy}{dx} = 0$$

Put $x = 1$, $y = 1$, and $\frac{dy}{dx} = 2$. Then,

$$2 + 1 + a + 2b = 0 \text{ or } a + 2b + 3 = 0 \quad (2)$$

Solving (1) and (2), we get $b = -2$ and $a = 1$.

9. $y = f(x) = \sin x$

$$\therefore \quad f'(x) = \frac{dy}{dx} = \cos x$$

Tangent to the curve at $P(x_1, y_1)$ is normal to the curve $Q(x_2, y_2)$. Thus,

$$\cos x_1 = -\frac{1}{\cos x_2} = \frac{\sin x_2 - \sin x_1}{x_2 - x_1} \text{ (slope of chord } PQ)$$

$$\therefore \quad \cos x_1 \cos x_2 = -1$$

$$\therefore \quad \sin x_1 = \sin x_2 = 0$$

Therefore, there is no such line.

10. Let the line be normal to the curve at point $P(x_1, y_1)$ on the curve. Then,

$$Ax_1 + By_1 = 1 \quad (1)$$

and $a^{n-1} y_1 = x_1^n \quad (2)$

Differentiating $a^{n-1} y = x^n$ w.r.t. x, we get

$$\frac{dy}{dx} = \frac{nx^{n-1}}{a^{n-1}} = \frac{nx^n}{xa^{n-1}} = \frac{ny}{x}$$

Now, slope of line = slope of normal to the curve at $P(x_1, y_1)$

$$\text{or} \quad -\frac{A}{B} = -\frac{x_1}{ny_1} \quad \text{or} \quad nAy_1 = Bx_1 \quad (3)$$

From equations (1) and (3), $A(nAy_1)/B + By_1 = 1$

$$\text{or} \quad (nA^2 + B^2)y_1 = B \quad (4)$$

$$\text{and} \quad (nA^2 + B^2)(Bx_1/nA) = B$$

$$\text{or} \quad B(nA^2 + B^2)x_1 = nAB \quad (5)$$

Now, substituting the values of y_1 and x_1 from equations (4) and (5), respectively, in equation (2), we get

$$a^{n-1} B(nA^2 + B^2)^{n-1} = n^n A^n$$

11. Given curve is $x^a y^b = K^{a+b}$

Differentiating w.r.t. x, we get

$$ax^{a-1} y^b + x^a by^{b-1} \frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{ay}{bx}$$

Consider any point on curve as $P(h, k)$.

Tangent at P is

$$y - k = -\frac{ak}{bh}(x - h)$$

x-intercept is $Q\left(\frac{(a+b)h}{a}, 0\right)$.

y-intercept is $R\left(0, \frac{(a+b)k}{b}\right)$.

Clearly, P divides QR in ratio $\frac{b}{a}$.

Exercise 5.2

1. Here, the curves are

$$y = a^x \text{ and } y = b^x$$

Solving the curves, $a^x = b^x$ or $\left(\frac{a}{b}\right)^x = 1$

or $x = 0$ and $y = 1$

Therefore, point of intersection is (0, 1).

$$\left(\frac{dy}{dx}\right)_{(0,1)} = (a^x \log a)_{(0,1)} = \log a \quad (\text{for } y = a^x)$$

$$\left(\frac{dy}{dx}\right)_{(0,1)} = (b^x \log b)_{(0,1)} = \log b \quad (\text{for } y = b^x)$$

Thus, angle between the curves,

$$\tan\theta = \left|\frac{\log a - \log b}{1 + (\log a)(\log b)}\right|$$

$$\text{or} \quad \theta = \tan^{-1}\left|\frac{\log a/b}{1 + \log a \log b}\right|$$

2. The two curves are

$$xy = a^2 \quad (1)$$

$$\text{and} \quad x^2 + y^2 = 2a^2 \quad (2)$$

Solving equations (1) and (2), the points of intersection are (a, a) and $(-a, -a)$.

Differentiating equation (1), $dy/dx = -y/x = m_1$ (say)

Differentiating equation (2), $dy/dx = -x/y = m_2$ (say)

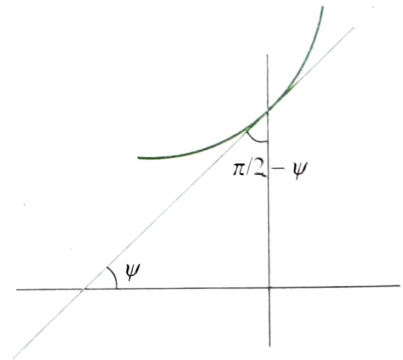
At both points, $m_1 = -1 = m_2$.

Hence, the two curves touch each other.

3. $\left[\frac{dy}{dx}\right]_{x=0} = K^2$

$$\text{or} \quad \tan\psi = K^2 \text{ or } \cot\left(\frac{\pi}{2} - \psi\right) = K^2$$

$$\text{or} \quad \left(\frac{\pi}{2} - \psi\right) = \cot^{-1} K^2$$



4. For first curve,

$$2x - \frac{2y}{3}\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = \frac{3x}{y} = m_1$$

For second curve,

$$3xy^2 \frac{dy}{dx} + y^3 = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{y}{3x} = m_2$$

Now, $m_1 \times m_2 = -1$

Therefore, angle between the curves is 90°.

5. We have the curves

$$x^3 - 3xy^2 + 2 = 0 \quad (1)$$

$$\text{and} \quad 3x^2 y - y^3 - 2 = 0 \quad (2)$$

Differentiating (1) and (2) w.r.t. x, we get

$$\frac{dy}{dx}=\frac{x^2-y^2}{2xy} \text{ and } \frac{dy}{dx}=\frac{-2xy}{x^2-y^2}$$

So, $m_1=\frac{x^2-y^2}{2xy}$ and $m_2=\frac{-2xy}{x^2-y^2}$

Since, $m_1 \times m_2=-1$, angle between curves is 90°.

6. $ay+x^2=7$, and $x^3=y$ cut orthogonally. Now

$$\left(\frac{dy}{dx}\right)_1=-\frac{2x}{a} \text{ and } \left(\frac{dy}{dx}\right)_2=3x^2$$

$$\therefore \quad \left[\left(-\frac{2x}{a}\right)(3x^2)\right]_{(1,1)}=-1$$

or $$-\frac{2}{a}\times 3=-1 \text{ or } a=6$$

Exercise 5.3

1. When $x=0, y=-1$.

$$\frac{dy}{dx}=3x^2+6x+4 \quad \text{or} \quad \left.\frac{dy}{dx}\right|_{x=0}=4$$

$$\therefore \quad \text{Length of tangent}=\left|y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\right|$$

$$=\left|-1\sqrt{1+\frac{1}{16}}\right|$$

$$=\frac{\sqrt{17}}{4}$$

2. Let point of tangency be (x_1, y_1). Then,

$$m=\left.\frac{dy}{dx}\right|_{x_1}=\frac{2ax_1}{x_1^2-a^2}$$

$$\text{tangent + subtangent}=y_1\sqrt{1+\frac{1}{m^2}}+\frac{y_1}{m}$$

$$=y_1\sqrt{1+\frac{(x_1^2-a^2)^2}{4a^2x_1^2}}+\frac{y_1(x_1^2-a^2)}{2ax_1}$$

$$=y_1\frac{\sqrt{x_1^4+a^4+2a^2x_1^2}}{2ax_1}+\frac{y_1(x_1^2-a^2)}{2ax_1}$$

$$=\frac{y_1(x_1^2+a^2)}{2ax_1}+\frac{y_1(x_1^2-a^2)}{2ax_1}$$

$$=\frac{2y_1(x_1^2)}{2ax_1}$$

$$=\frac{x_1y_1}{a}\propto x_1y_1$$

3. $$\left(\frac{\text{length of normal}}{\text{length of tangent}}\right)^2=\frac{\left(y\sqrt{1+\frac{dy}{dx}}\right)^2}{\left(y\sqrt{1+\frac{dx}{dy}}\right)^2}$$

$$=\left(\frac{dy}{dx}\right)^2=\left(\frac{y\frac{dy}{dx}}{y\frac{dx}{dy}}\right)=\frac{\text{sub-normal}}{\text{sub-tangent}}$$

4. $y=a^{1-n}x^n$ or $\frac{dy}{dx}=a^{1-n}nx^{n-1}$

$$\text{Sub-normal}=\left|y\frac{dy}{dx}\right|$$

$$=\left|ya^{1-n}nx^{n-1}\right|$$

$$=\left|a^{1-n}x^na^{1-n}nx^{n-1}\right|$$

$$=\left|a^{2-2n}x^{2n-1}\right|$$

which is constant if $2n-1=0$ or $n=1/2$.

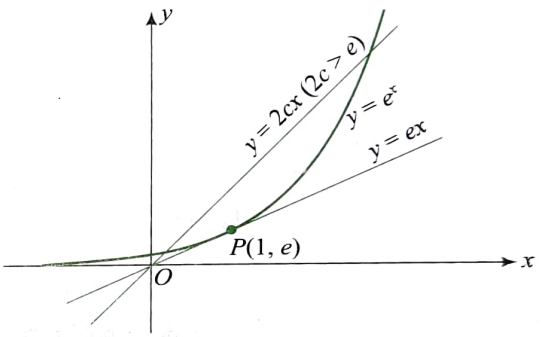
Exercise 5.4

1. We have equation $f(x)=e^x-cx^2=0$.

If equation has three distinct real roots, then $f'(x)=0$ must have two distinct roots as between two roots of the equation $f(x)=0$, there lies at least one root of $f'(x)=0$.

$$f'(x)=e^x-2cx$$

Roots of $f'(x)=0$ occur for values of x where graphs of $y=e^x$ and $y=2cx$ intersect.



For the curve, $\frac{dy}{dx}=e^x$.

Also, slope of the line is $2c$.

Let the line touch the curve at $P(x_1, y_1)$.

Then $2c=e^{x_1}$, $y_1=e^{x_1}$ and $y_1=2cx_1$.

$\therefore \quad x_1=1$ and $y_1=e$

So, point P lies at $(1, e)$.

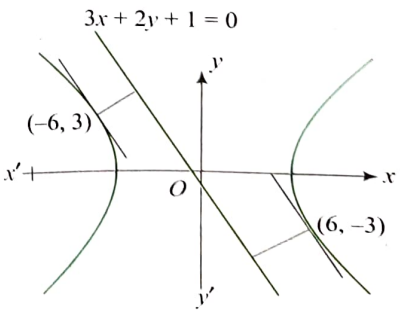
The line intersects the curve at two points if slope of line is greater than e.

$\Rightarrow \quad 2c>e$

$\therefore \quad c>e/2$

Therefore, least positive integral value of c is 2.

2.



Slope of the given line $3x+2y+1=0$ is $(-3/2)$.

Let us locate the point on the curve at which the tangent is parallel to given line.

Differentiating the curve on both sides w.r.t. to x, we get

$$6x - 8y\frac{dy}{dx} = 0$$

or $$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{3x_1}{4y_1} = -\frac{3}{2} \text{ [since parallel to } 3x + 2y + 1 = 0]$$

or $$\frac{x_1}{y_1} = -2 \qquad (1)$$

Also the point (x_1, y_1) lies on $3x^2 - 4y^2 = 72$. Therefore,

$$3x_1^2 - 4y_1^2 = 72$$

or $$3\frac{x_1^2}{y_1^2} - 4 = \frac{72}{y_1^2} \qquad (2)$$

or $$3(4) - 4 = \frac{72}{y_1^2} \qquad \text{[From (1)]}$$

or $$y_1^2 = 9 \quad \text{or} \quad y_1 = \pm 3.$$

The required points are $(-6, 3)$ and $(6, -3)$.

Distance $(-6, 3)$ from the given line

$$= \frac{|-18+6+1|}{\sqrt{13}} = \frac{11}{\sqrt{13}}$$

and distance of $(6, -3)$ from the given line

$$= \frac{|18-6+1|}{\sqrt{13}} = \frac{13}{\sqrt{13}} = \sqrt{13}$$

Thus, $(-6, 3)$ is the required point.

3. We have

$$3 - x^2 > |x - a| \qquad (1)$$

The graph of $y = 3 - x^2$ is downward parabola with vertex at $(0, 3)$.

Now, we have following cases:

Case I: If $a \le 0$

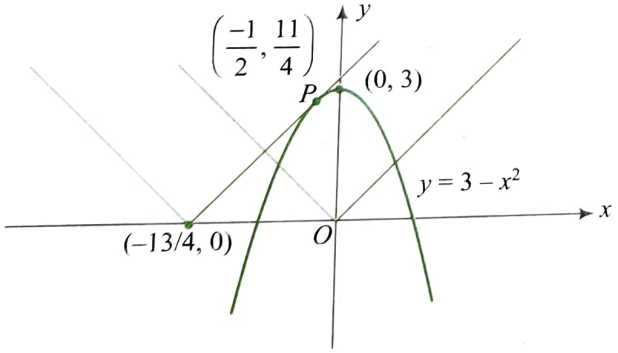
If the line $y = x - a$ is tangent to the curve $y = 3 - x^2$, then comparing slopes, we get

$$1 = -2x$$

$$\therefore \quad x = -1/2$$

Putting this value in curve, we get $y = 11/4$.

So, point of contact is $P(-1/2, 11/4)$.



Point P lies on the line $y = x - a$.

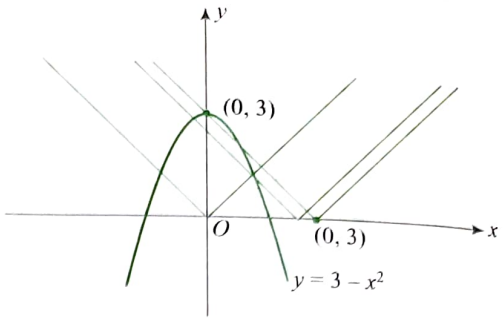
So, $a = -13/4$.

Therefore, for inequality (1) to have at least one negative solution, $a > -13/4$.

Clearly, for $3 - x^2 > |x|$ has negative solutions. (2)

Thus, $a \in (-13/4, 0]$

Case II: If $a > 0$



If the line $y = a - x$ passes though the point $(0, 3)$, then $a = 3$.

So, for inequality (1) to have at least one negative solution, $a < 3$.

Thus, $a \in (0, 3)$ (3)

From (2) and (3), $a \in (-13/4, 3)$.

4. Let $P(h, k)$ be a point of contact of tangents from the origin $(0, 0)$ on the curve $y = \sin x$.

Since P lies on the curve, $k = \sin h$ (1)

Also, $$\frac{dy}{dx} = \cos x \text{ or } \left(\frac{dy}{dx}\right)_{(h, k)} = \cos h \qquad (2)$$

Slope of the line joining $O(0, 0)$ and $P(h, k)$ is $\frac{k}{h}$.

Given that $\cos h = \frac{k}{h}$ (3)

Squaring and adding (1) and (3), we get $k^2 + \frac{k^2}{h^2} = 1$

or $$h^2k^2 + k^2 = h^2$$

or $$k^2 = \frac{h^2}{1+h^2}$$

or $$y^2 = \frac{x^2}{1+x^2}$$

5. $y = x^4 + 3x^2 + 2x$

$$\therefore \quad \frac{dy}{dx} = 4x^3 + 6x + 2$$

Point on curve which is nearest to the line $y = 2x - 1$ is the point where tangent to curve is parallel to given line. Therefore,

$$4x^3 + 6x + 2 = 2$$

or $$2x^3 + 3x = 0$$

or $$x = 0, y = 0$$

Therefore, point on the curve at the least distance from the line $y = 2x - 1$ is $(0, 0)$.

Distance of this point from line is $\frac{1}{\sqrt{5}}$.

6. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two of these points.

Given $y = x^3 + 2x - 1$ and $y = 2x^3 - 4x + 2$

$$\therefore \quad y_1 = 2x_1^3 - 4x_1 + 2 \qquad (1)$$

and $$2y_1 = 2x_1^3 + 4x_1 - 2 \qquad (2)$$

Subtracting (1) from (2), we get

$$y_1 = 8x_1 - 4 \qquad (3)$$

Similarly, we get $y_2 = 8x_2 - 4$ (4)

From (4) − (3), we get

$$y_2 - y_1 = 8(x_2 - x_1)$$

$$\therefore \quad \frac{y_2 - y_1}{x_2 - x_1} = 8, \text{ which is slope of the required line}$$

Exercise 5.5

1. Given $s^2 = (at^2 + 2bt + c)$ or $s = \sqrt{(at^2+2bt+c)}$ (1)

$$\therefore \quad \frac{ds}{dt} = \frac{(at+b)}{\sqrt{(at^2+2bt+c)}}$$

$$= \frac{(at+b)}{s} = V \quad \text{(say)} \quad \text{[From equation (1)]} \qquad (2)$$

Again differentiating both sides w.r.t. t,

$$\frac{d^2s}{dt^2} = \frac{s(a) - (at+b)\frac{ds}{dt}}{s^2}$$

$$= -\frac{as - (at+b)\frac{(at+b)}{s}}{s^2} \quad \text{[from equation (2)]}$$

$$= \frac{as^2 - (at+b)^2}{s^3}$$

$$= \frac{a(at^2+2bt+c) - (a^2t^2+2abt+b^2)}{s^3}$$

$$= \frac{(ac-b^2)}{s^3}$$

$$\therefore \quad \text{acceleration} \propto \frac{1}{s^3}$$

2. At time t, the distance z between the cyclists is given by

$$z^2 = (3ut)^2 + (4ut)^2$$

or $\quad z = 5ut$

or $\quad \frac{dz}{dt} = 5u$

3. $V = \frac{4\pi}{3}(x+10)^3$, where x is the thickness of ice

$$\therefore \quad \frac{dV}{dt} = 4\pi(x+10)^2 \frac{dx}{dt}$$

$$\therefore \quad \left[\frac{dx}{dt}\right]_{x=5} = \frac{-50}{4\pi(5+10)^2} = \frac{-50}{900\pi} = -\frac{1}{18\pi}$$

Hence, the rate at which thickness decreases is $\frac{1}{18\pi}$ cm/s

4. Given x and y are the sides of two squares. Thus, the area of two squares is x^2 and y^2.

We have to obtain $\frac{d(y^2)}{d(x^2)} = \frac{2y\frac{dy}{dx}}{2x} = \frac{y}{x} \times \frac{dy}{dx}$ (1)

where the given curve is $y = x - x^2$

$$\therefore \quad \frac{dy}{dx} = 1 - 2x \qquad (2)$$

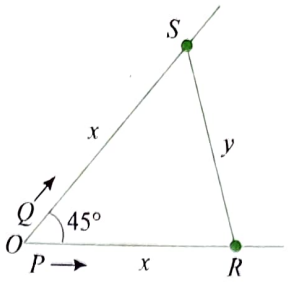
Thus, $\frac{d(y^2)}{d(x^2)} = \frac{y}{x}(1-2x)$ [From equations (1) and (2)]

or $\quad \frac{d(y^2)}{d(x^2)} = \frac{(x-x^2)(1-2x)}{x} = (2x^2 - 3x + 1)$

Thus, the rate of change of the area of second square with respect to first square is $(2x^2 - 3x + 1)$.

5. Let R and S be the positions of men P and Q at any time t. Since velocities are same,

$OR = OS = x$ (say) and given $\frac{dx}{dt} = u$ (1)



Let $SR = y$.

Now, in triangle ORS, applying cosine rule,

$$y^2 = x^2 + x^2 - 2x \times x \cos 45° = 2x^2 - x^2\sqrt{2}$$

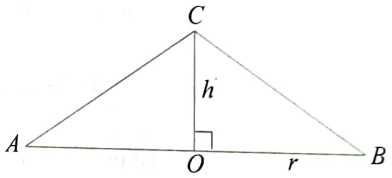
or $\quad y = x\sqrt{(2-\sqrt{2})}$

or $\quad \frac{dy}{dt} = \left[\sqrt{(2-\sqrt{2})}\right]\frac{dx}{dt}$

$$= u\sqrt{(2-\sqrt{2})} \quad \text{[From equation (1)]}$$

Hence, the required rate at which they are being separated is $u\sqrt{(2-\sqrt{2})}$.

6. Let h be the height and r the radius of the cone. Then,



$$h = \frac{1}{6}r \quad \text{(Given)}$$

or $\quad r = 6h \qquad (1)$

$$\therefore \quad \text{Volume } V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi(6h)^2 h = 12\pi h^3 \quad \text{[From (1)]}$$

$$\therefore \quad \frac{dV}{dt} = 36\pi h^2 \frac{dh}{dt}$$

or $\quad 12 = 36\pi h^2 \frac{dh}{dt} \quad \left(\because \frac{dV}{dt} = 12 \text{ cm}^3/\text{s}\right)$

or $\quad \frac{dh}{dt} = \frac{1}{3\pi h^2}$

When $h = 4$ cm, $\frac{dh}{dt} = \frac{1}{3\pi(4)^2} = \frac{1}{48\pi}$ cm/s.

Hence, the rate at which the height of the sand cone increases when the height is 4 cm is $\frac{1}{48\pi}$ cm/s.

7. We have,

$$L = 200\,(10 - t)^2$$

$$\therefore \quad \text{Rate at which the water is running out} = -\frac{dL}{dt}$$

$$\therefore \quad \frac{dL}{dt} = -200 \times 2\,(10-t) \times (-1) = 400\,(10-t)$$

Rate at which the water is running out at the end of 5 s

$$= 400\,(10-5) = 2000 \text{ L/s}$$

= Final rate

$$\text{Initial rate} = -\left(\frac{dL}{dt}\right)_{t=0}$$

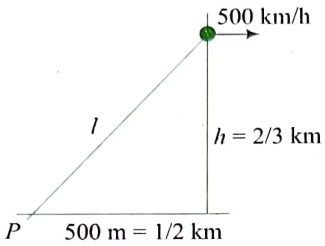
$$= 4000 \text{ L/s}$$

$\therefore$ Average rate during 5 s $= \dfrac{\text{Initial rate + Final rate}}{2}$

$= \dfrac{4000 + 2000}{2}$

$= 3000$ L/s

8. $l^2 = h^2 + x^2$



$\therefore \quad 2l\dfrac{dl}{dt} = 0 + 2x\dfrac{dx}{dt}$

or $\dfrac{dl}{dt} = \dfrac{x}{l}\cdot\dfrac{dx}{dt}$

where $x = \dfrac{1}{2}$ km, $h = \dfrac{2}{3}$ km

Then $l = \sqrt{\dfrac{1}{4} + \dfrac{4}{9}} = \dfrac{5}{6}$ km

$\therefore \quad \dfrac{dl}{dt} = \dfrac{1}{2}\cdot\dfrac{6}{5}\cdot 500 = 300$ km/h

Exercise 5.6

1. Consider $y = (x)^{1/3}$. Let $x = 27$ and $\Delta x = -1$.

Then,

$\Delta y = (x + \Delta x)^{1/3} - (x)^{1/3} = (26)^{1/3} - (27)^{1/3} = (26)^{1/3} - 3$

or $(26)^{1/3} = 3 + \Delta y$

Now, dy is approximately equal to Δy and is given by

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{3(x)^{\frac{2}{3}}}(\Delta x) \qquad \left[\text{as } y = (x)^{\frac{1}{3}}\right]$$

$$= \frac{1}{3(27)^{\frac{2}{3}}}(-1) = \frac{-1}{27}$$

$$= -0.0370$$

Hence, the approximate value of $(26)^{1/3}$ is $3 + (-0.0370) = 2.963$.

2. Consider function $y = x^6$.

Let $x = 2$ and $\Delta x = -0.001$.

Then $\Delta y = (x + \Delta x)^6 - x^6$.

Now, dy is approximately equal to Δy and is given by

$$\Delta y = \left(\frac{dy}{dx}\right)_{x=2}\Delta x = 6(2)^5(-0.001)$$

or $f(1.999) = f(2) - 6 \times 32 \times 0.001$

$= 64 - 64 \times 0.003$

$= 63.808$ (approx.)

3. Let $y = \cos x$

$\therefore \quad \dfrac{dy}{dx} = -\sin x$

Then $\Delta y = \cos(x + \Delta x) - \cos x$

$= \cos(60° 1') - \cos 60°$

Now, dy is approximately equal to Δy and is given by

$$\Delta y = \left(\frac{dy}{dx}\right)_{x=60°}\Delta x = -\frac{\sqrt{3}}{2}\times 1' = -\frac{\sqrt{3}}{2}\times\frac{1}{60}$$

or $\cos(60° 1') = \dfrac{1}{2} - \dfrac{\sqrt{3}}{120}$

4. Let $x = 3$ and $\Delta x = 0.02$. Then

$f(3.02) = f(x + \Delta x) = 3(x + \Delta x)^2 + 5(x + \Delta x) + 3$

Now, $f(x + \Delta x) = f(x) + \Delta y$

$\approx f(x) + f'(x)\,\Delta x$ (As $dx = \Delta x$)

or $f(3.02) \approx (3x^2 + 5x + 3) + (6x + 5)\,\Delta x$

$= (3(3)^2 + 5(3) + 3) + (6(3) + 5)(0.02)$ (As $x = 3, \Delta x = 0.02$)

$= (27 + 15 + 3) + (18 + 5)(0.02)$

$= 45 + 0.46 = 45.46$

Hence, approximate value of $f(3.02)$ is 45.46.

5. Let r be the radius of the sphere and Δr be the error in measuring the radius.

Then $r = 9$ cm and $\Delta r = 0.03$ cm. Now, the volume V of the sphere is given by

$$V = \frac{4}{3}\pi r^3$$

or $\dfrac{dV}{dr} = 4\pi r^2$

$\therefore \quad dV = \left(\dfrac{dV}{dr}\right)\Delta r = (4\pi r^2)\,\Delta r$

$= 4\pi(9)^2(0.03) = 9.72\pi$ cm^3

Thus, the approximate error in calculating the volume is $9.72\,\pi$ cm^3.

Exercise 5.7

1. $f(x) = \begin{vmatrix} \tan x & \tan a & \tan b \\ \sin x & \sin a & \sin b \\ \cos x & \cos a & \cos b \end{vmatrix}$

$f(a) = f(b) = 0$, $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b).

$f'(x) = 0$ has at least one root in (a, b).

2. Consider the function $f(x) = x^3 + 2ax^2 + bx$.

$f(x)$ being a polynomial function is continuous in $[0, 1]$ and differentiable in $(0, 1)$.

Also, $f(0) = 0$. If $f(1) = 0$, then all the three conditions of Rolle's theorem will be satisfied. Therefore,

$f'(c) = 0$, for at least one c in $(0, 1)$. Hence,

$f'(x) = 3x^2 + 4ax + b = 0$ for at least once in $(0, 1)$

i.e., the equation $3x^2 + 4ax + b = 0$ has at least one root in $(0, 1)$ if $f(1) = 0$, i.e., $1 + 2a + b = 0$.

3. Let $h(x) = f(x) - 3g(x)$.

Since $f(x)$ and $g(x)$ are differentiable, $h(x)$ is also differentiable.

Since there exists a real number c in $(0, 2)$, such that $h'(c) = 0$, we must have

$h(0) = h(2)$

$\therefore \quad f(0) - 3g(0) = f(2) - 3g(2)$

$\Rightarrow \quad 2 - 3 \times 1 = 8 - 3 \times g(2)$

$\Rightarrow \quad g(2) = 3$

4. Let $f(x) = x^5 - a_0x^4 + 3ax^3 + bx^2 + cx + d = 0$

$f'(x) = 5x^4 - 4a_0x^3 + 9ax^2 + 2bx + c$

$f''(x) = 20x^3 - 12a_0x^2 + 18ax + 2b$

$f'''(x) = 60x^2 - 24a_0x + 18a$

or $f'''(x) = 6(10x^2 - 4a_0x + 3a)$

Now, discriminant $= 36(16a_0^2 - 4 \times 10 \times 3a)$

or $D = 288(2a_0^2 - 15a) < 0$ [as $2a_0^2 - 15a < 0$, given]

Hence, the roots of $f'''(x) = 0$ cannot be real. Therefore, all the roots of $f(x) = 0$ will not be real.

5. Let $g(x) = (a - x)(b - x)f(x)$ on $[a, b]$.

Then $g(a) = g(b) = 0$.

Using the Rolle's theorem, there exists $c \in (a, b)$ such that $g'(c) = 0$.

Therefore, $-(b-c)f(c) - (a-c)f(c) + (a-c)(b-c)f'(c) = 0$

$$\therefore \quad \frac{f'(c)}{f(c)} = \frac{1}{a-c} + \frac{1}{b-c}$$

6. Consider function $h(x) = f(x)\, e^{g(x)}$.

Here, $h(x)$ is continuous on $[a, b]$ and differentiable on (a, b).

Since $h(a) = h(b) = 0$, by Rolle's theorem, there exists $c \in (a, b)$ such that $h'(c) = 0$.

Now, $h'(x) = f'(x)\, e^{g(x)} + f(x)\, e^{g(x)} g'(x)$

For $h'(c) = 0$, we have

$f'(c)\, e^{g(c)} + f(c)\, e^{g(c)} g'(c) = 0$

$\therefore \quad g'(c) f(c) + f'(c) = 0$

7. Given equation is

$\phi'(x + a) - \phi'(x) = 0$ or $f'(x) = 0$,

where $f(x) = \phi(x + a) - \phi(x) + k$

Now, $f(0) = \phi(a) - \phi(0) + k$

And $f(2a) = \phi(3a) - \phi(2a) + k = \phi(a) - \phi(0) + k$

Thus, $f(0) = f(2a)$.

So, by Rolle's theorem, we have $f'(c) = 0$ for at least one $c \in (0, 2a)$.

Hence, equation $\phi'(x + a) = \phi'(x)$ has at least one root in $(0, 2a)$.

8. Consider the function

$g(x) = f^2(x) - x^2, x \in [a, b]$

$g(a) = f^2(a) - a^2$

and $g(b) = f^2(b) - b^2$

Since $f^2(a) - a^2 = f^2(b) - b^2$,

$g(a) = g(b)$

Hence, Rolle's theorem is applicable to $g(x)$.

So, there exists at least one $c \in (a, b)$, where $g'(c) = 0$.

Now, $g'(x) = 2f(x) \cdot f'(x) - 2x = 0$

$\therefore \quad f(x) \cdot f'(x) = x$

i.e., $f(c) \cdot f'(c) - c = 0$

Exercise 5.8

1. Given $f(x) = 3x^2 + 5x + 7$ (1)

or $f(1) = 3 + 5 + 7 = 15$

and $f(3) = 27 + 15 + 7 = 49$

Also, $f'(x) = 6x + 5$.

Here, $a = 1, b = 3$.

Now, from Lagrange's mean value theorem,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or $$6c + 5 = \frac{f(3) - f(1)}{3 - 1}$$

$$= \frac{49 - 15}{2} = 17 \text{ or } c = 2$$

2. We have to prove

$$(b^3 - a^3)\, f'(c) - [f(b) - f(a)]\, (3c^2) = 0$$

Let us assume a function

$$F(x) = (b^3 - a^3)\, f(x) - [f(b) - f(a)]x^3$$

which is continuous in $[a, b]$ and differentiable in (a, b) as both $f(x)$ and x are continuous and differentiable. Also,

$$F(a) = b^3 f(a) - a^3 f(b) = F(b)$$

So, according to Rolle's theorem, there exists at least one $c \in (a, b)$ such that $F(c) = 0$, which proves the required result.

3. Consider $f(x) = x^4$

Using LMVT in $[a, b]$, we get

$$4c^3 = \frac{b^4 - a^4}{b - a}, \text{ where } c \in (a, b)$$

$$\Rightarrow \quad \frac{c}{a + b} = \frac{b^2 + a^2}{4c^2}$$

4. $f(x)$ and $g(x)$ are continuous and differentiable functions.

Now, let $H(x) = \begin{vmatrix} f(a) & f(x) \\ g(a) & g(x) \end{vmatrix}$ (1)

Then $H(a) = 0$ and $H(b) = \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix}$.

So, $H(x)$ satisfies the condition of mean value theorem.

Therefore, $\frac{H(b) - H(a)}{b - a} = H'(c)$, where $a < c < b$

or $$H'(c) = \frac{1}{(b-a)} \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} \quad (2)$$

From (1),

$$H'(x) = \begin{vmatrix} 0 & f(x) \\ 0 & g(x) \end{vmatrix} + \begin{vmatrix} f(a) & f'(x) \\ g(a) & g'(x) \end{vmatrix} = \begin{vmatrix} f(a) & f'(x) \\ g(a) & g'(x) \end{vmatrix}$$

or $$H'(c) = \begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix} \quad (3)$$

From equations (2) and (3), we get

$$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b - a) \begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}$$

5. Let $f(x) = \tan^{-1} x$.

Then for some $\alpha \in (x, y)$,

$$f'(\alpha) = \frac{\tan^{-1} y - \tan^{-1} x}{y - x}$$

$$\left| \frac{1}{1 + \alpha^2} \right| = \left| \frac{\tan^{-1} x - \tan^{-1} y}{x - y} \right|$$

Since $\left| \frac{1}{1 + \alpha^2} \right| \le 1$,

$|\tan^{-1} x - \tan^{-1} y| \le |x - y|$

6. Let $f(x) = \log_e x, x \in [a, b]$, and $0 < a < b$.

Clearly, $f(x)$ is continuous and differentiable.

Hence, according to LMVT, there exists at least one $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or $$\frac{1}{c} = \frac{\log b - \log a}{b - a} = \frac{\log\left(\frac{b}{a}\right)}{b - a}$$

Now, $a < c < b$

or $$\frac{1}{a} > \frac{1}{c} > \frac{1}{b}$$

or $$\frac{1}{a} > \frac{\log\left(\frac{b}{a}\right)}{b - a} > \frac{1}{b}$$

or $$\frac{b - a}{b} < \log\left(\frac{b}{a}\right) < \frac{b - a}{a}$$

7. Let $f(x) = x^n$, $n > 1$, $x \in [b, a]$.

By LMVT,

$$\frac{f(a) - f(b)}{a - b} = f'(c),\ b < c < a$$

$$\therefore \quad \frac{a^n - b^n}{a - b} = nc^{n-1} \qquad (1)$$

$\because$ $b < c < a$ and $n > 1$,

$b^{n-1} < c^{n-1} < a^{n-1}$

or $nb^{n-1} < nc^{n-1} < na^{n-1}$

or $nb^{n-1} < \frac{a^n - b^n}{a - b} < na^{n-1}$ [using (1)]

or $nb^{n-1}(a - b) < a^n - b^n < na^{n-1}(a - b)$

Also if $0 < n < 1$, $b < c < a$

or $b^{n-1} > c^{n-1} > a^{n-1}$

or $nb^{n-1}(a - b) > a^n - b^n > na^{n-1}(a - b)$

8. Consider $\phi(x) = f(x) - g(x)$

$\therefore$ $\phi'(x) = f'(x) - g'(x)$

$\phi(x)$ is also continuous and derivable in $[x_0, x]$.

Using LMVT for $Q(x)$ in $[x_0, x]$,

$$\phi'(x) = \frac{\phi(x) - \phi(x_0)}{x - x_0}$$

Since $\phi'(x) = f'(x) - g'(x)$ and $f'(x) - g'(x) > 0$,

$\phi'(x) > 0$

Hence, $\phi(x) - \phi(x_0) > 0$

$\phi(x) > \phi(x_0)$

$f(x) - g(x) > 0$ [$\because$ $\phi(x_0) = f(x_0) - g(x_0) = 0$]

9. Consider $g(x) = x^2$.

Since $f(x)$ is differentiable function in (a, b), there exists at least one $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$$\therefore \quad \frac{f(b) - f(a)}{b^2 - a^2} = \frac{f'(c)}{2c}$$

$$\Rightarrow \quad (a^2 - b^2) f'(c) = 2c\,(f(a) - f(b))$$

Exercises

Single Correct Answer Type

1. (2) Given curve is $x^{3/2} + y^{3/2} = 2a^{3/2}$ (1)

$$\therefore \quad \frac{3}{2}\sqrt{x} + \frac{3}{2}\sqrt{y}\,\frac{dy}{dx} = 0 \qquad \text{(Differentiate w.r.t. } x\text{)}$$

$$\text{or} \quad \frac{dy}{dx} = -\frac{\sqrt{x}}{\sqrt{y}}$$

Since the tangent is equally inclined to the axes,

$$\frac{dy}{dx} = \pm 1$$

$$\therefore \quad -\frac{\sqrt{x}}{\sqrt{y}} = \pm 1 \text{ or } -\frac{\sqrt{x}}{\sqrt{y}} = -1 \qquad [\because \sqrt{x} > 0, \sqrt{y} > 0]$$

$$\therefore \quad \sqrt{x} = \sqrt{y}$$

Putting $\sqrt{y} = \sqrt{x}$ in (1), we get

$$2x^{3/2} = 2a^{3/2} \text{ or } x^3 = a^3$$

Therefore, $x = a$ and, so, $y = a$.

2. (1) $\frac{dx}{dt} = a + \frac{a}{2}\,2\cos 2t = a\,[1 + \cos 2t] = 2a\cos^2 t$

and $\frac{dy}{dt} = 2a\,(1 + \sin t)\cos t$

$$\therefore \quad \frac{dy}{dx} = \frac{2a(1 + \sin t)\cos t}{2a\cos^2 t} = \frac{(1 + \sin t)}{\cos t}$$

Then, the slope of the tangent,

$$\tan\theta = \frac{\left(\cos(t/2) + \sin(t/2)\right)^2}{\cos^2(t/2) - \sin^2(t/2)}$$

$$= \frac{1 + \tan\frac{t}{2}}{1 - \tan\frac{t}{2}} = \tan\left(\frac{\pi}{4} + \frac{t}{2}\right)$$

or $\theta = \frac{\pi + 2t}{4}$

3. (4) Differentiating w.r.t. x, we get $e^y \frac{dy}{dx} = 2x$

or $\frac{dy}{dx} = \frac{2x}{1 + x^2}$ ($\because e^y = 1 + x^2$)

or $m = \frac{2x}{1 + x^2}$ or $|m| = \frac{2|x|}{1 + |x|^2}$

But $1 + |x|^2 - 2|x| = (1 - |x|)^2 \geq 0$

or $1 + |x|^2 \geq 2|x|$,

$\therefore$ $|m| \leq 1$

4. (3) $\frac{dy}{dx} = 3x^2 - 2ax + 1$

Given that $\frac{dy}{dx} \geq 0$ for all x.

or $3x^2 - 2ax + 1 \geq 0$ for all x

Therefore, $D \leq 0$ or $4a^2 - 12 \leq 0$

or $-\sqrt{3} \leq a \leq \sqrt{3}$

5. (1) Here, $y > 0$. Putting $y = x$ in $y = \sqrt{4 - x^2}$, we get $x = \sqrt{2}, -\sqrt{2}$.

So, the point is $(\sqrt{2}, \sqrt{2})$.

Differentiating $y^2 + x^2 = 4$ w.r.t. x, we get

$$2y\frac{dy}{dx} + 2x = 0 \text{ or } \frac{dy}{dx} = -\frac{x}{y}$$

$\therefore$ At $(\sqrt{2}, \sqrt{2})$, $\frac{dy}{dx} = -1$

6. (2) Differentiating w.r.t. x, we get

$$1 + \frac{dy}{dx} = e^{xy}\left(y + x\frac{dy}{dx}\right) \text{ or } \frac{dy}{dx} = \frac{ye^{xy} - 1}{1 - xe^{xy}}$$

If $\frac{dy}{dx} = \infty$ then $1 - xe^{xy} = 0$.

This holds for $x = 1$, $y = 0$.

7. (3) The equation of the line is

$$y - 3 = \frac{3 + 2}{0 - 5}(x - 0), \text{ i.e., } x + y - 3 = 0$$

$$y = \frac{c}{x + 1} \text{ or } \frac{dy}{dx} = \frac{-c}{(x + 1)^2}$$

Let the line touches the curve at (α, β). Then

$$\alpha + \beta - 3 = 0,\ \left(\frac{dy}{dx}\right)_{(\alpha, \beta)} = \frac{-c}{(\alpha + 1)^2} = -1, \text{ and } \beta = \frac{c}{\alpha + 1}$$

$\therefore \quad \frac{c}{(c/\beta)^2} = 1$ or $\beta^2 = c$ or $(3-\alpha)^2 = c = (\alpha+1)^2$

or $3-\alpha = \pm(\alpha+1)$ or $3-\alpha = \alpha+1$ or $\alpha = 1$

So, $c = (1+1)^2 = 4$.

8. (3) $f(0) = 1$.

So, the point where curve crosses y-axis is $(0, 1)$.

Slope of tangent at $(0, 1)$ is $f'(0) = (f(0))^2 + 5 = 6$.

Therefore, the equation of tangent at $(0, 1)$ is

$y - 1 = 6(x - 0)$

or $6x - y + 1 = 0$

9. (1) Putting $x = 0$ in the given curve, we obtain $y = 1$.

So, the given point is $(0, 1)$. Now,

$$y = e^{2x} + x^2 \text{ or } \frac{dy}{dx} = 2e^{2x} + 2x \text{ or } \left(\frac{dy}{dx}\right)_{(0,1)} = 2$$

The equation of the tangent at $(0, 1)$ is

$y - 1 = 2(x - 0)$ or $2x - y + 1 = 0$ (1)

Required distance = length of $\perp$ from $(0, 0)$ on (1)

$$= \frac{1}{\sqrt{5}}$$

10. (1) Let the required point be (x_1, y_1). Now,

$3y = 6x - 5x^3$

or $3\frac{dy}{dx} = 6 - 15x^2$

or $\frac{dy}{dx} = 2 - 5x^2$

or $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 2 - 5x_1^2$

The equation of the normal at (x_1, y_1) is

$$y - y_1 = \frac{-1}{2 - 5x_1^2}(x - x_1)$$

If it passes through the origin, then

$$0 - y_1 = \frac{-1}{2 - 5x_1^2}(0 - x_1)$$

or $y_1 = \frac{-x_1}{2 - 5x_1^2}$ (1)

Since (x_1, y_1) lies on the given curve,

$3y_1 = 6x_1 - 5x_1^3$ (2)

Solving equations (1) and (2), we obtain $x_1 = 1$ and $y_1 = 1/3$.

Hence, the required point is $(1, 1/3)$.

11. (1) $2x^2 + y^2 = 12$ or $\frac{dy}{dx} = -\frac{2x}{y}$.

Slope of normal at point $A(2, 2)$ is $\frac{1}{2}$.

Also, point $B\left(-\frac{22}{9}, -\frac{2}{9}\right)$ lies on the curve and slope of AB is

$$\frac{2 - (-2/9)}{2 - (-22/9)} = \frac{1}{2}.$$

Hence, the normal meets the curve again at point $\left(-\frac{22}{9}, -\frac{2}{9}\right)$.

12. (1) $y = \frac{2}{3}x^3 + \frac{1}{2}x^2$

$\therefore \quad \frac{dy}{dx} = \frac{2}{3}3x^2 + \frac{1}{2}2x = 2x^2 + x$

Since the tangent makes equal angles with the axes,

$\frac{dy}{dx} = \pm 1$

or $2x^2 + x = \pm 1$

or $2x^2 + x - 1 = 0$ $(2x^2 + x + 1 = 0$ has no real roots)

or $(2x - 1)(x + 1) = 0$

i.e., $x = \frac{1}{2}$ or $x = -1$

13. (4) $y = b\,e^{-x/a}$ meets the y-axis at $(0, b)$. Also,

$$\frac{dy}{dx} = b\,e^{-x/a}\left(-\frac{1}{a}\right)$$

At $(0, b)$, $\frac{dy}{dx} = be^0\left(-\frac{1}{a}\right) = -\frac{b}{a}$

Therefore, required tangent is

$$y - b = -\frac{b}{a}(x - 0) \text{ or } \frac{x}{a} + \frac{y}{b} = 1$$

14. (2) $x^2 - y^2 = 8$

$\therefore \quad \frac{dy}{dx} = \frac{x}{y}$ or $-\frac{1}{dy/dx} = -\frac{y}{x}$

At the point $\left(-\frac{5}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$, $-\frac{1}{dy/dx} = \frac{-3/\sqrt{2}}{-5/\sqrt{2}} = \frac{3}{5}$

Also, $9x^2 + 25y^2 = 225$

$\therefore \quad 18x + 50y\frac{dy}{dx} = 0$

or $\frac{dy}{dx} = -\frac{9x}{25y}$ or $-\frac{dx}{dy} = \frac{25y}{9x}$

At the point $\left(-\frac{5}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$,

$$-\frac{dx}{dy} = \frac{25 \times 3/\sqrt{2}}{9(-5/\sqrt{2})} = -\frac{15}{9} = -\frac{5}{3}$$

Since the product of the slopes is -1, the normals cut orthogonally, i.e., the required angle is equal to $\frac{\pi}{2}$.

15. (2) We have $f''(x) = 6(x - 1)$

Integrating, we get $f'(x) = 3(x - 1)^2 + c$ (1)

At $(2, 1)$, $y = 3x - 5$ is tangent to $y = f(x)$.

Thus, $f'(2) = 3$.

From equation (1),

$3 = 3(2 - 1)^2 + c$ or $3 = 3 + c$ or $c = 0$

$\therefore \quad f'(x) = 3(x - 1)^2$

Integrating, we get $f(x) = (x - 1)^3 + c'$.

Since the curve passes through $(2, 1)$,

$1 = (2 - 1)^3 + c'$ or $c' = 0$

$\therefore \quad f(x) = (x - 1)^3$

$\therefore \quad f(0) = -1$

16. (2) We have curve

$x + y - \log_e(x + y) = 2x + 5$

Differentiating w.r.t. x, we get

$$1 + y' - \frac{1}{x + y}(1 + y') = 2$$

$\Rightarrow \quad (1 + y')(x + y) - (1 + y') = 2x + 2y$

$\Rightarrow \quad (x + y - 1) + y'(x + y - 1) = 2x + 2y$

$\Rightarrow \quad y' = \dfrac{x+y+1}{x+y-1}$

For vertical tangent at (α, β),

$\alpha + \beta - 1 = 0$

$\therefore \quad \alpha + \beta = 1$

17.(1) $\dfrac{dx}{dt} = \dfrac{e^{\sqrt{t}}}{2\sqrt{t}}$ and $\dfrac{dy}{dt} = 3 - \dfrac{2}{t}$.

At $t = 1$, $x = e$ and $y = 3$

$\therefore \quad \dfrac{dx}{dt} = \dfrac{e}{2}, \dfrac{dy}{dt} = 1.$

So, the slope of the tangent is $\dfrac{2}{e}$.

Hence, the equation of tangent is $y = \dfrac{2}{e}x + 1.$

18. (1) $y^2 = \alpha x^3 - \beta$ or $\dfrac{dy}{dx} = \dfrac{3\alpha x^2}{2y}$

Therefore, slope of the normal at (2, 3) is

$$\left(-\frac{dx}{dy}\right)_{(2,3)} = -\frac{2\times 3}{3\alpha(2)^2} = -\frac{1}{2\alpha} = -\frac{1}{4}$$

or $\alpha = 2$

Also, (2, 3) lies on the curve. Therefore,

$9 = 8\alpha - \beta$ or $\beta = 16 - 9 = 7$ or $\alpha + \beta = 9.$

19. (4) $x = 2 \ln \cot t + 1, y = \tan t + \cot t$

Slope of the tangent,

$$\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}} = \left(\frac{\sec^2 t - \operatorname{cosec}^2 t}{-\frac{2}{\cot t}\operatorname{cosec}^2 t}\right)_{t=\frac{\pi}{4}} = 0$$

20. (2) $y = e^x + e^{-x}$ or $\dfrac{dy}{dx} = e^x - e^{-x} = \tan\theta,$

where θ is the angle of the tangent with the x-axis.

For $\theta = 60°$, we have

$\tan 60° = e^x - e^{-x}$

or $e^{2x} - \sqrt{3}e^x - 1 = 0$

or $e^x = \dfrac{\sqrt{3} \pm \sqrt{7}}{2}$

or $x = \log_e\left(\dfrac{\sqrt{3}+\sqrt{7}}{2}\right)$

21. (3) $x^2y = c^3$

Differentiating w.r.t. x, we have

$$x^2\frac{dy}{dx} + 2xy = 0$$

or $\dfrac{dy}{dx} = -\dfrac{2y}{x}$

Equation of the tangent at (h, k) is

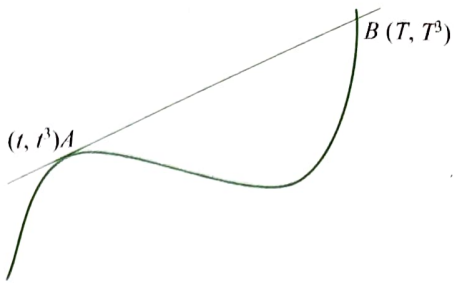
$$y - k = -\frac{2k}{h}(x - h)$$

$y = 0$ gives $x = \dfrac{3h}{2} = a,$

and $x = 0$ gives $y = 3k = b.$

Now, $a^2b = \dfrac{9h^2}{4}3k = \dfrac{27}{4}h^2k = \dfrac{27}{4}c^3.$

22. (1)



$\dfrac{dy}{dx} = 3x^2 = 3t^2$ at A

$\therefore \quad 3t^2 = \dfrac{T^3 - t^3}{T - t} = T^2 + Tt + t^2$

or $T^2 + Tt - 2t^2 = 0$

or $(T - t)(T + 2t) = 0$, i.e., $T = t$ or $T = -2t$

($T = t$ is not possible)

Now, $m_A = 3t^2$ and $m_B = 3T^2$

$\therefore \quad \dfrac{m_B}{m_A} = \dfrac{T^2}{t^2} = \dfrac{4t^2}{t^2}$ (Using $T = -2t$)

23. (4) $x \sin y + y \sin x = \pi$

Differentiating w.r.t. x, we get

$$\sin y + x\cos y\frac{dy}{dx} + \frac{dy}{dx}\sin x + y\cos x = 0$$

When $x = \pi/2$ and $y = \pi/2$, we have

$$1 + \frac{dy}{dx} = 0$$

$\Rightarrow \quad y' = -1$

Therefore, the equation of the tangent line is

$$y - \frac{\pi}{2} = -\left(x - \frac{\pi}{2}\right) \quad \text{or} \quad x + y = \pi$$

24. (3) $\dfrac{a}{x^2} + \dfrac{b}{y^2} = 1$ or $ay^2 + bx^2 = x^2y^2$ (1)

$\therefore \quad -\dfrac{2a}{x^3} - \dfrac{2b}{y^3}\dfrac{dy}{dx} = 0$ or $\dfrac{dy}{dx} = -\dfrac{ay^3}{bx^3}$

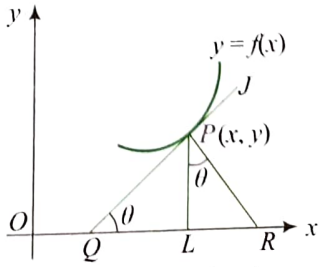
Equation of the tangent at (h, k) is $y - k = -\dfrac{ak^3}{bh^3}(x - h).$

For x-intercept, put $y = 0$. Then

$$x = \frac{bh^3}{ak^2} + h = h\left[\frac{bh^2 + ak^2}{ak^2}\right] = h\left[\frac{h^2k^2}{ak^2}\right] = \frac{h^3}{a}$$

Therefore, x-intercept is proportional to the cube of abscissa.

25. (1)



(1)

Given curve is $2x^2y^2 - x^4 = c$ (2)

Sub-normal at $P(x, y) = y\dfrac{dy}{dx}$

From (1), we get $2\left(x^2 2y\dfrac{dy}{dx} + 2xy^2\right) - 4x^3 = 0$

or $\frac{dy}{dx} = \frac{x(x^2 - y^2)}{x^2 y}$ (3)

Now, $x(x - yy') = x^2 - xy\frac{dy}{dx}$

$= x^2 - (x^2 - y^2)$ [From (3)]

$= y^2$

$\therefore$ Mean proportion $= \sqrt{x(x - yy')} = y$

26. (1) $g(x) = \frac{x+2}{x-1}$

$\therefore \quad g'(x) = \frac{-3}{(x-1)^2}$

Slope of given line $= -3$

or $\frac{-3}{(x-1)^2} = -3$ or $x = 2$

Also, $g(2) = 4$.

$(2, 4)$ also lies on given line.

Hence, the given line is tangent to the curve.

27. (2) Length of sub-normal = length of sub-tangent or $\frac{dy}{dx} = \pm 1$

If $\frac{dy}{dx} = 1$, equation of the tangent is

$y - 4 = x - 3$ or $y - x = 1$

area of $\Delta OAB = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

If $\frac{dy}{dx} = -1$, equation of the tangent is

$y - 4 = -x + 3$ or $y + x = 7$,

area $= \frac{1}{2} \times 7 \times 7 = \frac{49}{2}$

28. (1) $y = x + \sin x$

If $\frac{dy}{dx} = 1 + \cos x = 0$, then $\cos x = -1$ or $x = \pm \pi, \pm 3\pi \ldots$.

Also, $y = \pm \pi, \pm 3\pi \ldots$.

But for the given constraint on x and y, no such y exists.

Hence, no such tangent exists.

29. (3) Solving $y = |x^2 - 1|$ and $y = \sqrt{7 - x^2}$, we have

$|x^2 - 1| = \sqrt{7 - x^2}$

or $x^4 - 2x^2 + 1 = 7 - x^2$

or $x^4 - x^2 - 6 = 0$

or $(x^2 - 3)(x^2 + 2) = 0$

or $x = \pm \sqrt{3}$

Points of intersection of the curves $y = |x^2 - 1|$ and

$y = \sqrt{7 - x^2}$ are $(\pm \sqrt{3}, 2)$.

Since both the curves are symmetrical about the y-axis, points of intersection are also symmetrical.

Now, $y = x^2 - 1$ or $\frac{dy}{dx} = 2x$

or $m_1 = \left.\frac{dy}{dx}\right|_{(\sqrt{3}, 2)} = 2\sqrt{3}$

and $y = \sqrt{7 - x^2}$ or $\frac{dy}{dx} = -\frac{x}{y}$

or $m_2 = \left.\frac{dy}{dx}\right|_{(\sqrt{3}, 2)} = -\frac{\sqrt{3}}{2}$

Therefore, $\tan\theta = \left|\frac{5\sqrt{3}}{4}\right|$

30. (4) Differentiating $y^3 - x^2 y + 5y - 2x = 0$ w.r.t. x, we get

$3y^2 y' - 2xy - x^2 y' + 5y' - 2 = 0$

or $y' = \frac{2xy + 2}{3y^2 - x^2 + 5}$ or $y'_{(0, 0)} = \frac{2}{5}$

Differentiating $x^4 - x^3 y^2 + 5x + 2y = 0$ w.r.t. x, we get

$4x^3 - 3x^2 y^2 - 2x^3 yy' + 5 + 2y' = 0$

or $y' = \frac{3x^2 y^2 - 4x^3 - 5}{2 - 2x^3 y}$ or $y'_{(0, 0)} = -\frac{5}{2}$.

Thus, both the curves intersect at right angle.

31. (4) Solving the curves, we get point of intersection (a^2, a).

For $x = y^2$, $\frac{dy}{dx} = \frac{1}{2y}$.

At (a^2, a), $\frac{dy}{dx} = \frac{1}{2a}$.

For $xy = a^3$, $\frac{dy}{dx} = -\frac{y}{x}$.

At (a^2, a), $\frac{dy}{dx} = -\frac{a}{a^2} = -\frac{1}{a}$.

Since the curves cut orthogonally,

$\frac{1}{2a} \times -\frac{1}{a} = -1$ or $2a^2 = 1$ or $a^2 = \frac{1}{2}$

32. (4) $\frac{dy}{dx} = ke^{kx} = k$ at $(0, 1)$. Equation of the tangent is $y - 1 = kx$.

At point of intersection with x-axis, $x = -\frac{1}{k}$, where

$-2 \le -\frac{1}{k} \le -1$ or $k \in \left[\frac{1}{2}, 1\right]$

33. (2) $4x^2 + 9y^2 = 72$

Differentiating w.r.t. x, we have

$8x + 18y\frac{dy}{dx} = 0$ or $\frac{dy}{dx} = -\frac{4}{9}\frac{x}{y}$

At $(3, 2)$, $\frac{dy}{dx} = -\frac{4}{9} \times \frac{3}{2} = -\frac{2}{3}$

Also, $x^2 - y^2 = 5$ or $\frac{dy}{dx} = \frac{x}{y}$. At $(3, 2)$, $\frac{dy}{dx} = \frac{3}{2}$.

Therefore, the curves cut orthogonally.

34. (2) Any point on the parabola $y^2 = 8x$ $(4a = 8$ or $a = 2)$ is $(at^2, 2at)$ or $(2t^2, 4t)$.

If its distance from the circle is minimum then its distance from the center, $(0, -6)$, of the circle is minimum.

Let D be the distance. Then

$z = D^2 = (2t^2)^2 + (4t + 6)^2 = 4(t^4 + 4t^2 + 12t + 9)$

$\therefore \quad \frac{dz}{dt} = 4(4t^3 + 8t + 12) = 0$

or $16(t^3 + 2t + 3) = 0$

or $16(t + 1)(t^2 - t + 3) = 0$

or $t = -1$

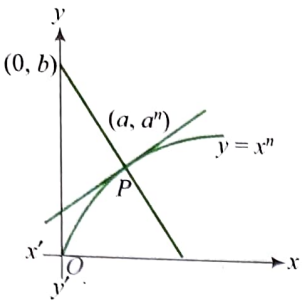
$\frac{d^2 z}{dt^2} = 16(3t^2 + 2) = +\text{ve}$. Hence, minimum.

Therefore, point is $(2, -4)$.

35. (3) $y = x^n$

$\frac{dy}{dx} = n x^{n-1} = na^{n-1}$ at P

Slope of the normal at $P = -\dfrac{1}{na^{n-1}}$



Equation of the normal at P is $y - a^n = -\dfrac{1}{na^{n-1}}(x - a)$

Put $x = 0$ to get y-intercept.

$$y = a^n + \frac{1}{na^{n-2}};$$

Hence, $b = a^n + \dfrac{1}{na^{n-2}}$

$$\lim_{a\to 0} b = \begin{cases} 0, & \text{if } n < 2 \\ \dfrac{1}{2}, & \text{if } n = 2 \\ \infty, & \text{if } n > 2 \end{cases}$$

36. (1) Since the same line is tangent at one point $x = a$ and normal at other point $x = b$, tangent at $x = b$ will be perpendicular to tangent at $x = a$.

Also, slope of tangent changes from positive to negative or negative to positive. Therefore, it takes the value zero somewhere. Thus, there exists a point $c \in (a, b)$ where $f'(c) = 0$.

37. (4) We have $y^2 = 18x$ (1)

$$\therefore \quad 2y\frac{dy}{dx} = 18 \text{ or } \frac{dy}{dx} = \frac{9}{y}$$

Given that $\dfrac{dy}{dx} = 2$ or $\dfrac{9}{y} = 2$ or $y = \dfrac{9}{2}$

Putting in (1), we get $\dfrac{81}{4} = 18x$ or $x = \dfrac{9}{8}$.

Hence, the point is $\left(\dfrac{9}{8}, \dfrac{9}{2}\right)$.

38. (1) $V = \dfrac{4}{3}\pi r^3$, $S = 4\pi r^2$

$$\frac{dV}{dr} = 4\pi r^2, \quad \frac{dS}{dr} = 8\pi r$$

$$\therefore \quad \frac{dV}{dS} = \frac{dV/dr}{dS/dr} = \frac{4\pi r^2}{8\pi r} = \frac{r}{2}$$

When $r = 2$, $\dfrac{dV}{dS} = \dfrac{2}{2} = 1$

39. (2) $V = x^3$ and the percent error in measuring x is $\dfrac{dx}{x} \times 100 = k$

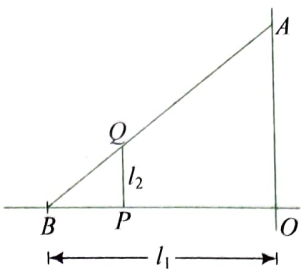
The percent error in measuring volume $= \dfrac{dV}{V} \times 100$

Now, $\dfrac{dV}{dx} = 3x^2$

or $dV = 3x^2 dx$ or $\dfrac{dV}{V} = \dfrac{3x^2 dx}{x^3} = 3\dfrac{dx}{x}$

$$\therefore \quad \frac{dV}{V} \times 100 = 3\frac{dx}{x} \times 100 = 3k$$

40. (2)



Let $BP = x$. From similar triangle property, we get

$$\frac{AO}{l_1} = \frac{l_2}{x} \quad \text{or } AO = \frac{l_1 l_2}{x} \quad \text{or } \frac{d(AO)}{dt} = \frac{-l_1 l_2}{x^2}\frac{dx}{dt}$$

When $x = \dfrac{l_1}{2}$, $\dfrac{d(AO)}{dt} = -\dfrac{2l_2}{5}$ m/s.

41. (1) $f'(x) = 0$

$\Rightarrow \quad (x+3)e^{-x/2} + xe^{-x/2} - (x/2)(x+3)e^{-x/2} = 0$

$\Rightarrow \quad x^2 - x - 6 = 0$

$\Rightarrow \quad x = -2, 3$

$\Rightarrow \quad c = -2 \in (-3, 0)$

42. (4) Given $V = \pi r^2 h$.

Differentiating both sides, we get

$$\frac{dV}{dt} = \pi\left(r^2\frac{dh}{dt} + 2r\frac{dr}{dt}h\right) = \pi r\left(r\frac{dh}{dt} + 2h\frac{dr}{dt}\right)$$

$$\frac{dr}{dt} = \frac{1}{10} \quad \text{and} \quad \frac{dh}{dt} = -\frac{2}{10}$$

$$\therefore \quad \frac{dV}{dt} = \pi r\left(r\left(-\frac{2}{10}\right) + 2h\left(\frac{1}{10}\right)\right) = \frac{\pi r}{5}(-r + h)$$

Thus, when $r = 2$ and $h = 3$,

$$\frac{dV}{dt} = \frac{\pi(2)}{5}(-2+3) = \frac{2\pi}{5}$$

43. (2) Given $\dfrac{dV}{dt} = -4$ cm^3/min; we have to find $\dfrac{dS}{dt}$ when $V = 125$ cm^3.

$$V = x^3; \quad S = 6x^2$$

$$\therefore \quad \frac{dV}{dt} = 3x^2\frac{dx}{dt}$$

or $\quad -4 = 3x^2\dfrac{dx}{dt}$ (1)

Also, $\dfrac{dS}{dt} = 12x\dfrac{dx}{dt} = -\dfrac{16}{x}$

When $V = 125 = x^3$, $x = 5$

or $\quad \left(\dfrac{dS}{dt}\right)_{x=5} = -\dfrac{16}{5}$ cm^2/min

44. (1) Let r, l, and h denote, respectively, the radius, slant height, and height of the cone at any time t. Then,

$$l^2 = r^2 + h^2$$

or $\quad 2l\dfrac{dl}{dt} = 2r\dfrac{dr}{dt} + 2h\dfrac{dh}{dt}$

or $l\frac{dl}{dt} = r\frac{dr}{dt} + h\frac{dh}{dt}$

$= 7 \times 3 + 24 \times (-4) \qquad \left[\because \frac{dh}{dt} = -4 \text{ and } \frac{dr}{dt} = 3\right]$

$= -75$

Where $r = 7$ and $h = 24$, we have

$l^2 = 7^2 + 24^2$

or $l = 25$

$\therefore \quad l\frac{dl}{dt} = -75$ or $\frac{dl}{dt} = -3$

Let S denote the lateral surface area. Then

$$\frac{dS}{dt} = \frac{d}{dt}(2\pi rl) = 2\pi\left\{\frac{dr}{dt}l + r\frac{dl}{dt}\right\}$$

$= 2\pi\{3 \times 25 + 7 \times (-3)\}$

$= 108\pi \text{ cm}^2/\text{min}$

45. (3) $f(0) = -1; f(1) = 7$. So, $f(0)$ and $f(1)$ have opposite signs.

46. (4) $f(x)$ vanishes at points where

$$\sin\frac{\pi}{x} = 0, \text{ i.e., } \frac{\pi}{x} = k\pi, k = 1, 2, 3, 4, \ldots$$

Hence, $x = \frac{1}{k}$.

Also, $f'(x) = \sin\frac{\pi}{x} - \frac{\pi}{x}\cos\frac{\pi}{x}$ if $x \neq 0$.

Since the function has a derivative at any interior point of the interval (0, 1), it is continuous in [0,1], and $f(0) = f(1)$, Rolle's theorem is applicable to any one of the intervals

$$\left[\frac{1}{2}, 1\right], \left[\frac{1}{3}, \frac{1}{2}\right] = \ldots, \left[\frac{1}{k+1}, \frac{1}{k}\right].$$

Hence, there exists at least one c in each of these intervals where $f'(c) = 0$. Therefore, there are infinite points.

47. (2) Applying Rolle's theorem to $F(x) = f(x) - 2g(x)$, we get

$F(0) = 0$

$F(1) = f(1) - 2g(1)$

or $0 = 6 - 2g(1)$

or $g(1) = 3$

48. (2) Let $f(x) = \frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx$,

which is continuous and differentiable.

$f(0) = 0$

$$f(-1) = \frac{a}{4} - \frac{b}{3} + \frac{c}{2} - d = \frac{1}{4}(a + 2c) - \frac{1}{3}(b + 3d) = 0$$

So, according to Rolle's theorem, there exists at least one root of $f'(x) = 0$ in $(-1, 0)$.

49. (4) Here, $f(x) = \log_e x$

$$\therefore \quad f'(c) = \frac{f(b) - f(a)}{b - a} \text{ or } \frac{1}{c} = \frac{\log_e 3 - \log_e 1}{3 - 1}$$

or $\frac{1}{c} = \frac{1}{2}\log_e 3$ or $c = 2\log_3 e$

50. (3) $f'(x) = 12x^2 - 6x - 1$

$$= 12\left(x - \frac{1}{4}\right)^2 - \frac{7}{4} \geq -\frac{7}{4}$$

Hence, $\frac{f(x_1) - f(x_2)}{x_1 - x_2} \geq -\frac{7}{4}$.

51. (4) Let $g(x) = f(x) - x^2$. We have $g(1) = 0, g(2) = 0, g(3) = 0$ [$\because f(1) = 1, f(2) = 4, f(3) = 9$].

From Rolle's theorem on $g(x)$, $g'(x) = 0$ for at least one $x \in (1, 2)$. Let $g'(c_1) = 0$ where $c_1 \in (1, 2)$.

Similarly, $g(x) = 0$ for at least one $x \in (2, 3)$. Let $g'(c_2) = 0$ where $c_2 \in (2, 3)$. Therefore,

$g'(c_1) = g'(c_2) = 0$

By Rolle's theorem, at least one $x \in (c_1, c_2)$ such that $g''(x) = 0$ or $f''(x) = 2$ for some $x \in (1, 3)$.

52. (2) $f\left(\frac{5\pi}{6}\right) = \log\sin\left(\frac{5\pi}{6}\right) = \log\sin\frac{\pi}{6} = \log\frac{1}{2} = -\log 2$

$$f\left(\frac{\pi}{6}\right) = \log\sin\frac{\pi}{6} = -\log 2$$

$$f'(c) = \frac{1}{\sin x}\cos x = \cot x$$

By Lagrange's mean value theorem,

$$\frac{f(5\pi/6) - f(\pi/6)}{(5\pi/6) - (\pi/6)} = \cot c$$

or $\cot c = 0$ or $c = \frac{\pi}{2}$

Thus, $c = \frac{\pi}{2} \in (\pi/6, 5\pi/6)$.

53. (4) (1) Discontinuous at $x = 1 \Rightarrow$ not applicable.

(2) $F(x)$ is not continuous (jump discontinuity) at $x = 0$.

(3) Discontinuity(missing point) at $x = 1 \Rightarrow$ not applicable.

(4) Notice that $x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 - x - 6)$.

Hence, $f(x) = x^2 - x - 6$ if $x \neq 1$ and $f(1) = -6$.

Thus, f is continuous at $x = 1$. So, $f(x) = x^2 - x - 6$ is continuous in the interval $[-2, 3]$.

Also, note that $f(-2) = f(3) = 0$. Hence, Rolle's theorem implies $f'(x) = 2x - 1 = 0$ for at least one $x \in [-2, 3]$.

So, $x = 1/2$ which lies between -2 and 3.

54. (2) Applying LMVT in [0, 1] to the function $y = f(x)$, we get

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}, \text{ for some } c \in (0, 1)$$

or $e^{c^2} = \frac{f(1) - f(0)}{1}$

or $f(1) - 10 = e^{c^2}$ for some $c \in (0, 1)$

But $1 < e^{c^2} < e$ in (0, 1)

or $1 < f(1) - 10 < e$

or $11 < f(1) < 10 + e$

or $A = 11, B = 10 + e$

or $A - B = 1 - e$

55. (2) We know that there exists at least one x in (0, 1) for which

$$\frac{f(1) - f(0)}{g(1) - g(0)} = \frac{f'(x)}{g'(x)}$$

or $\frac{2 - 10}{4 - 2} = \frac{f'(x)}{g'(x)}$

or $f'(x) + 4g'(x) = 0$

for at least one x in (0, 1).

56. (3) From LMVT, there exists at least $(n-1)$ points where $f'(x) = m$.

Therefore, there exist at least $(n-2)$ points where $f''(x) = 0$ (using Rolle's theorem).

57. (4) We have $f'(1) = 1$

Now, $f(2x) = f(x)$ (1)

$\therefore \quad 2f'(2x) = f'(x)$ (2)

$\Rightarrow \quad f'(2) = \frac{1}{2}$ (Putting $x = 1$)

In (2), putting $x = 2$, we get

$$2f'(4) = f'(2) = \frac{1}{2}$$

$$\therefore \quad f'(4) = \frac{1}{4}$$

Since $f'(x)$ is differentiable, there exists at least one $c \in (2, 4)$, such that

$$f''(c) = \frac{f'(4) - f'(2)}{4-2} = \frac{\frac{1}{4} - \frac{1}{2}}{2} = -\frac{1}{8}.$$

58. (4) Let $g(x) = f(x) + Ax^3$ and choose A such that $g(a) = g(b)$. Therefore,

$$A = \frac{-f(b) + f(a)}{b^3 - a^3} = \frac{-4}{b^3 - a^3}$$

Since $g(x)$ satisfies condition of Rolle's theorem,

$g'(c) = 0$ for some $c \in (a, b)$

or $\quad f'(c) - 12\,\frac{c^2}{b^3 - a^3} = 0$

or $\quad (b^3 - a^3) f'(c) = 12c^2$

Multiple Correct Answers Type

1. (1), (2), (4)

$$f(x) = \frac{x}{1-x^2}$$

$$\therefore \quad f'(x) = \frac{1+x^2}{(1-x^2)^2} = 1, \text{ i.e., } x = 0, -\sqrt{3}, \sqrt{3}$$

Therefore, the points are $(0, 0)$, $\left(\pm\sqrt{3}, \mp\frac{\sqrt{3}}{2}\right)$.

2. (1), (2), (3), (4)

We have $y = ce^{x/a}$

$$\therefore \quad \frac{dy}{dx} = \frac{c}{a}e^{x/a} = \frac{1}{a}y$$

or $\quad \frac{y}{dy/dx} = a = \text{constant}$

or $\quad$ Sub-tangent = constant

Length of the sub-normal $= y\frac{dy}{dx} = y\,\frac{y}{a} = \frac{y^2}{a}$

$\propto$ (square of the ordinate)

Equation of the tangent at (x_1, y_1) is $y - y_1 = \frac{y_1}{a}(x - x_1)$.

This meets the x-axis at a point given by

$$-y_1 = \frac{y_1}{a}(x - x_1) \text{ or } x = x_1 - a$$

The curve meets the y-axis at $(0, c)$. Therefore,

$$\left(\frac{dy}{dx}\right)_{(0,c)} = \frac{c}{a}$$

So, the equation of the normal at $(0, c)$ is

$$y - c = -\frac{1}{c/a}(x - 0) \text{ or } ax + cy = c^2$$

3. (1), (3), (4)

$y = x(c - x)$ (1)

$y = x^2 + ax + b$ (2)

Slope of curve (1) $= c - 2x$

At $(1, 0)$, $c - 2 = m_1$ (say)

Slope of curve (2) $= 2x + a$

At $(1, 0)$, $2 + a = m_1$ (say)

Curves are touching at $(1, 0)$. Therefore,

$m_1 = m_2$

or $\quad 2 + a = c - 2$ (3)

Also, $(1, 0)$ lies on both the curves. Therefore,

$0 = c - 1$ and $0 = 1 + a + b$ (4)

Solving (3) and (4), we get

$a = -3, b = 2, c = 1$

4. (2), (3)

$y = x^2 + 4x - 17$

or $\quad \frac{dy}{dx} = 2(x+2)$ or $\left(\frac{dy}{dx}\right)_{x=\frac{5}{2}} = 9$

or $\quad \tan\theta = 9$

where θ is the angle with positive direction of x-axis.

Therefore, angle with y-axis is $\frac{\pi}{2} \pm \theta = \frac{\pi}{2} \pm \tan^{-1}9$.

5. (1), (2), (3), (4)

a. $y^2 = 4ax$ or $m_1 = y' = \frac{2a}{y}$

$y = e^{-x}/2a$ or $m_2 = y' = -\frac{1}{2a}e^{-x/2a} = -\frac{1}{2a}y$

$m_1 m_2 = -1$.

Hence, orthogonal.

b. $y^2 = 4ax$

or $\quad y' = \frac{4a}{2y_1} = \frac{2a}{y_1}$, [Not defined at $(0, 0)$]

$x^2 = 4ay$

or $\quad y' = \frac{2x_1}{4a} = \frac{x_1}{2a} = 0$ at $(0, 0)$

Therefore, the two curves are orthogonal at $(0, 0)$.

c. $xy = a^2, x^2 - y^2 = b^2$

$$\therefore \quad m_1 = y'' = -\frac{y}{x}$$

and $\quad m_2 = y' = \frac{x}{y}$

$m_1 m_2 = -1$

Hence, orthogonal.

d. $y = ax \Rightarrow y' = a = m_1$

$x^2 + y^2 = c^2 \Rightarrow y' = -\frac{x_1}{y_1} = m_2$

$$m_1 m_2 = -\frac{ax_1}{y_1} = -\frac{y_1}{y_1} = -1$$

Hence, orthogonal.

6. **(1), (2)**

Since the intercepts are equal in magnitude but opposite in sign,

$$\left.\frac{dy}{dx}\right|_P = 1$$

Now, $\frac{dy}{dx} = x^2 - 5x + 7 = 1$

or $x^2 - 5x + 6 = 0$

i.e., $x = 2$ or 3

7. **(1), (3)**

$$xy = (a + x)^2$$

or $y + xy' = 2(a + x)$

Now, $y' = \pm 1$

or $y \pm x = 2(a + x)$

$$\frac{(a+x)^2}{x} \pm x = 2(a + x)$$

or $$\pm x = 2(a + x) - \frac{(a+x)^2}{x}$$

or $\pm x^2 = (a + x)(x - a)$

or $\pm x^2 = x^2 - a^2$

or $2x^2 = a^2$ or $x = \pm \frac{a}{\sqrt{2}}$

8. **(1), (2)**

Let $P(x, y)$ be a point on the curve $\ln(x^2 + y^2) = c \tan^{-1}\frac{y}{x}$.

Differentiating both sides with respect to x, we get

$$\frac{2x + 2yy'}{(x^2 + y^2)} = \frac{c(xy' - y)}{(x^2 + y^2)} \text{ or } y' = \frac{2x + cy}{cx - 2y} = m_1 \text{ (say)}$$

Slope of $OP = \frac{y}{x} = m_2$ (say) (where O is origin)

Let the angle between the tangents at P and OP be θ. Then,

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{2x + cy}{cx - 2y} - \frac{y}{x}}{1 + \frac{2xy + cy^2}{cx^2 - 2xy}}\right| = \frac{2}{c}$$

or $\theta = \tan^{-1}\left(\frac{2}{c}\right)$

which is independent of x and y.

9. **(1), (2), (3)**

At any point t on the given curve, we have

$$\frac{dy}{dx} = \frac{3a\cos^2 t(-\sin t)}{3a\sin^2 t(\cos t)} = -\cot t$$

The equations of the tangent and normal at t are

$$x\cot t + y\sin t - (a/2)\sin 2t = 0 \quad (1)$$

and $$x\sin t - y\cos t - a\cos 2t = 0 \quad (2)$$

$$\therefore \quad OT = \frac{|(a/2)\sin 2t|}{\sqrt{\cos^2 t + \sin^2 t}} = \frac{a}{2}\sin 2t$$

or $2OT = a\sin 2t$

and $$ON = \frac{|a\cos 2t|}{\sqrt{\sin^2 t + \cos^2 t}} = a\cos 2t$$

$$\therefore \quad 4OT^2 + ON^2 = a^2\sin^2 2t + a^2\cos^2 2t = a^2$$

Length of the tangent at $t = \frac{\left|y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right|}{|dy/dx|}$

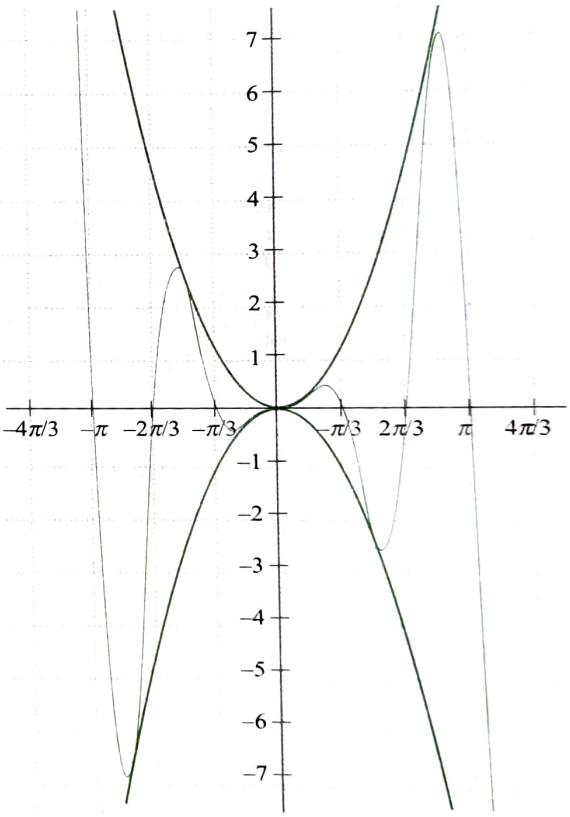
$$= \left|\frac{y\,\text{cosec}\,t}{-\cot t}\right|$$

$$= \left|\frac{y}{\cos t}\right|$$

Length of the normal at $t = \left|y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right|$

$$= \left|y\sqrt{1 + \cot^2 t}\right| = \left|\frac{y}{\sin t}\right|$$

10. **(1), (2), (3)**



Verify from the graph.

11. **(2), (4)**

Here, $y = \frac{1}{x + a}$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{(x + a)^2}$$

So, the slope of normal is $(x + a)^2 > 0$ for all x.

Thus, the line $x\cos\theta + y\sin\theta = P$ is normal to the curve if

$$-\frac{\cos\theta}{\sin\theta} > 0$$

or $\cot\theta < 0$

Therefore, θ lies in II or IV quadrant.

12. (3), (4)

Equation of tangent to $y = x^3$ at (x_1, y_1) is

$$y - x_1^3 = 3x_1^2(x - x_1) \quad (1)$$

Equation of tangent to $y = x^{1/3}$ at (x_1, y_1) is

$$y - x_2^{1/3} = \frac{1}{3x_2^{2/3}}(x - x_2) \quad (2)$$

Equations (1) and (2) represent the same line if

$$\frac{1}{1} = \frac{9x_1^2x_2^{2/3}}{1} = \frac{-2x_1^3}{\frac{2}{3}x_2^{1/3}} \Rightarrow x_1 = \pm\frac{1}{\sqrt{3}}$$

13. (1), (2), (4)

f is not differentiable at $x = \frac{1}{2}$

g is not continuous in $[0, 1]$ at $x = 0$

h is not continuous in $[0, 1]$ at $x = 1$

$k(x) = (x+3)^{\ln_2 5} = (x+3)^p$, where $2 < p < 3$,

which is continuous and differentiable.

14. (1), (2), (3)

Clearly, $f(0) = 0$. So, $f(x) = 0$ has two real roots 0, α_0 (> 0).

Therefore, $f'(x) = 0$ has a real root α_1 lying between 0 and α_0.

So, $0 < \alpha_1 < \alpha_0$.

Again, $f'(x) = 0$ is a fourth-degree equation. As imaginary roots occur in conjugate pairs, $f'(x) = 0$ will have another real root α_2. Therefore, $f''(x) = 0$ will have a real root lying between α_1 and α_2. As $f(x) = 0$ is an equation of the fifth degree, it will have at least three real roots and, so, $f'(x)$ will have at least two real roots.

15. (1), (2), (3)

(1) Let $f(x) = e^x \cos x - 1$

$\therefore$ $f'(x) = e^x(\cos x - \sin x) = 0$

$\therefore$ $\tan x = 1$, which has a root between two roots of $f(x) = 0$

(2) Let $f(x) = e^x \sin x - 1$

$\therefore$ $f'(x) = e^x(\sin x + \cos x) = 0$

$\therefore$ $\tan x = -1$, which has a root between two roots of $f(x) = 0$

(3) Let $f(x) = e^{-x} - \cos x$

$\therefore$ $f'(x) = -e^{-x} + \sin x = 0$

$\therefore$ $e^{-x} = \sin x$, which has a root between two roots of $f(x) = 0$

16. (2), (3), (4)

Function $f(x) = x^{\frac{1}{3}}$ is non-differentiable at $x = 0$.

But the functions given in the options (2), (3) and (4) are differentiable in the given domain.

17. (1), (2), (3)

Consider $g(x) = e^x f(x)$.

Then $g(a) = g(b)$.

Using Rolle's theorem, there exists at least one $c \in (a, b)$ for which

$g'(c) = 0$

or $e^c f(c) + e^c f'(c) = 0$

or $f(c) + f'(c) = 0$

Thus, $f(x) + f'(x) = 0$ has at least one root in (a, b).

Consider $g(x) = e^{-x} f(x)$.

Then $g(a) = g(b)$.

Using Rolle's theorem, there exists at least one $c \in (a, b)$ for which

$g'(c) = 0$

or $-e^{-c} f(c) + e^c f'(c) = 0$

or $-f(c) + f'(c) = 0$

Thus, $f'(x) - f(x) = 0$ has at least one root in (a, b).

Now, consider $g(x) = (f(x))^2$.

Then $g(a) = g(b)$.

Using Rolle's theorem, there exists at least one $c \in (a, b)$ for which

$g'(c) = 0$

or $2f'(c) f(c) = 0$

or $f'(c) f(c) = 0$

Thus, $f'(x) f(x) = 0$ has at least one root in (a, b).

Linked Comprehension Type

1. (1) Let $P_1(t_1, t_1^3)$ be a point on the curve $y = x^3$. Therefore

$$\left.\frac{dy}{dx}\right|_{(t_1, t_1^3)} = 3t_1^2$$

Tangent at P_1 is $y - t_1^3 = 3t_1^2(x - t_1)$

The intersection of (1) and $y = x^3$ is

$x^3 - t_1^3 = 3t_1^2(x - t_1)$

or $(x - t_1)(x^2 + xt_1 + t_1^2) - 3t_1^2(x - t_1) = 0$

or $(x - t_1)^2(x + 2t_1) = 0$

If $P_2(t_2, t_2^3)$, then

$(t_2 - t_1)^2 (t_2 + 2t_1) = 0$

$\therefore$ $t_2 = -2t_1 \quad (t_2 \neq t_1)$

Similarly, the tangent at P_2 will meet the curve at the point $P_3(t_3, t_3^3)$ when $t_3 = -2t_2 = 4t_1$ and so on.

The abscissae of $P_1, P_2, \ldots, P_n$ are

$t_1, -2t_1, 4t_1, \ldots, (-2)^{n-1} t_1$

These are in G.P. Therefore,

$$\frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3} = \cdots = -2 \ (r \text{ say})$$

$\therefore$ $t_2 = t_1 r$, $t_3 = t_2 r$, and $t_4 = t_3 r$

If $x_1 = 1$, then $x_2 = -2$, $t_3 = 4$, ...Then,

$$\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{x_n} = \text{sum of infinite G.P. with common ratio } (-1$$

with first term 1

$$= \frac{1}{1 - \left(-\frac{1}{2}\right)} = \frac{2}{3}$$

2. (3) Then $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{y_n}$ = sum of infinite G.P. with common ratio $(-1/8)$ with first term 1

$$= \frac{1}{1 - \left(-\frac{1}{8}\right)} = \frac{8}{9}$$

3. (4) $\therefore$ Area of $\Delta P_2P_3P_4 = \frac{1}{2}\begin{vmatrix} t_2 & t_2^3 & 1 \\ t_3 & t_3^3 & 1 \\ t_4 & t_4^3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} rt_1 & r^3t_1^3 & 1 \\ rt_2 & r^3t_2^3 & 1 \\ rt_3 & r^3t_3^3 & 1 \end{vmatrix}$

$= r^4$ (Area of $(\Delta P_1P_2P_3)$

$\therefore$ $\frac{\text{Area of } (\Delta P_1P_2P_3)}{\text{Area of } (\Delta P_2P_3P_4)} = \frac{1}{r^4} = \frac{1}{(-2)^4} = \frac{1}{16}$

4. (1) $\frac{dy}{dx} = \frac{1-9t^2}{-6t} = \tan\theta$

or $9t^2 - 6\tan\theta . t - 1 = 0$

or $3t = \tan\theta \pm \sec\theta$

or $\tan\theta + \sec\theta = 3t$

5. (2) $P \equiv (-2, 2)$ or $t = -1$ or $\left.\frac{dy}{dx}\right|_{t=-1} = -\frac{4}{3} = m_P$

Equation of the tangent is $y - 2 = -\frac{4}{3}(x+2)$. Therefore,

$$t - 3t^3 - 2 = -\frac{4}{3}(1 - 3t^2 + 2)$$

or $9t^3 + 12t^2 - 3t - 6 = 0$

or $3t^3 + 4t^2 - t - 2 = 0$

or $(3t^2 + t - 2)(t+1) = 0$

or $(3t-2)(t+1)^2 = 0$

or $t = \frac{2}{3}$

or $Q \equiv \left(-\frac{1}{3}, -\frac{2}{9}\right)$

6. (3) $\left.\frac{dy}{dx}\right|_{t=2/3} = \frac{3}{4} = m_Q$

$m_P m_Q = -1$ or angle $= 90°$

7. (2), **8. (1)**

Let V be the volume and r the radius of the balloon at any time. Then,

$$V = \left(\frac{4}{3}\right)\pi r^3$$

$$\therefore \quad \frac{dV}{dt} = \left(\frac{4}{3}\right)(3\pi r^2)\frac{dr}{dt}$$

$$= 4\pi r^2 \frac{dr}{dt} = 40 \qquad \text{(Given)}$$

$$\therefore \quad \frac{dr}{dt} = \frac{10}{\pi r^2} \qquad (1)$$

Now, let S be the surface area of the balloon when its radius is r. Then

$$S = 4\pi r^2$$

$$\therefore \quad \frac{dS}{dt} = 8\pi r \frac{dr}{dt} \qquad (2)$$

From (1) and (2), $\frac{dS}{dt} = 8\pi r \frac{10}{\pi r^2} = \frac{80}{r}$

When $r = 8$, the rate of increase of $S = \frac{80}{8} = 10\,\text{cm}^2/\text{min}$.

Therefore, increase of S in $\frac{1}{2}$ min $= 10 \times \left(\frac{1}{2}\right) = 5$ cm^2/min.

If r_1 is the radius of the balloon after (1/2) min, then $4\pi r_1^2 = 4\pi(8)^2 + 5$

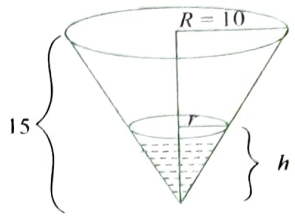
or $r_1^2 - 8^2 = \frac{5}{4\pi} = 0.397$ nearly

or $r_1^2 = 64.397$ or $r_1 = 8.025$ nearly.

$\therefore$ Required increase in the radius $= r_1 - 8 = 8.025 - 8$

$= 0.025$ cm

9. (2)



$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{2}{3}h\right)^2 h = \frac{4}{27}\pi h^3$$

$$\frac{dV}{dt} = \frac{4}{9}\pi h^2 \frac{dh}{dt}$$

$$\Rightarrow \quad 5 = \frac{4}{9}\pi \times (7.5)^2 \frac{dh}{dt}$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{45}{4\pi \times (7.5)^2} = \frac{1}{5\pi} \text{ cm/min}$$

10. (1) $V(\text{at } h = 3) = 4\pi$

11. (4) $\frac{dV}{dt} = \frac{4}{9}\pi h^2 \frac{dh}{dt}$

$$\Rightarrow \quad \frac{d^2V}{dt^2} = \frac{4}{9}\pi\left[2h\left(\frac{dh}{dt}\right)^2 + h^2\frac{d^2h}{dt^2}\right]$$

$$\Rightarrow \quad -\frac{4}{9} = \frac{4}{9}\pi(7.5)\left[2\left(\frac{1}{5\pi}\right)^2 + 7.5\frac{d^2h}{dt^2}\right]$$

$$\Rightarrow \quad \frac{d^2h}{dt^2} = \frac{1}{7.5}\left(\frac{-1}{7.5\pi} - \frac{2}{25\pi^2}\right)$$

12. (3), 13. (4)

$$\frac{dy}{dt} = 2 \text{ cm/sec}, \; y^3 = -x$$

$$\Rightarrow \quad 3y^2\frac{dy}{dt} = -\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dx}{dt} = -6y^2$$

When ΔABC is right triangle, then $m_{AC} \times m_{AB} = -1$

$$\Rightarrow \quad \frac{y}{x} \times \frac{2}{8} = -1$$

$$\Rightarrow \quad y = -4x$$

Solving with $y^3 = -x$, we get

$$y = \frac{1}{2} \text{ and } x = -\frac{1}{8}$$

Now, $dy = 2dt$

$$\Rightarrow \quad \int_0^{1/2} dy = 2\int_0^{t_0} dt$$

$$\Rightarrow \quad 1/2 = 2t_0$$

$$\Rightarrow \quad t_0 = \frac{1}{4}\text{sec}$$

From the figure, we get

$$\frac{\beta - \frac{1}{2}}{\alpha + \frac{1}{8}} = \left.\frac{dy}{dx}\right|_C$$

$$\Rightarrow \quad \frac{(2\beta-1)4}{8\alpha+1} = -\frac{1}{3y^2}\bigg|_{y=\frac{1}{2}} = -\frac{4}{3}$$

$$\Rightarrow \quad \frac{2\beta-1}{8\alpha+1} = -\frac{1}{3}$$

$$\Rightarrow \quad 6\beta - 3 = -8\alpha - 1$$

$$\Rightarrow \quad 6\beta + 8\alpha = 2$$

$$\Rightarrow \quad 3\beta + 4\alpha = 1$$

Matrix Match Type

1. a → p, q; b → r, s; c → r, q; d → p, s.

a. Given $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ (say)

$\therefore \quad da = 2R\cos A\ dA$

$db = 2R\cos B\ dB$

$dc = 2R\cos C\ dC$

$$\therefore \quad \frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 2R\,(dA + dB + dC) \qquad (1)$$

Also, $A + B + C = \pi$

So, $dA + dB + dC = 0$ (2)

From equations (1) and (2), we get

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} + 1 = 1$$

or $m = \pm 1$

b. $x^2y^2 = 16$ or $xy = \pm 4$ (1)

$$L_{ST} = \left|\frac{y}{dy/dx}\right|$$

Differentiating (1) w.r.t. x, we get

$$y + xy' = 0 \text{ or } y' = \frac{-y}{x}$$

$$L_{ST} = \left|\frac{y}{y/x}\right| = |x| \text{ or } L_{ST} = 2$$

or $k = \pm 2$.

c. $y = 2e^{2x}$ intersects y-axis at (0, 2)

$$\frac{dy}{dx} = 4e^{2x}$$

$$\therefore \quad \frac{dy}{dx}\bigg|_{\text{at } x=0} = 4$$

$\therefore$ Angle of intersection with y-axis $= \frac{\pi}{2} - \tan^{-1} 4 = \cot^{-1} 4$

$\therefore \quad n = 2 \text{ or } -1$

d. $\frac{dx}{dy} = e^{\sin y}\cos y$

Slope of the normal at $(1, 0) = -1$

Equation of the normal is $x + y = 1$.

$$\text{Area} = \frac{1}{2}$$

$\therefore \quad t = 1, -2.$

2. a → q; b → r; c → p; d → s

a. $r = 6$ cm $\delta r = 0.06$

$A = \pi r^2$ or $\delta A = 2\pi r\delta r = 2\pi(6)(0.06) = 0.72\pi$

b. $V = x^3$, $\delta V = 3x^2\,\delta x$

$$\frac{\delta V}{V} \times 100 = 3\frac{\delta x}{x} \times 100 = 3 \times 2 = 6$$

c. $(x-2)\frac{dx}{dt} = 3\frac{dx}{dt}$

or $x = 5$

d. $A = \frac{\sqrt{3}}{4}x^2$

or $\frac{dA}{dt} = \frac{\sqrt{3}}{2}\left(x\frac{dx}{dt}\right) = \frac{\sqrt{3}}{2} \times 30 \times \frac{1}{10} = \frac{3\sqrt{3}}{2}$

3. a → p, q; b → p, s; c → r; d → q

a. $y^2 = 4x$ and $x^2 = 4y$ intersect at points (0, 0) and (4, 4)

$C_1: y^2 = 4x$ $C_2: x^2 = 4y$

$$\frac{dy}{dx} = \frac{2}{y}, \quad \frac{dy}{dx} = \frac{x}{2}$$

$$\frac{dy}{dx}\bigg|_{0,0} = \infty \quad \frac{dy}{dx}\bigg|_{0,0} = 0$$

Hence, $\tan\theta = 90°$ at point (0, 0).

$$\frac{dy}{dx}\bigg|_{(4,4)} = \frac{1}{2} \quad \frac{dy}{dx}\bigg|_{(4,4)} = 2$$

$$\tan\theta = \left|\frac{2 - \frac{1}{2}}{1 + 2 \times \frac{1}{2}}\right| = \frac{3}{4}$$

b. Solving I: $2y^2 = x^3$ and II: $y^2 = 32x$, we get (0, 0), (8, 16), and (8, –16).

At (0, 0), $\frac{dy}{dx}\Big|_{(0,0)} = 0$ for I

At (0, 0), $\frac{dy}{dx}\Big|_{(0,0)} = \infty$ for II

Hence, angle = 90°

Now, $\frac{dy}{dx}\Big|_{(8,16)} = \frac{3x^2}{4y} = \frac{3}{4}\frac{64}{16} = 3$ for I

$$\frac{dy}{dx}\bigg|_{(8,16)} = \frac{32}{2y} = \frac{16}{16} = 1 \text{ for II}$$

$$\therefore \quad \tan\theta = \frac{3-1}{1+3} = \frac{2}{4} = \frac{1}{2}$$

c. The two curves are

$xy = a^2$ (1)

$x^2 + y^2 = 2a^2$ (2)

Solving (1) and (2), the points of intersection are (a, a) and $(-a, -a)$

Differentiating (1), $dy/dx = -y/x = m_1$ (say).

Differentiating (2), $dy/dx = -x/y = m_2$ (say).

At both points, $m_1 = -1 = m_2$.

Hence, the two curves touch each other.

d. $y^2 = x$, $x^3 + y^3 = 3xy$

For the first curve, $2y\frac{dy}{dx} = 1$ or $\frac{dy}{dx}\Big|_P = \frac{1}{2y_1}$

For the second curve, $\frac{dy}{dx}\Big|_P = \frac{y_1 - x_1^2}{y_1^2 - x_1}$

Solving $y^2 = x$ and $x^3 + y^3 = 3xy$,

$y^6 + y^3 = 3y^3$ or $y^3 + 1 = 3$ or $y^3 = 2$

$\therefore \quad y_1 = 2^{1/3}$ and $x_1 = 2^{2/3}$.

Now, $m_1 = \dfrac{1}{2\times 2^{\frac{1}{3}}} = \dfrac{1}{2^{\frac{4}{3}}}$; $m_2 = \dfrac{2^{\frac{1}{3}} - 2^{\frac{4}{3}}}{2^{\frac{2}{3}} - 2^{\frac{2}{3}}} \to \infty$

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{1 - \frac{m_1}{m_2}}{\frac{1}{m_2} + m_1}\right| = \left|\frac{1}{m_1}\right| = 2^{\frac{4}{3}} = 16^{\frac{1}{3}}$$

$\therefore \quad \theta = \tan^{-1}(16^{1/3})$

4.(3) Since the curve $y = ax^3 + bx^2 + cx + 5$ touches the x-axis at $P(-2, 0)$, then x-axis is the tangent at $(-2, 0)$. The curve meets y-axis in $(0, 5)$.

We have

$$\frac{dy}{dx} = 3ax^2 + 2bx + c$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{(0,5)} = 0 + 0 + c = 3 \qquad \text{(Given)}$$

$$\Rightarrow \quad c = 3 \qquad (1)$$

and $\left.\dfrac{dy}{dx}\right|_{(-2,0)} = 0$

$$\Rightarrow \quad 12a - 4b + c = 0$$

$$\Rightarrow \quad 12a - 4b + 3 = 0 \quad \text{[From (1)]} \qquad (2)$$

Point $(-2, 0)$ lies on the curve. Then

$$0 = -8a + 4b - 2c + 5$$

$$\Rightarrow \quad 0 = -8a + 4b - 1 \qquad (\because \ c = 3)$$

$$\Rightarrow \quad 8a - 4b + 1 = 0 \qquad (3)$$

From (2) and (3), we get

$$a = -\frac{1}{2}, b = -\frac{3}{4}$$

Hence, $a = -\dfrac{1}{2}$, $b = -\dfrac{3}{4}$ and $c = 3$.

5.(2) (1) Let $f'(x) = \log x - \dfrac{3}{x} + 1$

$$\Rightarrow \quad f(x) = (x - 3)\log x + c$$

$$\Rightarrow \quad f(1) = f(3)$$

(2) $f'(x) = 4ax^3 + 3bx^2 + 2cx + d$

$$\Rightarrow \quad f(x) = ax^4 + bx^3 + cx^2 + dx + e$$

$$\Rightarrow \quad f(0) = f(3)$$

$$\Rightarrow \quad 27a + 9b + 3c + d = 0$$

(3) $\dfrac{f(b) - f(a)}{b - a} = f'\left(\sqrt{3}\right) = \dfrac{2}{3}$

$$\Rightarrow \quad \frac{ab - 1}{ab} = \frac{2}{3}$$

(4) $\dfrac{f(b) - f(a)}{b - a} = f'\left(\dfrac{1}{2}\right)$

$$\Rightarrow \quad a + b = 1$$

Numerical Value Type

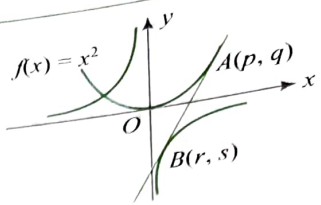
1.(5) $y = x^2$ and $y = -\dfrac{8}{x}$; $\quad q = p^2$ and $s = -\dfrac{8}{r}$ (1)

Equating $\dfrac{dy}{dx}$ at A and B, we get

$$2p = \frac{8}{r^2} \qquad (1)$$

or $\quad pr^2 = 4$

Now, $m_{AB} = \dfrac{q - s}{p - r}$



or $\quad 2p = \dfrac{p^2 + \frac{8}{r}}{p - r}$

or $\quad p^2 = 2pr + \dfrac{8}{r}$ or $p^2 = \dfrac{16}{r}$

or $\quad \dfrac{16}{r^4} = \dfrac{16}{r}$ = or $r = 1 \ (r \neq 0)$ or $p = 4$

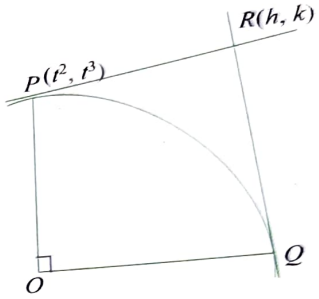
$\therefore \quad r = 1, \ p = 4$

Hence, $p + r = 5$.

2. (7) $x = t^2$; $y = t^3$

$$\frac{dx}{dt} = 2t; \ \frac{dy}{dt} = 3t^2$$

$$\frac{dy}{dx} = \frac{3t}{2}$$



Equation of tangent at P is

$$y - t^3 = \frac{3t}{2}(x - t^2)$$

$$2k - 2t^3 = 3th - 3t^3$$

$\therefore \quad t^3 - 3th + 2k = 0 \qquad (1)$

Product of roots, $t_1 t_2 t_3 = -2k$

Putting $t_1 t_2 = -1$, $t_3 = 2k$.

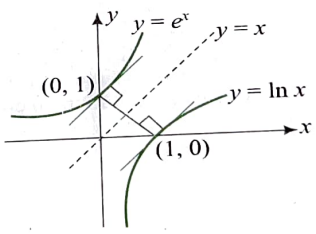
Now, t_3 must satisfy equation (1). Therefore,

$$(2k)^3 - 3(2k)h + 2k = 0$$

i.e., $\quad 4y^2 - 3x + 1 = 0$ or $4y^2 = 3x - 1$

or $\quad a + b = 7$

3. (8)



Since the graphs of $y = e^x$ and $y = \log_e x$ are symmetrical about the line $y = x$, minimum distance is the distance along the common normal to both the curves, i.e., $y = x$ must be parallel to the tangent as both the curves are inverse of each other.

$$\left.\frac{dy}{dx}\right|_{x_1} = e^{x_1} = 1$$

or $\quad x_1 = 0$ and $y_1 = 1$

or $\quad A \equiv (0, 1)$ and $B \equiv (1, 0)$

or $\quad AB = \sqrt{2}$

4. (12) $f(x) = f(6 - x)$ (1)

On differentiating (1) w.r.t. x, we get

$$f'(x) = -f'(6 - x) \qquad (2)$$

Putting $x = 0, 2, 3, 5$ in (2), we get

$$f'(0) = -f'(6) = 0$$

Similarly, $f'(2) = -f'(4) = 0$

$$f'(3) = 0$$

$$f'(5) = -f'(1) = 0$$

$\therefore \quad f'(0) = 0 = f'(2) = f'(3) = f'(5) = f'(1) = f'(4) = f'(6)$

Therefore, $f'(x) = 0$ has minimum seven roots in $[0, 6]$.

Now, consider a function $y = f'(x)$.

As $f'(x)$ satisfies Rolle's theorem in intervals $[0, 1]$, $[1, 2]$, $[2, 3]$, $[3, 4]$, $[4, 5]$, and $[5, 6]$, respectively, by Rolle's theorem, the equation $f''(x) = 0$ has minimum six roots.

Now, $g(x) = (f''(x))^2 + f'(x)\, f'''(x) = 0 = h'(x)$, where $h(x) = f'(x)\, f''(x)$.

Clearly, $h(x) = 0$ has minimum 13 roots in $[0, 6]$.

Hence, again by Rolle's theorem, $g(x) = h'(x)$ has minimum 12 zeroes in $[0, 6]$.

5. (2) $y = x^n$

$$\therefore \quad \frac{dy}{dx} = n x^{n-1} = na^{n-1}$$

$$\text{Slope of normal} = -\frac{1}{na^{n-1}}$$

Equation of normal is $y - a^n = -\dfrac{1}{na^{n-1}}(x - a)$.

Put $x = 0$ to get y-intercept. Then

$$y = a^n + \frac{1}{na^{n-2}}$$

Hence, $b = a^n + \dfrac{1}{na^{n-2}}$

$$\text{or,} \quad \lim_{a \to 0} b = \begin{cases} 0, & \text{if} \quad n < 2 \\ \dfrac{1}{2}, & \text{if} \quad n = 2 \\ \infty, & \text{if} \quad n > 2 \end{cases}$$

6. (3) $\dfrac{dy}{dx} = \dfrac{y}{x} = -\dfrac{1}{2}\cot^3\theta = -\dfrac{1}{2}$ at $\theta = \dfrac{\pi}{4}$

Also, the point P for $\theta = \pi/4$ is $(2, 1)$.

Equation of tangent is

$$y - 1 = -\frac{1}{2}(x - 2) \text{ or } x + 2y - 4 = 0 \qquad (1)$$

This meets the curve whose Cartesian equation on eliminating θ by $\sec^2\theta - \tan^2\theta = 1$ is

$$y^2 = \frac{1}{x - 1} \qquad (2)$$

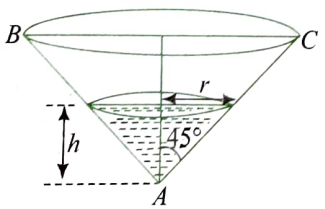
Solving (1) and (2), we get $y = 1, -\dfrac{1}{2}$

$$\therefore \quad x = 2, 5$$

Hence, P is $(2, 1)$ as given and Q is $\left(5, -\dfrac{1}{2}\right)$. Therefore,

$$PQ = \sqrt{\frac{45}{4}} = \frac{3\sqrt{5}}{2}$$

7. (5)



We have

$$\frac{dV}{dt} = 2 \text{ or } \frac{d}{dt}\left(\frac{1}{3}\pi r^3\right) = 2 \qquad [\text{Here, } r = h, \text{ as } \theta = 45°]$$

$$\text{or} \quad \pi r^2 \frac{dr}{dt} = 2 \quad \text{or} \quad \frac{dr}{dt} = \frac{2}{\pi r^2} \qquad (1)$$

Now, perimeter $= 2\pi r = p$ (let)

$$\therefore \quad \frac{d}{dt}(2\pi r) = 2\pi\left(\frac{2}{\pi r^2}\right) = \frac{4}{r^2} \qquad [\text{Using equation (1)}] \quad (2)$$

When $h = 2$ m, $r = 2$ m.

Hence, $\dfrac{dp}{dt} = \dfrac{4}{4} = 1$ m/s.

8. (13) $y = x^3 + x + 16$

$$\left(\frac{dy}{dx}\right)_{x_1, y_1} = 3x_1^2 + 1$$

$$\therefore \quad 3x_1^2 + 1 = \frac{y_1}{x_1}$$

$$\text{or} \quad 3x_1^3 + x_1 = x_1^3 + x_1 + 16$$

$$\text{or} \quad 2x_1^3 = 16$$

$$\text{or} \quad x_1 = 2 \quad \text{or} \quad y_1 = 26$$

$$\therefore \quad m = 13$$

9. (4) We have $f(0) = 2$

Now, $y - f(a) = f'(a)\,[x - a]$.

For x intercept, $y = 0$. So,

$$x = a - \frac{f(a)}{f'(a)} = a - 2 \quad \text{or} \quad \frac{f(a)}{f'(a)} = 2$$

$$\text{or} \quad \frac{f'(a)}{f(a)} = \frac{1}{2}$$

$\therefore$ On integrating both sides w.r.t. a, we get

$$\ln f(a) = \frac{a}{2} + \log_e C$$

$$f(a) = Ce^{a/2}$$

$$f(x) = Ce^{x/2}$$

$$f(0) = C \quad \text{or} \quad C = 2$$

$$\therefore \quad f(x) = 2e^{x/2}$$

Hence, $k = 2$, $p = \dfrac{1}{2}$ or $\dfrac{k}{p} = 4$

10. (9) $y = ax^2 + bx + c$, $\dfrac{dy}{dx} = 2ax + b$

When $x = 1$, $y = 0$

$$\therefore \quad a + b + c = 0 \qquad (1)$$

$$\left.\frac{dy}{dx}\right|_{x=1} = 3 \text{ and } \left.\frac{dy}{dx}\right|_{x=3} = 1$$

$$2a + b = 3 \qquad (2)$$

$$6a + b = 1 \qquad (3)$$

Solving (1), (2), and (3),

$$a = -\frac{1}{2},\ b = 4,\ c = -\frac{7}{2}$$

$$\therefore \quad 2a - b - 4c = -1 - 4 + 14 = 9$$

11. (5) $y = e^{a + bx^2}$ passes through $(1, 1)$

$$\therefore \quad 1 = e^{a+b}$$

$$\therefore \quad a + b = 0$$

$$\text{Also,} \left(\frac{dy}{dx}\right)_{(1,\,1)} = -2$$

$$\therefore \quad e^{a+bx^2} \cdot 2bx = -2$$

$$\text{or} \quad e^{a+b} \cdot 2b(1) = -2$$

$$\text{or} \quad b = -1 \text{ and } a = 1$$

$$\text{or} \quad 2a - 3b = 5$$

12. (4) Let $x = r\cos\theta, y = r\sin\theta$

$\therefore \quad r^2(1 + \cos\theta\sin\theta) = 1$

or $\quad r^2 = \dfrac{2}{2+\sin 2\theta}$

or $\quad r^2_{\max} = \dfrac{2}{1}$

13. (3) Let $f(x) = x^3, x \in [a, b]$.

$f(x)$ satisfies conditions of LMVT [as $f(x)$ is continuous and differentiable] Therefore,

$$\frac{f(b)-f(a)}{b-a} = f'(c),\ a < c < b$$

or $\quad \dfrac{b^3-a^3}{b-a} = 3c^2$

or $\quad b^2 + ba + a^2 = 3c^2$

14. (6) Let $F(x) = f^3(x)$.

$F(x)$ is continuous and differentiable function in [1, 3].

$$\therefore \quad \frac{F(3)-F(1)}{3-1} = F'(c) \text{ for at least one } c \in (1, 3)$$

$$\therefore \quad \frac{f^3(3)-f^3(1)}{2} = 3f^2(c)\times f'(c)$$

$\therefore \quad k = 6$

15. (-1) Equation of tangent to the curve $f(x, y) = 0$ at any point $P(x, y)$ on it is

$$Y - y = \frac{dy}{dx}(X - x)$$

x-intercept of the tangent $= y$ (given)

$$\Rightarrow \quad x - \frac{dx}{dy}y = y$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{x-y}{y}$$

$$\Rightarrow \quad \frac{d^2x}{dy^2} = \frac{y\dfrac{dx}{dy} - x}{y^2}$$

At $x = 1, y = 1, \dfrac{dx}{dy} = 0$

So, at (1, 1), $\dfrac{d^2x}{dy^2} = -1$.

16. (2) Let $g(x) = f(x) - \sin x^2$

$\therefore \quad g(1) = g(2) = g(3) = 0$

So, $g'(x)$ vanishes at least twice.

So, $f'(x) = 2x\cos x^2$ has at least 2 roots in the interval (1, 3).

17. (3) We must have

$f(1) = f(2)$

$\therefore \quad 3m + n + 7 = 0$ (1)

Also, $f'(x) = 0$ at $x = 4/3$.

$\therefore \quad 8m + 3n + 16 = 0$ (2)

Solving, we get $m = -5$ and $n = 8$.

18. (-2) Differentiating the curve w.r.t. x, we get

$$2x - xy' - y + 2yy' + \alpha = 0$$

$$\therefore \quad y' = \frac{2x - y + \alpha}{x - 2y}$$

Therefore, the equation of tangent at (2, 2) is

$$y - 2 = \frac{2+\alpha}{-2}(x - 2)$$

or $\quad \left(1 + \dfrac{\alpha}{2}\right)x + y - \alpha - 4 = 0$

It is given that the distance of (0, 0) from the line is 2.

$$\Rightarrow \quad \frac{|\alpha + 4|}{\sqrt{\left(1+\frac{\alpha}{2}\right)^2 + 1}} = 2$$

$$\Rightarrow \quad (\alpha + 4)^2 = 4\left[1 + \alpha + \frac{\alpha^2}{4} + 1\right]$$

$$\Rightarrow \quad \alpha^2 + 8\alpha + 16 = 8 + \alpha^2 + 4\alpha$$

$$\Rightarrow \quad 4\alpha = -8$$

$$\Rightarrow \quad \alpha = -2$$

19. (7) Given that, $f(x) = \begin{cases} x\log_e x, & x > 0 \\ 0, & x = 0 \end{cases}$

As conclusion of LMVT holds at $x = 1$ in [0, a], we have

$$f'(1) = \frac{f(a) - f(0)}{a - 0} \quad (1)$$

Now, $f'(x) = \left(x \times \dfrac{1}{x} + 1 \times \log_e x\right) = 1 + \log_e x$

$\therefore \quad f'(1) = 1 + \log_e 1 = 1$

So, from (1), we have $1 = \dfrac{a\log_e a - 0}{a}$

$\therefore \quad \log_e a = 1$

$\Rightarrow \quad a = e$

$\therefore \quad [a^2] = [e^2] = 7$

Archives

JEE Main

Single Correct Answer Type

1. (1) Shortest distance between line $x - y + 1 = 0$ and curve $x = y^2$ is distance of the point on the curve where tangent is parallel to given line from the given line.

$$\Rightarrow \quad 1 = 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2y} = \text{Slope of given line}$$

$$\Rightarrow \quad \frac{1}{2y} = 1 \Rightarrow y = \frac{1}{2} \Rightarrow x = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$\Rightarrow \quad (x, y) = \left(\frac{1}{4}, \frac{1}{2}\right)$$

Therefore the shortest distance

$$= \frac{\left|\frac{1}{4} - \frac{1}{2} + 1\right|}{\sqrt{1+1}} = \frac{3}{4\sqrt{2}} = \frac{3\sqrt{2}}{8}$$

2. (3) We have $y = x + \dfrac{4}{x^2}$. Tangent is parallel to x-axis

$$\Rightarrow \quad \frac{dy}{dx} = 0$$

$$\Rightarrow \quad 1 - \frac{8}{x^3} = 0$$

$$\Rightarrow \quad x = 2 \ \therefore \ y = 3$$

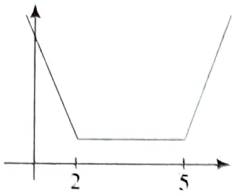
The equation of the tangent is

$y - 3 = 0\ (x - 2)$ or $y - 3 = 0$

3. (3) $f(x) = 3,\ 2 \le x \le 5$

$f'(x) = 0,\ 2 < x < 5$

$f'(4) = 0$



4. (4) Let $h(x) = f(x) - 2g(x)$

As $h(0) = h(1) = 2$

Hence, using Rolle's theorem

$h'(c) = 0$

$\Rightarrow \quad f'(c) = 2g'(c)$

5. (4) $x^2 + 2xy - 3y^2 = 0$

$\Rightarrow \quad 2x + 2xy' + 2y - 6yy' = 0$

$\Rightarrow \quad y' = \dfrac{x+y}{3y-x}$

At $x = 1, y = 1$ we have $\dfrac{dy}{dx} = 1$

$\therefore$ Equation of normal at (1, 1) is

$y - 1 = -(x - 1)$

$\Rightarrow \quad x + y = 2$

Solving with curve, we get

$x^2 + 2x(2 - x) - 3(2 - x)^2 = 0$

$\Rightarrow \quad x = 1, 3$

Thus, $P(1, 1)$ and $Q(3, -1)$.

So, normal meets curve again at $(3, -1)$ in fourth quadrant.

Aliternative Method:

$x^2 + 2xy - 3y^2 = 0$

$(x - y)(x + 3y) = 0$

$x - y = 0$ or $x + 3y = 0$

Equation of normal at (1, 1) to line $x - y = 0$ is $x + y - 2 = 0$, which intersects the line $x + 3y = 0$ at $(3, -1)$.

6. (1) $y = f(x) = \tan^{-1}\left(\sqrt{\dfrac{1+\sin x}{1-\sin x}}\right)$

At $x = \dfrac{\pi}{6}$, $y = \dfrac{\pi}{3}$

$$f(x) = \tan^{-1}\left(\left|\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right|\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right) \qquad \therefore\ x \in \left(0, \frac{\pi}{2}\right)$$

$\therefore \quad f(x) = \dfrac{\pi}{4} + \dfrac{x}{2}$

$\therefore \quad f'(x) = \dfrac{1}{2}$

$\therefore$ Slope of normal $= -2$

$\therefore$ Equation of normal is $y - \dfrac{\pi}{3} = -2\left(x - \dfrac{\pi}{6}\right)$

or $\quad y = -2x + \dfrac{2\pi}{3}$

7. (3) $y = \dfrac{x+6}{(x-2)(x-3)}$

Point of intersection with y-axis is (0, 1).

Now, $\dfrac{dy}{dx} = \dfrac{(x^2 - 5x + 6)(1) - (x+6)(2x-5)}{(x^2 - 5x + 6)^2}$

At (0, 1), $\dfrac{dy}{dx} = 1$

So, slope of normal is -1.

Hence, equation of normal is $x + y = 1$, which passes through the point (1/2, 1/2).

8. (1) Given curves are

$y^2 = 6x$ ...(i)

and $9x^2 + by^2 = 16$ ...(ii)

Let these curves intersect at right angles at point $P(x_1, y_1)$.

Differentiating curve (i) w.r.t. x, we get

$$2y\frac{dy}{dx} = 6$$

So, $\left(\dfrac{dy}{dx}\right)_P = \dfrac{3}{y_1}$ = slope of tangent to curve at point P

Differentiating curve (ii) w.r.t. x, we get

$$18x + 2by\frac{dy}{dx} = 0$$

So, $\left(\dfrac{dy}{dx}\right)_P = \dfrac{-9x_1}{by_1}$ = slope of tangent to curve at point P

At point P, tangents are perpendicular.

$\therefore \quad \left(\dfrac{3}{y_1}\right)\left(\dfrac{-9x_1}{by_1}\right) = -1$

$\Rightarrow \quad 27x_1 = by_1^2$

From curve (i), $y_1^2 = 6x_1$

$\therefore \quad 27x_1 = b(6x_1)$

$\Rightarrow \quad b = 27/6 = 9/2$

Multiple Correct Answers Type

1. (2), (3)

Let $h(x) = f(x) - 3g(x)$

$h(-1) = f(-1) - 3g(-1) = 3 - 0 = 3$

$h(0) = f(0) - 3g(0) = 6 - 3 = 3$

$h(2) = f(2) - 3g(2) = 0 - (-3) = 3$

Thus, $h'(x) = 0$ has at least one root in $(-1, 0)$ and at least one root in $(0, 2)$.

But since $h''(x) = 0$ has no root in $(-1, 0)$ and $(0, 2)$ therefore $h'(x) = 0$ has exactly 1 root in $(-1, 0)$ and exactly 1 root in $(0, 2)$.

2. (1), (2), (4)

(1) $f: R \to [-2, 2]$ is twice differentiable function and $(f(0))^2 + (f'(0))^2 = 85$

Now, $f(0) \in [-2, 2]$.

$\therefore \quad (f(0))^2 \in [0, 4]$

$\therefore \quad 81 \le (f'(0))^2 \le 85$ [from $(f(0))^2 + (f'(0))^2 = 85$]

So, there will be small interval $(0 - \delta, 0 + \delta)$ where $f'(x) > 0$ as $f(x)$ is continuous and differentiable.

Therefore, statement (1) is true.

(2) Using LMVT for $x \in (-4, 0)$, there exists at least one x_0 such that

$$f'(x_0) = \frac{f(-4) - f(0)}{-4 - 0} = \frac{f(0) - f(-4)}{4}$$

Now, $-2 \le f(0), f(-4) \le 2$

$\therefore \quad -4 \le f(0) - f(-4) \le 4$

$\Rightarrow \quad -1 \le \dfrac{f(0) - f(-4)}{4} \le 1$

$\therefore \quad -1 \le f''(x_0) \le 1$

So, $|f'(x_0)| \le 1$

(3) Clearly, range of $f(x) \subseteq [-2, 2]$.

So, $f(x)$ is bounded function.

Therefore, $\lim\limits_{x \to \infty} f(x) = 1$ cannot be confirmed.

(4) Consider $g(x) = f^2(x) + (f'(x))^2$

$|f'(x_1)| \le 1$ (given)

$|f(x_1)| \le 2$

So, $g(x_1) \le 5$ for at least one $x_1 \in (-4, 0)$

Similarly, $g(x_2) \le 5$ for at least one $x_2 \in (0, 4)$

$g(0) = 85$. So, $g(x)$ has maxima in (x_1, x_2) say at $x = \alpha$. (by LMVT)

$\therefore \quad g'(\alpha) = 0$ and $g(\alpha) \ge 85$

$\therefore \quad 2f(\alpha) f'(\alpha) + 2(f'(\alpha))(f''(\alpha)) = 0$

If $f'(\alpha) = 0$, then $g(\alpha) = (f(\alpha))^2 \ge 85$, which is not possible.

Thus, $f(\alpha) + f''(\alpha) = 0$ for at least one $\alpha \in (-4, 4)$.

Chapter 6

Concept Application Exercises

Exercise 6.1

1. (a) $f(x) = \cot^{-1} x + x$,

Differentiating w.r.t. x, we get

$$f'(x) = \frac{-1}{1+x^2} + 1 = \frac{-1+1+x^2}{1+x^2} = \frac{x^2}{1+x^2}$$

Clearly, $f'(x) \geq 0$ for all x. ($f'(x) = 0$, for $x = 0$ only)

So, $f(x)$ increases in $(-\infty, \infty)$.

(b) $f(x) = \log(1+x) - \dfrac{2x}{2+x}$

$$\therefore \quad f'(x) = \frac{1}{1+x} - \frac{2(2+x)-2x}{(2+x)^2}$$

$$= \frac{x^2}{(x+1)(x+2)^2}$$

Obviously, $f'(x) \geq 0$ for all $x > -1$. ($f'(x) = 0$, for $x = 0$ only)

Hence, $f(x)$ is increasing on $(-1, \infty)$.

2. (a) $f(x) = -2x^3 - 9x^2 - 12x + 1$

$$\therefore \quad f'(x) = -6x^2 - 18x - 12$$

$$= -6(x+2)(x+1)$$

or $f'(x) > 0$, if $x \in (-2, -1)$

and $f'(x) < 0$, if $x \in (-\infty, -2) \cup (-1, -\infty)$

Thus, $f(x)$ is increasing for $x \in (-2, -1)$ and $f(x)$ is decreasing for $x \in (-\infty, -2) \cup (-1, -\infty)$.

(b) Let $y = f(x) = x^2 e^{-x}$

$$\therefore \quad \frac{dy}{dx} = 2xe^{-x} - x^2 e^{-x}$$

$$= e^{-x}(2x - x^2)$$

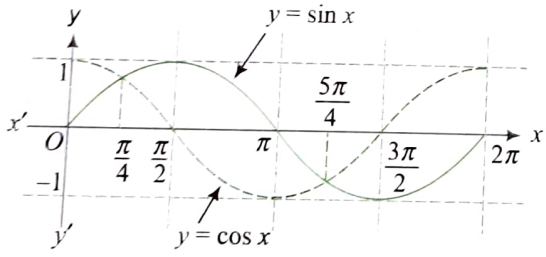
$$= e^{-x} x(2-x)$$

$f(x)$ is increasing if $f'(x) > 0$ or $x(2-x) > 0$ or $x \in (0, 2)$

$f(x)$ is decreasing if $f'(x) < 0$ or $x(2-x) < 0$

or $x \in (-\infty, 0) \cup (2, \infty)$

(c) We have $f'(x) = \cos x - \sin x$



$f(x)$ is increasing if $f'(x) > 0$ or $\cos x > \sin x$

or $x \in (0, \pi/4) \cup \left(\dfrac{5\pi}{4}, 2\pi\right)$ (see the graph)

$f(x)$ is decreasing if $f'(x) < 0$ or $\cos x < \sin x$

or $x \in \left(\pi/4, \dfrac{5\pi}{4}\right)$

(d) Given $f(x) = 3\cos^4 x + 10\cos^3 x + 6\cos^2 x - 3$

$$\therefore \quad f'(x) = 12\cos^3 x(-\sin x) + 30\cos^2 x(-\sin x) + 12\cos x(-\sin x)$$

$$= -3\sin 2x(2\cos^2 x + 5\cos x + 2)$$

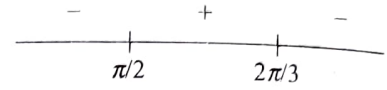
$$= -3\sin 2x(2\cos x + 1)(\cos x + 2)$$

$$f'(x) = 0 \Rightarrow \sin 2x = 0 \Rightarrow x = 0, \frac{\pi}{2}, \pi$$

or $2\cos x + 1 = 0 \Rightarrow x = \dfrac{2\pi}{3}$

as $\cos x + 2 \neq 0$.

Sign scheme of $f'(x)$ is as follows.



So, $f(x)$ decreases on $\left(0, \dfrac{\pi}{2}\right) \cup \left(\dfrac{2\pi}{3}, \pi\right)$ and increases on $\left(\dfrac{\pi}{2}, \dfrac{2\pi}{3}\right)$.

(e) $f(x) = (\log_e x)^2 + (\log_e x)$, $x > 0$

$$\therefore \quad f'(x) = 2\frac{(\log_e x)}{x} + \frac{1}{x} = \frac{2\log_e x + 1}{x}$$

$f(x)$ increases when $2\log_e x + 1 > 0$

or $\log_e x > -\dfrac{1}{2}$

or $x > e^{-\frac{1}{2}}$

or $f(x)$ increases when $x \in \left(\dfrac{1}{\sqrt{e}}, \infty\right)$

or $f(x)$ decreases when $x \in \left(0, \dfrac{1}{\sqrt{e}}\right)$

3. $f'(x) = \dfrac{\sin x - x\cos x}{\sin^2 x} = \dfrac{\cos x(\tan x - x)}{\sin^2 x}$

$0 < x \leq 1$ or $x \in$ first quadrant for which $\tan x > x$, $\cos x > 0$

or $f'(x) > 0$ for $0 < x \leq 1$

Thus, $f(x)$ is an increasing function.

$$g'(x) = \frac{\tan x - x\sec^2 x}{\tan^2 x} = \frac{\sin x\cos x - x}{\sin^2 x} = \frac{\sin 2x - 2x}{2\sin^2 x}$$

Now, $0 < 2x \leq 2$, for which $\sin 2x < 2x$

or $g'(x) < 0$

Thus, $g(x)$ is decreasing.

4. $x = \dfrac{1}{1+t^2}$ and $y = \dfrac{1}{t(1+t^2)}$

$$\therefore \quad \frac{dx}{dt} = -\frac{2t}{(1+t^2)^2}, \quad \frac{dy}{dt} = -\frac{1+3t^2}{t^2(1+t^2)^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1+3t^2}{2t^3}$$

$$\frac{dy}{dx} > 0 \text{ if } t > 0 \Rightarrow x = \frac{1}{1+t^2} \in (0, 1)$$

Hence, $f(x)$ is an increasing function.

5. If $f(x) = (a+2)x^3 - 3ax^2 + 9ax - 1$ decreases monotonically for all $x \in R$, then

$f'(x) \leq 0$ for all $x \in R$

$3(a+2)x^2 - 6ax + 9a \leq 0$ for all $x \in R$

$\therefore \quad a + 2 < 0$ and discriminant ≤ 0

or $a < -2$ and $-8a^2 - 24a \leq 0$

or $a < -2$ and $a(a+3) \geq 0$

i.e., $a < -2$ and $a \leq -3$ or $a \geq 0$

or $a \leq -3$

Since $f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$ is decreasing for all real values of x, $f'(x) \leq 0$ for all x

or $\sqrt{3}\cos x + \sin 4x - 2a \leq 0$ for all x

or $\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin 4x \leq a$ for all x

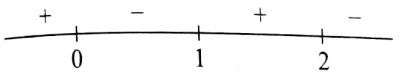
or $\sin\left(x + \frac{\pi}{3}\right) \leq a$ for all x

or $a \geq 1 \left[\because \sin\left(x+\frac{\pi}{3}\right) \leq 1\right]$

7. $f(x) = 2\log|x-1| - x^2 + 2x + 3$

$\therefore \quad f'(x) = \frac{2}{x-1} - 2x + 2 = 2\left[\frac{1-(x-1)^2}{x-1}\right] = \frac{-2x(x-2)}{x-1}$

Sign scheme for $\frac{-2x(x-2)}{(x-1)}$ is as follows:



$f'(x) > 0$ if $x \in (-\infty, 0)$ or $x \in (1, 2)$

Therefore, $f(x)$ is increasing in the interval $(-\infty, 0) \cup (1, 2)$ and decreases if $x \in (0, 1) \cup (2, \infty)$.

8. $f'(x) = \cos x + \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} - 2$

$= \cos x + \sec x - 2$

$= \frac{\cos^2 x - 2\cos x + 1}{\cos x}$

$= \frac{(\cos x - 1)^2}{\cos x} \geq 0$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore$ $f(x)$ is increasing function.

Also, $f'(x) = 0$ at $x = 0$ only.

$\therefore$ $f(x)$ is strictly increasing function.

9. $f(x) = \log_e\left(\frac{\log_e x}{x}\right)$

$f(x)$ is defined if $\frac{\log_e x}{x} > 0$

For $x \in (0, 1)$, $\frac{\log_e x}{x} < 0$

For $x \in (1, \infty)$, $\frac{\log_e x}{x} > 0$

$\therefore$ Domain of $f(x)$ is $(1, \infty)$.

Now $f'(x) = \frac{1}{\frac{\log_e x}{x}} \times \frac{1 - \log_e x}{x^2}$

$\therefore \quad f'(x) = \frac{1 - \log_e x}{x \log_e x}$

$f'(x) > 0$ if $1 - \log_e x > 0$ or $x < e$.

$f'(x) < 0$ if $1 - \log_e x < 0$ or $x > e$.

Thus $f(x)$ increases in $(1, e)$ and decrease in (e, ∞)

10. $g(x) = f(\log x) + f(2 - \log x)$

$\therefore \quad g'(x) = [f'(\log x) - f'(2 - \log x)]/x$

$g(x)$ increases if $g'(x) > 0$. Now, $x > 0$

or $f'(\log x) - f'(2 - \log x) > 0$

or $f'(\log x) > f'(2 - \log x)$

or $\log x < 2 - \log x$ [$f''(x) < 0$, $f'(x)$ is decreasing]

or $\log x < 1$

or $0 < x < e$

Exercise 6.2

1. $f(x) = \tan^{-1} x - \log_e x$

$\therefore \quad f'(x) = \frac{1}{1+x^2} - \frac{1}{x} = -\frac{(x^2 + 1 - x)}{x(x^2+1)} < 0 \ \forall \ x \in \left[\frac{1}{\sqrt{3}}, \sqrt{3}\right]$

So, $f_{\text{min.}} = f(\sqrt{3}) = \tan^{-1}\sqrt{3} - \log_e\sqrt{3} = \frac{\pi}{3} - \log_e\sqrt{3}$

$f_{\text{max.}} = f\left(\frac{1}{\sqrt{3}}\right) = \tan^{-1}\frac{1}{\sqrt{3}} - \log_e\sqrt{3} = \frac{\pi}{6} - \log_e\frac{1}{\sqrt{3}}$

Hence range is $\left[\frac{\pi}{3} - \log_e\sqrt{3}, \frac{\pi}{6} - \log_e\frac{1}{\sqrt{3}}\right]$

2. It can be proved that both the functions $\frac{\sin x}{x}$ and $\frac{x}{\tan x}$ are decreasing in $\left(0, \frac{\pi}{2}\right)$.

$\therefore$ Range of the function is $\left(\lim_{x \to \pi/2} f(x), \lim_{x \to 0} f(x)\right) \equiv \left(\frac{2}{\pi}, 2\right)$

3. $f(x) = \sin^{-1}x - \cot^{-1}x + x^2 + 2x + 6$

Domain of the $f(x)$ is $[-1, 1]$

Now $f'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{1}{1+x^2} + 2x + 2$

For $x \in [-1, 1]$, $2x + 2 > 0$.

$\therefore \quad f'(x) > 0$

So, f(x) is an increasing function.

$\therefore \quad f_{\min} = f(-1) = -\frac{\pi}{2} - \frac{3\pi}{4} + 5 = 5 - \frac{5\pi}{4}$

$f_{\max} = f(1) = \frac{\pi}{2} - \frac{\pi}{4} + 9 = 9 + \frac{\pi}{4}$

$\therefore \quad$ Range is $\left[5 - \frac{5\pi}{4}, 9 + \frac{\pi}{4}\right]$

4. Let $f(x) = x^3 + 2x + \cos x + \tan x$

$\therefore \quad f'(x) = 3x^2 + (2 - \sin x) + \sec^2 x > 0$

So, $f(x)$ is increasing function

$\therefore \quad f(0) = 1$ and $f(-1) < 0$

Therefore, equation has one solution in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

5. Let $f(x) = \ln(1+x) - \frac{x}{1+x}$

$\therefore \quad f'(x) = \frac{1}{(1+x)} - \frac{1+x-x}{(1+x)^2} = \frac{x}{(1+x)^2} > 0 \qquad (\because x > 0)$

Thus, $f(x)$ is an increasing function.

Since $x > 0$, we get $f(x) > f(0)$

or $\ln(1+x) - \frac{x}{1+x} > 0 \qquad [\because f(0) = 0]$

or $\frac{x}{1+x} < \ln(1+x)$.

6. Let $f(x) = \sin x - x$

$\therefore \quad f'(x) = \cos x - 1 = -(1 - \cos x) = -2\sin^2 x/2 < 0$

Thus, $f(x)$ is a decreasing function.

Now, $x > 0$

or $f(x) < f(0)$ or $\sin x - x < 0$ $[\because f(0) = 0]$

or $\sin x < x$ (1)

Now, let $g(x) = x - \frac{x^3}{6} - \sin x$

$\therefore \quad g'(x) = 1 - \frac{x^2}{2} - \cos x$

To find sign of $g'(x)$, we consider

$$\phi(x) = 1 - \frac{x^2}{2} - \cos x$$

$\therefore \quad \phi'(x) = -x + \sin x < 0$ [From equation (1)]

Thus, $\phi(x)$ is a decreasing function. Therefore,

$g'(x) < 0$

Thus, $g(x)$ is a decreasing function.

Since $x > 0$, we get $g(x) < g(0)$

or $x - \frac{x^3}{6} - \sin x < 0$ $[\because g(0) = 0]$

or $x - \frac{x^3}{6} < \sin x$ (2)

Combining equations (1) and (2), we get

$$x - \frac{-x^3}{6} < \sin x < x$$

7. Let us assume $f(x) = \tan^{-1}x - \frac{3x}{x^2+3}$

$$\therefore \quad f'(x) = \frac{1}{1+x^2} - \left[\frac{3(x^2+3) - 3x(2x)}{(x^2+3)^2}\right]$$

$$= \frac{x^4 + 6x^2 + 9 - (3x^2 + 9 - 6x^2)(1+x^2)}{(1+x^2)(x^2+3)^2}$$

$$= \frac{4x^4}{(1+x^2)(x^2+3)^2} > 0 \ \forall \ x \in (0, \infty)$$

Hence, $f(x)$ is increasing throughout.

Also, $f(0) = 0$

Hence, $f(x) > 0 \ \forall \ x > 0$

or $\tan^{-1}x > \frac{3x}{3+x^2}$

8. Let $f(x) = 2\sin x + \tan x - 3x$

$$\therefore \quad f'(x) = 2\cos x + \sec^2 x - 3$$

$$= \sec^2 x(2\cos^3 x - 3\cos^2 x + 1)$$

$$= \sec^2 x(1 - \cos x)^2(1 + 2\cos x)$$

$\therefore \quad f'(x) \geq 0, \ \forall \ x \in (0, \pi/2)$

So, $f(x)$ is an increasing function of x, $\forall \ x \in (0, \pi/2)$.

Therefore, for $x \geq 0$, $f(x) \geq f(0)$

$\Rightarrow \quad 2\sin x + \tan x - 3x \geq 0 \quad (\text{as } f(0) = 0)$

$\therefore \quad 2\sin x + \tan x \geq 3x$ where $0 \leq x < \frac{\pi}{2}$

9. Let $f(x) = \frac{\sin x}{x}$

$$\therefore \quad f'(x) = \frac{(x\cos x - \sin x)}{x^2} = \frac{\cos x(x - \tan x)}{x^2}$$

To find sign of $f'(x)$, we consider

$$g(x) = x - \tan x, \ 0 < x < \frac{\pi}{2}$$

$\therefore \quad g'(x) = 1 - \sec^2 x < 0$ $(\because \sec x >$

Thus, $g(x)$ is a decreasing function. Therefore,

$g(x) < g(0)$

or $x - \tan x < 0$

or $f'(x) < 0$

Thus, $f(x)$ is a decreasing function.

Also, $0 < x < \pi/2$

or $f(\pi/2) < f(x) < \lim_{x \to 0} f(x)$

or $\frac{2}{\pi} < \frac{\sin x}{x} < 1$

10. We have to prove that

$$\frac{x_2}{x_1} < \frac{\tan x_2}{\tan x_1} \text{ or } \frac{\tan x_1}{x_1} < \frac{\tan x_2}{x_2}$$

Now, consider $f(x) = \frac{\tan x}{x}$

$$\therefore \quad f'(x) = \frac{x\sec^2 x - \tan x}{x^2}$$

$$= \sec^2 x \frac{x - \sin x\cos x}{x^2}$$

$$= \frac{2x - \sin 2x}{2x^2\cos^2 x}$$

Now, $x \in (0, \pi/2)$

$\therefore \quad 2x \in (0, \pi)$, for which $2x > \sin 2x$

Thus, $f(x)$ is increasing. Now,

$x_1 < x_2$

or $f(x_1) < f(x_2)$

or $\frac{\tan x_1}{x_1} < \frac{\tan x_2}{x_2}$

or $\frac{x_2}{x_1} < \frac{\tan x_2}{\tan x_1}$

Exercise 6.3

1. $f(x) = \log_e(x-2) - \frac{1}{x}, \ x > 2$

$$\therefore \quad f'(x) = \frac{1}{x-2} + \frac{1}{x^2}$$

$$\Rightarrow \quad f''(x) = -\frac{1}{(x-2)^2} - \frac{2}{x^3} < 0$$

$\therefore \quad f(x)$ is always concave downward.

2. $y' = 1 + \log_e x - x$

$$\Rightarrow \quad y'' = \frac{1}{x} - 1 = \frac{1-x}{x}$$

$y'' > 0$ for $x \in (0, 1)$

and $y'' < 0$ for $x \in (1, \infty)$

$\therefore$ graph is concave upward for $x \in (0, 1)$ and concave downward for $x \in (1, \infty)$

3. $f(x) = \cos 2x + 2x\lambda^2 + (2\lambda + 1)(\lambda - 1)x^2, \ \lambda \in R.$

$$f\left(\frac{\alpha+\beta}{2}\right) < \frac{f(\alpha) + f(\beta)}{2} \text{ for all } \alpha \text{ and } \beta$$

$\therefore$ graph of $y = f(x)$ must be concave upward for all $x \in R$.

$\Rightarrow$ $f'(x) \geq 0 \ \forall \ x \in R$

$\Rightarrow$ $f''(x) = -4\cos 2x + 2(2\lambda + 1)(\lambda - 1) \geq 0 \ \forall \ x \in R$

$\Rightarrow$ $(2\lambda + 1)(\lambda - 1) \geq 2\cos 2x \ \forall \ x \in R$

$\Rightarrow$ $(2\lambda + 1)(\lambda - 1) \geq 2$

$\Rightarrow$ $2\lambda^2 - \lambda - 3 \geq 0$

$\Rightarrow$ $(2\lambda - 3)(\lambda + 1) \geq 0$

$\Rightarrow$ $\lambda \in (-\infty, -1] \cup \left[\frac{3}{2}, \infty\right)$

4. We have $f(x) = (\sin x + \cos x)\, e^x$

$\therefore$ $f'(x) = (\sin x + \cos x)\, e^x + e^x (\cos x - \sin x)$

$\Rightarrow$ $f'(x) = 2e^x \cos x$

$f''(x) = 2(e^x \cos x - e^x \sin x)$

$= 2e^x (\cos x - \sin x)$

If $f''(x) = 0$, then $\cos x - \sin x = 0$

$\therefore$ $\tan x = 1 \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$

5. Let $y = f(x) = x^m$, where $x > 0$ and $0 < m < 1$

$$\frac{d^2y}{dx^2} = m(m-1)x^{m-2} < 0$$

So, graph is concave downward.

$\therefore$ $$\frac{\sum_{i=1}^{n} f(x_i)}{n} < f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)$$

Therefore $$\frac{a_1^m + a_2^m + \ldots + a_n^m}{n} < \left(\frac{a_1 + a_2 + \ldots\ldots + a_n}{n}\right)^m$$

6. Triangle ABC is acute angled.

$\therefore$ $0 < A, B, C < \pi/2$

So, consider the graph of function $y = \sec x$ for $x \in (0, \pi/2)$

The graph of function is concave upward.

$\therefore$ $$\frac{\sec A + \sec B + \sec C}{3} \geq \sec\left(\frac{A + B + C}{3}\right)$$

$\Rightarrow$ $$\frac{\sec A + \sec B + \sec C}{3} \geq \sec\frac{\pi}{3}$$

$\Rightarrow$ $\sec A + \sec B + \sec C \geq 6$

Exercise 6.4

1. $f(x) = \frac{x^5}{20} - \frac{x^4}{12} + 5$

$\therefore$ $f'(x) = \frac{x^4}{4} - \frac{x^3}{3} = \frac{x^3}{12}(3x - 4)$

Sign scheme of $f'(x)$

From the sign scheme, $x = 0$ is point of maxima and $x = 4/3$ is point of minima.

Now $f''(x) = x^3 - x^2 = x^2(x - 1)$

Clearly $f''(x)$ changes sign at $x = 1$.

So, $x = 1$ is point of inflection.

2. $f(x) = \frac{x^2 + ax + b}{x - 10}$ has a stationary point at (4, 1). So, it must lie on the curve. Thus,

$16 + 4a + b = -6$ (1)

Also, $\left(\frac{dy}{dx}\right)_{(x=4)} = 0$

or $\left(\frac{x^2 - 20x - 10a - b}{(x-10)^2}\right)_{(x=4)} = 0$

or $10a + b = -64$ (2)

From (1) and (2), we have $a = -7$, $b = 6$.

Also, for these values of x,

$$y = \frac{x^2 - 7x + 6}{x - 10} \quad (3)$$

$\therefore$ $$\frac{dy}{dx} = \frac{(x-4)(x-16)}{(x-10)^2}$$

From $x = 4 - h$ to $x = 4 + h$, $\frac{dy}{dx}$ changes its sign from +ve to −ve.

Hence, $x = 4$ is point of maxima.

3. $f(x) = axe^{bx^2}$

According to the question,

$f(2) = 1$ and $f'(2) = 0$

$\therefore$ $2ae^{4b} = 1$ ...(1)

Also, $f'(x) = a[\, xe^{bx^2} \cdot 2bx + e^{bx^2}\,]$

$= ae^{bx^2}[2bx^2 + 1]$

$f'(2) = ae^{4b}(8b + 1) = 0$

$\therefore$ $b = -1/8$ ($\because$ $a \neq 0$)

$\therefore$ $a = \frac{\sqrt{e}}{2}$ (from (1))

$\therefore$ $f'(x) = \frac{\sqrt{e}}{2} e^{-\frac{1}{8}x^2}\left[-\frac{1}{4}x^2 + 1\right]$

$f'(x) = 0$ if $x = \pm 2$.

Clearly, $x = 2$ is point of maxima.

4. $f(x) = \frac{1}{3}\left(x + \frac{1}{x}\right)$

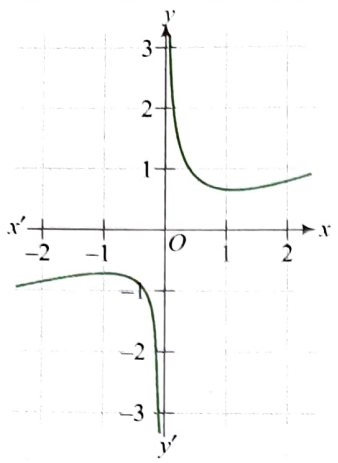
$\therefore$ $f'(x) = \frac{1}{3}\left(1 - \frac{1}{x^2}\right)$

Let $f'(x) = 0$ or $x^2 = 1$ or $x = \pm 1$. Also,

$f''(x) = \frac{1}{3}\left(\frac{2}{x^3}\right)$ or $f''(1) > 0$ and $f''(-1) < 0$

Thus, $x = 1$ is point of minima and $x = -1$ is point of maxima.

Here, $f(1) = \frac{2}{3}$ and $f(-1) = -\frac{2}{3}$.



Thus, local maximum value is less than local minimum value.

5. $f(x) = 1 + 2\sin x + 3\cos^2 x,\ 0 \le x \le 2\pi/3$

$\therefore \quad f'(x) = 2\cos x - 6\sin x\cos x$

$= 2\cos x\,(1 - 3\sin x)$

$f'(x) = 0 \Rightarrow \cos x = 0$ or $1 - 3\sin x = 0$

i.e., $x = \pi/2$ or $\sin x = 1/3$

$f''(x) = -2\sin x - 6\cos 2x$

$$f''\left(\frac{\pi}{2}\right) = -2\sin\frac{\pi}{2} - 6\cos\left(2\times\frac{\pi}{2}\right) = -2 + 6 = 4 > 0$$

Hence, $x = \pi/2$ is point of minima.

$f''(\sin^{-1} 1/3) = -2\,(1/3) - 6(1 - 2\times 1/9) = -2/3 - 14/3 < 0$

Hence, $x = \sin^{-1} 1/3$ is point of maximum.

$$f_{\min} = f\left(\frac{\pi}{2}\right) = 1 + 2\sin\frac{\pi}{2} + 3\cos^2\frac{\pi}{2} = 1 + 2 = 3$$

$$f_{\max} = f\left(\sin^{-1}\frac{1}{3}\right) = 1 + 2\left(\frac{1}{3}\right) + 3\left(1 - \frac{1}{9}\right) = \frac{5}{3} + \frac{8}{3} = \frac{13}{3}$$

6. $f(x) = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x,\ 0 \le x \le \pi$

$f'(x) = \cos x + \cos 2x + \cos 3x = 2\cos 2x\cos x + \cos 2x$

$= \cos 2x\,(2\cos x + 1)$

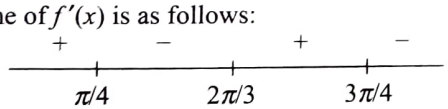
Let $f'(x) = 0$

or $\quad \cos 2x = 0$ or $2\cos x + 1 = 0$

or $\quad 2x = \pi/2, 3\pi/2$ or $\cos x = -1/2$

or $\quad x = \pi/4, 3\pi/4$ or $x = 2\pi/3$

Sign scheme of $f'(x)$ is as follows:



Hence, $x = \pi/4, 3\pi/4$ are points of maxima.

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \frac{1}{2}\sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{3\sqrt{2}} = \frac{4}{3\sqrt{2}} + \frac{1}{2} = \left(\frac{4\sqrt{2}+3}{6}\right)$$

$$f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} + \frac{1}{2}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{3}\sin\left(\frac{9\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{2} + \frac{1}{3}\frac{1}{\sqrt{2}} = \frac{4\sqrt{2}-3}{6}$$

$$f\left(\frac{2\pi}{3}\right) = \sin\frac{2\pi}{3} + \frac{1}{2}\sin\frac{4\pi}{3} + \frac{1}{3}\sin\left(\frac{9\pi}{3}\right) = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4}$$

$$= \frac{\sqrt{3}}{4}$$

$f(0) = 0, f(\pi) = 0$

Thus, $x = \frac{\pi}{4}$ is the point of global maxima,

$x = \frac{3\pi}{4}$ is the pont of local maxima, and

$x = \frac{2\pi}{3}$ is the point of local minima.

$x = 0$ is the point of global maxima.

7. $f(x) = -\sin^3 x + 3\sin^2 x + 5$

$\therefore \quad f'(x) = -3\cos x\sin^2 x + 6\sin x\cos x$

$= -3\sin x\cos x\,(\sin x - 2)$

Now, $\sin x - 2 < 0\ \forall\ x \in \left[0, \frac{\pi}{2}\right]$

$\sin x, \cos x \ge 0\ \forall\ x \in \left[0, \frac{\pi}{2}\right]$

$\therefore \quad f'(x) \ge 0$

Thus, $f(x)$ is a strictly increasing function $\forall\ x \in [0, \pi/2]$.

Hence, $f(x)$ is minimum when $x = 0$ and maximum when $x = \pi/2$.

$f_{\min} = f(0) = 5$

$f_{\max} = f(\pi/2) = 7$

8. $f(\theta) = \sin^p\theta\cos^q\theta,\ p, q > 0,\ 0 < \theta < \pi/2$

$f'(\theta) = p\sin^{p-1}\theta\cos\theta\cos^q\theta - q\cos^{q-1}\theta\sin\theta\sin^p\theta$

$= \sin^{p-1}\theta\cos^{q-1}\theta\,[p\cos^2\theta - q\sin^2\theta]$

Let $f'(\theta) = 0$

i.e., $\sin\theta = 0$ or $\cos\theta = 0$ (not possible)

or $\quad p\cos^2\theta - q\sin^2\theta = 0$ or $\tan^2\theta = p/q$ or $\tan\theta = \sqrt{\frac{p}{q}}$

$\left(\tan\theta \ne -\sqrt{\frac{p}{q}}, \text{as } 0 < \theta < \pi/2\right)$

Check for extremum.

When $\theta \to 0, f(\theta) \to 0$ (as $\sin\theta \to 0$)

When $\theta \to \pi/2, f(\theta) \to 0$ (as $\cos\theta \to 0$)

Also, for $\theta \in (0, \pi/2), f(\theta)$ is +ve.

Hence, the only point of extremum is point of maxima.

Thus, $\theta = \tan^{-1}\sqrt{\frac{p}{q}}$ is point of maxima when

$$\tan\theta = \sqrt{\frac{p}{q}},\ \cos\theta = \frac{\sqrt{q}}{\sqrt{p+q}},\ \text{and}\ \sin\theta = \frac{\sqrt{p}}{\sqrt{p+q}}$$

Hence, maximum value, $f_{\max} = \left(\frac{\sqrt{q}}{\sqrt{p+q}}\right)^q\left(\frac{\sqrt{p}}{\sqrt{p+q}}\right)^p$

$$= \left(\frac{p^p q^q}{(p+q)^{p+q}}\right)^{1/2}$$

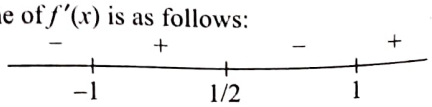
9. $f(x) = \log_e(3x^4 - 2x^3 - 6x^2 + 6x + 1),\ x \in (0, 2)$

$$f'(x) = \frac{12x^3 - 6x^2 - 12x + 6}{(3x^4 - 2x^3 - 6x^2 + 6x + 1)}$$

$$= \frac{6(2x^3 - x^2 - 2x + 1)}{(3x^4 - 2x^3 - 6x^2 + 6x + 1)}$$

$$= \frac{6(x^2 - 1)(2x - 1)}{(3x^4 - 2x^3 - 6x^2 + 6x + 1)}$$

Sign scheme of $f'(x)$ is as follows:



But $x \in (0, 2)$.

Hence, $x = 1/2$ is point of maxima, and $x = 1$ is point of minima.

Hence, $f_{min} = f(1) = \ln 2$

and $f_{max} = f(1/2) = \ln(39/16)$

10. $f(x) = x(x^2-4)^{-1/3}$

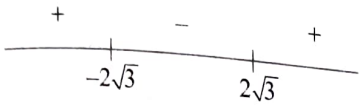
$$f'(x) = (x^2-4)^{-1/3} - \frac{1}{3}(x^2-4)^{-4/3}(2x)\,x$$

$$= \frac{1}{(x^2-4)^{1/3}} - \frac{2x^2}{3(x^2-4)^{4/3}}$$

$$= \frac{3(x^2-4)-2x^2}{3(x^2-4)^{4/3}}$$

$$= \frac{(x^2-12)}{3(x^2-4)^{4/3}}$$

Sign scheme of $f'(x)$ is as follows:



Thus, $x = 2\sqrt{3}$ is point of minima and $x = -2\sqrt{3}$ is point of maxima.

$$f_{max} = f(-2\sqrt{3}) = (-2\sqrt{3})(12-4)^{-1/3} = \frac{-2\sqrt{3}}{2} = -\sqrt{3}$$

$$f_{min} = f(2\sqrt{3}) = (2\sqrt{3})(12-4)^{-1/3} = \sqrt{3}$$

Here, maximum value is less than minimum value.

This is because $f(x)$ is discontinuous at $x = \pm 2$.

Since $f(x)$ is unbounded function, both the extreme values are local.

11. $f(x) = x^{2/3} - x^{4/3}$

$$f'(x) = (2/3)x^{-1/3} - (4/3)x^{1/3}$$

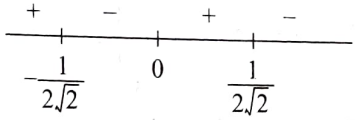
$$= \frac{2}{3}\frac{(1-2x^{2/3})}{x^{1/3}}$$

Critical points: $f'(x) = 0$ at $x = \pm\frac{1}{2\sqrt{2}}$.

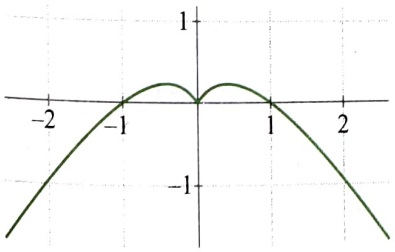
Also, $f'(x)$ does not exist at $x = 0$.

As $f(x)$ is continuous at $x = 0$, it is also a critical point.

Sign scheme of $f'(x)$ is as follows.



Thus, $x = 0$ is point of minima and $x = \pm\frac{1}{2\sqrt{2}}$ are points of maxima.



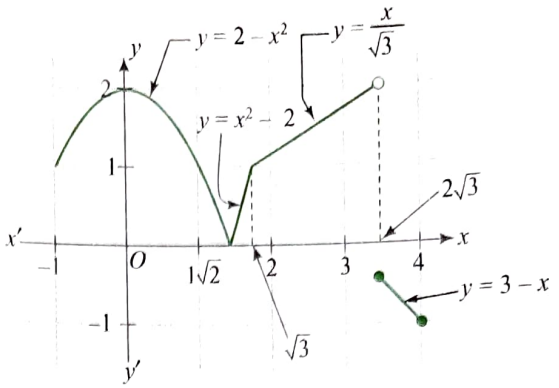
Also, range of $f(x)$ is $\left(-\infty, f\left(\pm\frac{1}{2\sqrt{2}}\right)\right]$.

12. Clearly, from the graph given in Figure, $x = \sqrt{2}$ is point of minima.

$x = \sqrt{3}$ is not a point of extremum.

$x = 2\sqrt{3}$ is also not a point of extremum.

$x = 0$ is point of maxima.



13. $f(0) = 1$

$$f(0^+) = \lim_{x\to 0^+}(x^2 - x + 1) \to 1^-$$

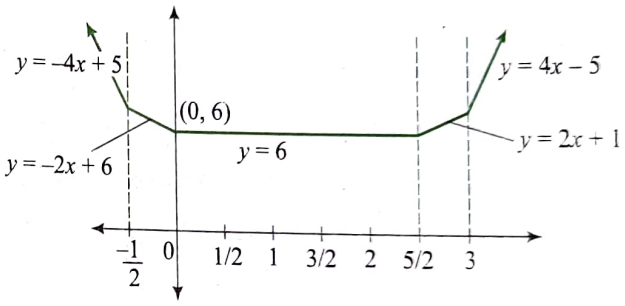
as $x^2 - x + 1$ is decreasing for $(-\infty, 1/2)$.

$$f(0^-) \to \lim_{x\to 0^-}(1 + \sin x) \to 1^-$$

Thus, $f(0^-) < f(0)$ and $f(0) > f(0^+)$.

Then at $x = 0$, $f(x)$ is the point of maxima.

14. $f(x) = |x| + \left|x + \frac{1}{2}\right| + |x-3| + \left|x - \frac{5}{2}\right|$



From the graph, minimum value is 6.

15. $f(x) = \begin{cases} \tan^{-1}\alpha - 5x^2, & 0 < x < 1 \\ -6x, & x \geq 1 \end{cases}$

$f(1) = -6$

For maximum at $x = 1$,

$$\lim_{x\to 1^-} f(x) = \tan^{-1}\alpha - 5 < -6$$

or $\tan^{-1}\alpha < -1$ or $\alpha < -\tan 1$

16. For $x < 1$, $f'(x) = 3x^2 - 2x + 10 > 0$.

Thus, $f(x)$ is an increasing function for $x < 1$.

For $x > 1$, $f'(x) = -2$.

Thus, $f(x)$ is a decreasing function for $x > 1$. Now, $f(x)$ will have greatest value at $x = 1$ if

$$\lim_{x\to(1^+)} f(x) \leq f(1)$$

or $-2 + \log_2(b^2 - 2) \leq 5$

or $0 < b^2 - 2 \leq 128$ or $2 \leq b^2 \leq 130$

or $b \in [-\sqrt{130}, -\sqrt{2}] \cup [\sqrt{2}, \sqrt{130}]$

Exercise 6.5

1. The given function is continuous for all $x \in R$.

$$f'(x) = \frac{2(x^2-1)}{(x^2+x+1)^2}$$

$f'(x) = 0 \Rightarrow x = \pm 1$

Sign scheme of $f'(x)$ is as follows.

From the sign scheme $x = 1$ is point of minima and $x = -1$ is point of maxima.

Also $f(1) = \frac{1}{3}$ and $f(-1) = 3$, $f(0) = 1$

Further $\lim\limits_{x\to\pm\infty} \frac{x^2 - x + 1}{x^2 + x + 1} = 1$

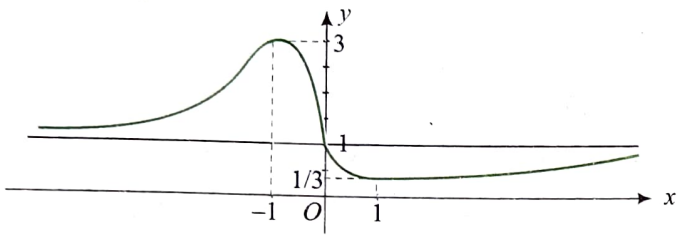
Now mark points $(-1, 3)$ and $(1, 1/3)$ on the coordinate axis.

In $(-1, 1)$, $f(x)$ decreases.

In $(1, \infty)$, $f(x)$ increases and approaches to '1' at infinity.

In $(-\infty, -1)$, $f(x)$ increases from '1^+' at negative infinity to '3' at $x = -1$.

From the above information graph of $y = f(x)$ is shown in the following figure.



2. We have $e(k - x \log_e x) = 1$

$$\Rightarrow \quad k - \frac{1}{e} = x \log_e x \qquad \ldots(1)$$

Then equation (1) has solution where graphs of $y = x \log e\, x$ and $y = k - \frac{1}{e}$ intersect

Consider the function $f(x) = x \log_e x$

$f'(x) = 1 + \log_e x$

$f'(x) = 0 \quad \Rightarrow x = 1/e$

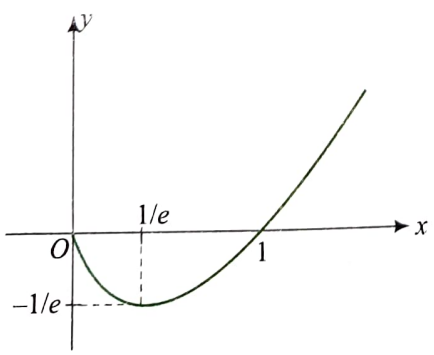
$f''(x) = 1/x \quad \Rightarrow f''(1/e) = e > 0$

$\Rightarrow$ $x = 1/e$ is point of minima.

Also $\lim\limits_{x\to 0} x \log_e x = \lim\limits_{x\to 0} \frac{\log_e x}{\frac{1}{x}} = \lim\limits_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -\lim\limits_{x\to 0} x = 0$

Also $f\left(\frac{1}{e}\right) = -\frac{1}{e}$

From the above information the graph of $f(x) = x \log_e x$ is as follows



From the graph, equation (1) has two has two distinct roots

if line $y = k - \frac{1}{e}$ cuts the graph between $-\frac{1}{e}$ and 0.

i.e., $-\frac{1}{e} < k - \frac{1}{e} < 0$ or $0 < k < \frac{1}{e}$

Equation has no roots if $k - \frac{1}{e} < -\frac{1}{e} \Rightarrow k < 0$

Equation has one root if $k - \frac{1}{e} = -\frac{1}{e}$

or $\quad k - \frac{1}{e} \geq 0$, i.e., $k = 0$ or $k \geq \frac{1}{e}$

3. We have $y = f(x) = (x+1)^{\frac{2}{3}} + (x-1)^{\frac{2}{3}}$

Domain of function is R.

$f(0) = 2$

also curve never intersect x-axis.

$f(x) = f(-x)$, so function is even hence graph is symmetrical about y-axis.

$$f'(x) = \frac{2}{3}(x+1)^{-\frac{1}{3}} + \frac{2}{3}(x-1)^{-\frac{1}{3}}$$

$$= \frac{2}{3}\left(\frac{1}{(x+1)^{\frac{1}{3}}} + \frac{1}{(x-1)^{\frac{1}{3}}}\right)$$

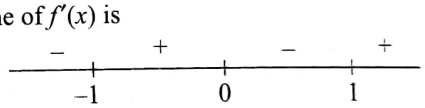
$$= \frac{2}{3}\left(\frac{(x+1)^{\frac{1}{3}} + (x-1)^{\frac{1}{3}}}{(x+1)^{\frac{1}{3}}(x-1)^{\frac{1}{3}}}\right)$$

Critical points are $x = -1, 0, 1$

$f'(x)$ does not exists at $x = -1$ and $x = 1$, hence function is non-differentiable at these points.

$f'(0) = 0$

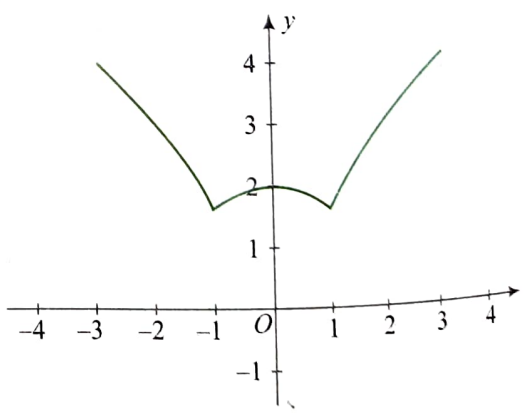
Sign scheme of $f'(x)$ is



$f(1) = f(-1) = 2^{2/3}$

$\lim\limits_{x\to\pm\infty} \left[(x+1)^{\frac{2}{3}} + (x-1)^{\frac{2}{3}}\right] = \infty$

Thus graph of the function is



4. $f(x) = \log_e (1 - \log_e x)$

$f(x)$ is defined if $1 - \log_e x > 0$

or $\quad \log_e x < 1$

or $\quad 0 < x < e$

$\therefore$ $\quad$ domain is $(0, e)$

Also, $f'(x) = -\frac{1}{(1 - \log_e x)} \cdot \frac{1}{x} < 0$

$\Rightarrow$ $\quad f(x)$ is continuous and decreasing $\forall\, x$ in its domain

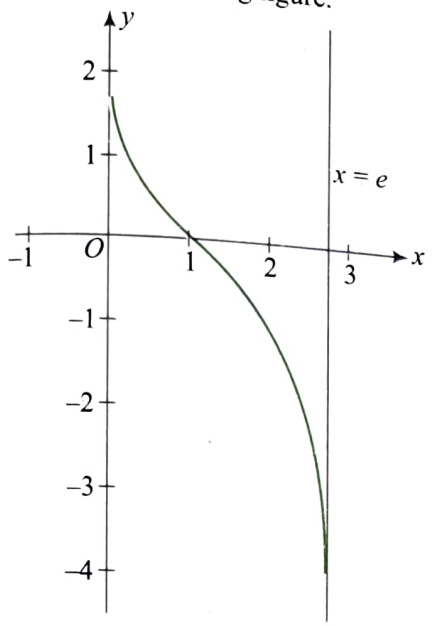
Further $\lim_{x\to 0^+} \log_e(1-\log_e x) = \log_e(1+\infty) = \infty$

$\lim_{x\to e^-} \log_e(1-\ln x) = \log_e(1-1^-) = \log_e 0^+ = -\infty$

For point of inflection $f''(x) = \dfrac{-\log_e x}{x^2(1-\log_e x)^2}$

Clearly $f''(1) = 0$. So, $x = 1$ which is a point of inflection.

Graph is as shown in the following figure.



From the graph $y = 0$ and $y = e$ are asymptotes.

5. $f(x) = \dfrac{4\log_e x}{x^2}$

The domain is $(0, \infty)$.

$$f'(x) = 4\frac{\frac{1}{x}x^2 - 2x\log_e x}{x^4} = 4\frac{1-2\log_e x}{x^3}$$

$f'(x) = 0 \Rightarrow 1 - 2\log_e x = 0$ or $x = e^{1/2}$

$$\lim_{x\to 0^+} \frac{4\log_e x}{x^2} = -\infty$$

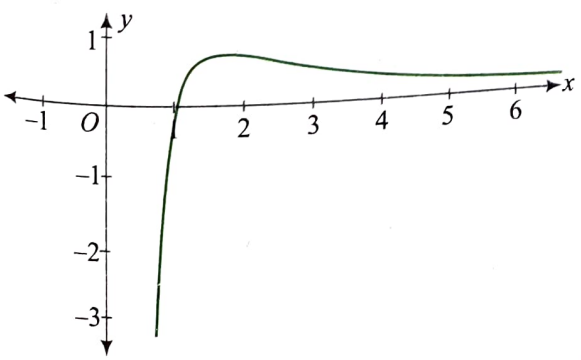
and $$\lim_{x\to\infty} \frac{4\log_e x}{x^2} = \lim_{x\to\infty} \frac{\frac{4}{x}}{2x} = \lim_{x\to\infty} \frac{2}{x^2} = 0$$

$f(x) < 0$ for $0 < x < 1$ and $f(x) > 0$ for $x > 1$

$f(x) = 0$, when $x = 1$

Derivative changes sign from +ve to −ve in neighbourhood of $x = e^{1/2}$.

From the above discussion, the graph of $y = f(x)$ is as shown in the following figure.



From the graph $x = 0$ (y-axis) and $x = 0$ (y-axis) are asymptotes to the curve.

Also range of the function is $(-\infty, f(e^{1/2})]$ or $(-\infty, 2/e]$

Exercise 6.6

1. Let the additional number of subscribers be x. So, the number of subscribers becomes $725 + x$, and then the profit per subscriber is ₹$(12 - x/100)$.

If P is the total profit in rupees, then

$$P = (725 + x)\left(12 - \frac{x}{100}\right)$$

$$= -\frac{x^2}{100} + \frac{19}{4}x + 8700$$

$$= \frac{1}{100}\left[870000 + \left(\frac{475}{2}\right)^2 - \left(x - \frac{475}{2}\right)^2\right]$$

P is maximum (greatest) when $x - 475/2 = 0$, i.e., $x = 237.5$. But x is +ve integer. So, x can be taken as 237 or 238.

Since $P(237) = P(238)$, for maximum profit, total number of subscribers should be $725 + 237 = 962$ or $725 + 238 = 963$.

2. Let r be the radius of the circle and a be the side of the square.

Then, we have

$$2\pi r + 4a = k \quad \text{(where } k \text{ is constant)}$$

or $$a = \frac{k - 2\pi r}{4}$$

The sum of the areas of the circle and the square (A) is given by

$$A = \pi r^2 + a^2$$

$$= \pi r^2 + \frac{(k - 2\pi r)^2}{16}$$

$$\therefore \quad \frac{dA}{dr} = 2\pi r + \frac{2(k - 2\pi r)(-2\pi)}{16} = 2\pi r - \frac{\pi(k - 2\pi r)}{4}$$

Now, $\dfrac{dA}{dr} = 0$

or $$2\pi r = \frac{\pi(k - 2\pi r)}{4}$$

or $$8r = k - 2\pi r$$

or $$(8 + 2\pi)r = k$$

or $$r = \frac{k}{8 + 2\pi} = \frac{k}{2(4 + \pi)}$$

Now, $\dfrac{d^2A}{dr^2} = 2\pi + \dfrac{\pi^2}{2} > 0$

Therefore, when $r = \dfrac{k}{2(4+\pi)}$, $\dfrac{d^2A}{dr^2} > 0$.

Thus, the sum of the areas is least when $r = \dfrac{k}{2(4+\pi)}$.

When $r = \dfrac{k}{2(4+\pi)}$,

$$a = \frac{k - 2\pi\left[\frac{k}{2(4+\pi)}\right]}{4} = \frac{k(4+\pi) - \pi k}{4(4+\pi)} = \frac{k}{4+\pi}$$

Hence, it has been proved that the sum of their areas is least when the side of the square is double the radius of the circle.

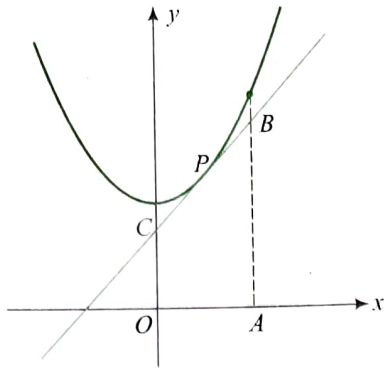
3. Equation of the curve is $y = x^2 + 1$.

Tangent at $P(a, b)$ is $y - b = 2a(x - a)$

i.e., $y - (a^2 + 1) = 2a(x - a)$

$x = 0 \Rightarrow y = 1 - a^2$, which is positive for $0 < a < 1$

and $x = 1 \Rightarrow y = 1 + 2a - a^2$



$\therefore$ $OC = 1 - a^2$ and $AB = 1 + 2a - a^2$

Z = Area of trapezium $OABC$

$$= \frac{1}{2}(OC + AB)OA = 1 + a - a^2,\ 0 < a < 1$$

$$\frac{dZ}{da} = [1 - 2a] = 0$$

or $a = 1/2$ and $\dfrac{d^2Z}{da^2} = -4 < 0$

Thus, at $a = 1/2$, area of trapezium is maximum (greatest). Thus, the required point is (1/2, 5/4).

4. $\dfrac{a^2}{x^2} + \dfrac{b^2}{y^2} = 1$

Let $x = r\cos\phi, y = r\sin\phi$

$\therefore$ $\dfrac{a^2}{r^2\cos^2\phi} + \dfrac{b^2}{r^2\sin^2\phi} = 1$

or $r^2 = a^2\sec^2\phi + b^2\operatorname{cosec}^2\phi$

$$\frac{dr^2}{d\phi} = a^2 2\sec\phi\sec\phi\tan\phi - b^2 2\operatorname{cosec}\phi\operatorname{cosec}\phi\cot\phi = 0$$

$\therefore$ $a^2\sec^2\phi\tan\phi = b^2\operatorname{cosec}^2\phi\cot\phi$

$\therefore$ $\tan^4\phi = \dfrac{b^2}{a^2}$

i.e., $\tan\phi = \sqrt{\dfrac{b}{a}}$ or $-\sqrt{\dfrac{b}{a}}$

$\therefore$ $r^2_{\min} = a^2\left(1 + \dfrac{b}{a}\right) + b^2\left(1 + \dfrac{a}{b}\right) = (a + b)^2$

$\therefore$ $r_{\min} = |a + b|$

5. Here, $f(x) = e^x\cos x$

$\therefore$ $f'(x) = e^x\cos x - e^x\sin x = e^x(\cos x - \sin x)$

where $f'(x)$ is slope of tangent (which is to be minimized).

So, let $f'(x) = g(x)$. Therefore,

$g(x) = e^x(\cos x - \sin x)$

$\therefore$ $g'(x) = e^x\{\cos x - \sin x\} + e^x\{-\sin x - \cos x\}$

$= e^x\{-2\sin x\}$

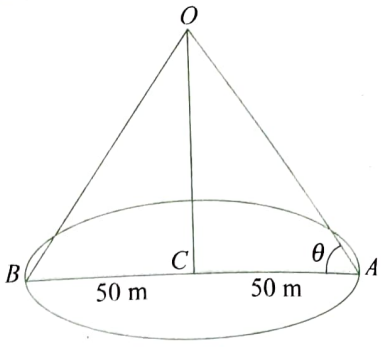
which is +ve when $x \in [\pi, 2\pi]$ and –ve when $x \in [0, \pi]$.

Thus, $g(x)$ is decreasing in $(0, \pi)$ and increasing in $(\pi, 2\pi)$.

So, at $x = \pi$, slope of tangent of the function $f(x)$ attains minima.

6. Let C be the center of the circular plot of lawn with a diameter of 100 m, i.e.,

$CA = CB = 50$ m



Let O be the position of light which is directly above C. Also, let $\angle OAC = \theta$, $(0 < \theta < \pi/2)$.

Then according to the question, the intensity I of the light at the circumcenter of the plot is given by

$$I = \frac{k\sin\theta}{(OA)^2} = \frac{k\sin\theta}{(50\sec\theta)^2}$$

or $I = [k/(2500)]\sin\theta\cos^2\theta$

$\therefore$ $dI/d\theta = [k/(2500)](\cos^3\theta - 2\sin^2\theta\cos\theta)$

$= (k/2500)\cos\theta(\cos^2\theta - 2\sin^2\theta)$

For maximum or minimum of I, $dI/d\theta = 0$, $\tan\theta = 1/\sqrt{2}$.

Now, we have

$d^2I/d\theta^2 = (k/2500)(-7\sin\theta\cos^2\theta + 2\sin^3\theta)$

$= (k/2500)\sin\theta\cos^2\theta(-7 + 2\tan^2\theta)$

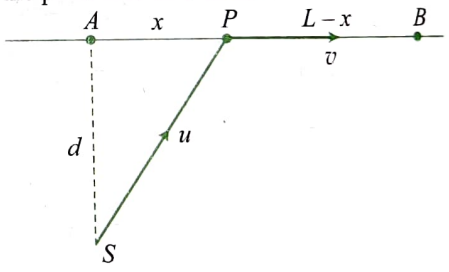
When $\tan\theta = 1/\sqrt{2}$, $d^2I/d\theta^2$ is –ve.

Hence, I (intensity of light) is maximum when $\tan\theta = 1/\sqrt{2}$, which is the only point of extrema, so gives the greatest intensity.

Thus, the required height of the light is

$OC = AC\tan\theta = 50/\sqrt{2} = 25\sqrt{2}$ m

7. Let the swimmer lands at the point P, x km from A and then walks from P to the point B to be reached.



Given that $AB = L$ km. Then $PB = (L - x)$ km.

t = Total time from S to B

= (Time taken from S to P) + (Time taken from P to B)

$= SP/u + PB/v$

$= \sqrt{(d^2 + x^2)}/u + (L - x)/v$

$\therefore$ $\dfrac{dt}{dx} = \dfrac{x}{u\sqrt{(d^2 + x^2)}} - \dfrac{1}{v}$

and $\dfrac{d^2t}{dx^2} = \dfrac{1}{u\sqrt{(d^2 + x^2)}} - \dfrac{x^2}{u(d^2 + x^2)^{3/2}}$

$$= \frac{d^2}{u(d^2 + x^2)^{3/2}}$$ which is +ve

For maximum or minimum of t, $dt/dx = 0$

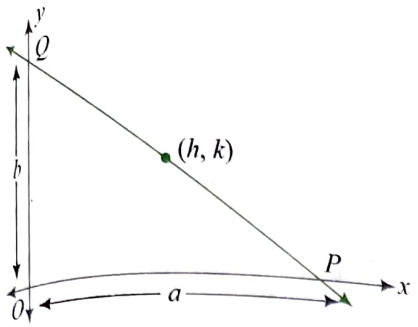
or $v^2x^2 = u^2(d^2 + x^2)$

or $x = \dfrac{ud}{\sqrt{v^2 - u^2}}$

Therefore, t is minimum for this value of x $\left(\text{since } \dfrac{d^2t}{dx^2} \text{ is +ve}\right)$.

Hence, the swimmer will reach his house in the shortest possible time if he lands at a distance $L - x = L - \dfrac{ud}{\sqrt{v^2 - u^2}}$ from his house to be reached.

8.



Let the line in intercept form be $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

It passes through (h, k). Then $\dfrac{h}{a} + \dfrac{k}{b} = 1$

or $\dfrac{k}{b} = 1 - \dfrac{h}{a} = \dfrac{a-h}{a}$ or $b = \dfrac{ak}{a-h}$

$$\Delta = \frac{1}{2}ab = \frac{1}{2}a\frac{ak}{a-h} = \frac{1}{2}\frac{k}{\frac{a-h}{a^2}} \qquad (1)$$

Δ is minimum when $y = \dfrac{a-h}{a^2} = \dfrac{1}{a} - \dfrac{h}{a^2}$ is maximum. Thus,

$$\frac{dy}{da} = -\frac{1}{a^2} + \frac{2h}{a^3} = 0 \text{ or } a = 2h \qquad (2)$$

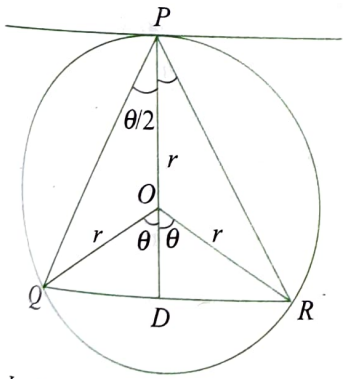
$$\frac{d^2y}{da^2} = \frac{2}{a^3} - \frac{6h}{a^4} = \frac{2}{a^3} - \frac{3}{a^3} \qquad \text{[by (2)]}$$

or $\dfrac{d^2y}{da^2} = -\dfrac{1}{a^3} = -\text{ve}$

Therefore, y is maximum.

Now, put $a = 2h$ in (1). Then $\Delta = \dfrac{1}{2}4h^2\dfrac{k}{h} = 2hk$.

9.



Let O be the center and r the radius of the circle.

Let QR be the chord parallel to the tangent at the point P on the circle.

Let $\angle QPR = \theta$. Then $\angle QOD = \angle ROD = \theta$.

Area of $\Delta PQR = A = \dfrac{1}{2}(QR)(PD) = QD(OP + OD)$

$$= r\sin\theta(r + r\cos\theta)$$

$$= \frac{1}{2}r^2(2\sin\theta + \sin 2\theta),\ 0 < \theta \le \pi/2$$

$\therefore$ $\dfrac{dA}{d\theta} = r^2(\cos\theta + \cos 2\theta)$

and $\dfrac{d^2A}{d\theta^2} = -r^2(\sin\theta + 2\sin 2\theta)$

For maximum or minimum of A, $dA/d\theta = 0$

or $\cos 2\theta + \cos\theta = 0$

or $2\cos^2\theta + \cos\theta - 1 = 0$

or $(2\cos\theta - 1)(\cos\theta + 1) = 0$

or $\cos\theta = 1/2$ or $\cos\theta = -1$

or $\theta = \pi/3$ $(\because 0 < \theta < \pi/2)$

When $\theta = \dfrac{\pi}{3}$, $\dfrac{d^2A}{d\theta^2} = -\dfrac{3\sqrt{3}}{2}$ (−ve).

Thus, A is maximum when $\theta = \pi/3$, the only critical point. Thus,

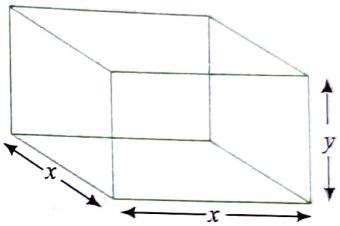
Maximum (greatest) area $A = \dfrac{1}{2}r^2[2\sin(\pi/3) + \sin(2\pi/3)]$

$= \dfrac{1}{4}(3\sqrt{3})r^2$.

Exercise 6.7

1. Given the volume of the box = 1024 cm^3

Let length of the side of square base be x cm and height of the box be y cm.



$\therefore$ Volume of the box $V = x^2y = 1024$

$\Rightarrow$ $y = \dfrac{1024}{x^2}$

Let C denote the cost of the box.

$\therefore$ $C = 2x^2 \times 5 + 4xy \times 2.50$

$= 10x^2 + 10xy$

$= 10x(x + y)$

$= 10x\left(x + \dfrac{1024}{x^2}\right)$

$= 10x^2 + \dfrac{10240}{x}$

On differentiating both sides w.r.t. x, we get

$$\frac{dC}{dx} = 20x - \frac{10240}{x^2}$$

$$\frac{dC}{dx} = 0$$

$\Rightarrow$ $20x = \dfrac{10240}{x^2}$

$\Rightarrow$ $20x^3 = 10240$

$\Rightarrow$ $x^3 = 512 = 8^3$

$\Rightarrow$ $x = 8$

Also $\dfrac{d^2C}{dx^2} = 20 + \dfrac{20480}{x^3} > 0$

$\therefore$ $x = 8$ is point of minima

Thus for $x = 8$, cost is minimum and the corresponding least cost of the box,

$$C(8) = 10(8)^2 + \frac{10240}{8} = 640 + 1280 = 1920$$

$\therefore$ Least cost = Rs. 1920

2. Let r and h be the radius and height of the cylinder respectively.

Then, volume (V) of the cylinder is given by,

$$V = \pi r^2 h = 100 \quad \text{(given)}$$

$$\therefore \quad h = \frac{100}{\pi r^2}$$

Surface (S) of the cylinder is given by,

$$S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + \frac{200}{r}$$

$$\therefore \quad \frac{dS}{dr} = 4\pi r - \frac{200}{r^2}, \quad \frac{d^2S}{dr^2} = 4\pi + \frac{400}{r^3}$$

$$\frac{dS}{dr} = 0 \Rightarrow 4\pi r = \frac{200}{r^2} \Rightarrow r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$$

For $r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$, $\frac{d^2S}{dr^2} > 0$.

$\therefore$ By second derivative test, the surface area is the minimum when the radius of the cylinder is $\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm.

When $r = \left(\frac{50}{\pi}\right)^{1/3}$, $h = \frac{100}{\pi\left(\frac{50}{\pi}\right)^{2/3}} = 2\left(\frac{50}{\pi}\right)^{1/3}$ *cm.*

Hence, the required dimensions of the can which has the minimum surface area is given by radius $= \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm and height $= 2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm.

3. $h = a \sin\alpha$ and $x = a\cos\alpha$;

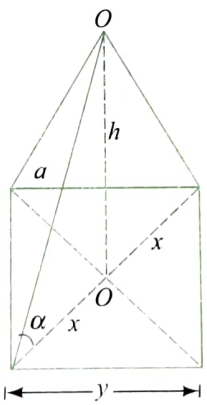
$x^2 + h^2 = a^2$

$$\therefore \quad V = \frac{1}{3}y^2h = \frac{1}{3}2x^2h$$

(Note: $4x^2 = 2y^2 \Rightarrow y^2 = 2x^2$)

$$\therefore \quad V(\alpha) = \frac{2}{3}a^2\cos^2\alpha\, a\sin\alpha$$

$$= \frac{2}{3}a^3\sin\alpha\cos^2\alpha$$

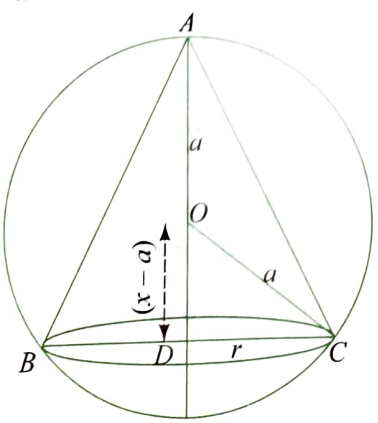


Now $V'(\alpha) = 0$ then $\tan\alpha = \frac{1}{\sqrt{2}}$;

$$\therefore \quad V_{max} = \frac{4\sqrt{3}a^3}{27}$$

4. Let $AD = x$ be the height of the cone ABC inscribed in a sphere of radius a. Therefore,

$OD = x - a$



Then radius of its base (r) $= CD = \sqrt{(OC^2 - OD^2)}$

$$= \sqrt{[a^2 - (x-a)^2]}$$

$$= \sqrt{(2ax - x^2)}$$

Thus, volume V of the cone is given by

$$V = \frac{1}{3}\pi r^2 x = \frac{1}{3}\pi(2ax - x^2)x = \frac{1}{3}\pi(2ax^2 - x^3)$$

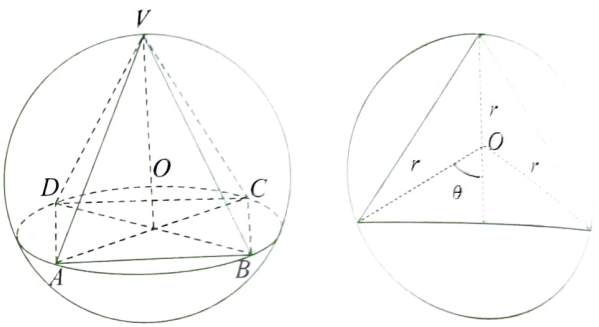
$$\therefore \quad \frac{dV}{dx} = \frac{1}{3}\pi(4ax - 3x^2) \text{ and } \frac{d^2V}{dx} = \frac{1}{3}\pi(4a - 6x)$$

For max. or min. of V, $dV/dx = 0$ or $x = 4a/3$ $(\because x \neq 0)$

For this value of V, $\frac{d^2V}{dx^2} = -(4\pi a/3) = (-\text{ve})$

Thus, V is maximum (i.e., greatest) when $x = 4a/3 = (2/3)(2a)$. i.e., when the height of cone is (2/3)rd of the diameter of sphere

5. If we cut the sphere from the center along the diagonal of the square base, we get the view shown in Figure.



Let r be the radius of the sphere. Then,

Height of the pyramid $= r + r\cos\theta$

Diagonal of the square base $= 2r\sin\theta$

$$\therefore \quad \text{Side of the square base} = \frac{2r\sin\theta}{\sqrt{2}}$$

[$\because$ diagonal $= \sqrt{2}$(side)]

Let V be the volume of the pyramid. Then,

$$V = \frac{1}{3}\text{ (Area of the base} \times \text{Height)}$$

$$= \frac{1}{3}\left(\frac{2r\sin\theta}{\sqrt{2}}\right)^2 \times (r + r\cos\theta)$$

$$= \frac{2}{3}r^3\sin^2\theta(1 + \cos\theta)$$

$$\therefore \quad \frac{dV}{d\theta} = \frac{2}{3}r^3[\sin 2\theta(1 + \cos\theta) - \sin^3\theta]$$

$$= \frac{2}{3}r^3[2\sin\theta\cos\theta + 2\sin\theta\cos^2\theta - \sin^3\theta]$$

$$= \frac{2}{3}r^3 \times \sin\theta(2\cos\theta + 2\cos^2\theta - \sin^2\theta)$$

$$= \frac{2}{3}r^3\sin\theta(3\cos^2\theta + 2\cos\theta - 1)$$

$$= \frac{2}{3}r^3\sin\theta(\cos\theta + 1)(3\cos\theta - 1)$$

For maximum or minimum values of V, we must have $\frac{dV}{d\theta} = 0$

or $\sin\theta(\cos\theta + 1)(3\cos\theta - 1) = 0$

or $3\cos\theta - 1 = 0$ [$\because \sin\theta = 0$ and $\cos\theta + 1 = 0$ are not possible]

or $\cos\theta = \frac{1}{3}$

Also, for $\cos\theta < \frac{1}{3}$ or $\theta > \cos^{-1}\frac{1}{3}$, $\frac{dV}{d\theta} < 0$

and for $\cos\theta > \frac{1}{3}$, $\theta < \cos^{-1}\frac{1}{3}$, $\frac{dV}{d\theta} > 0$

Thus, $\theta = \cos^{-1}\frac{1}{3}$ is point of maxima.

Hence, V is maximum when $\cos\theta = \frac{1}{3}$ and $\sin\theta = \frac{2\sqrt{2}}{3}$.

The maximum value of V is given by

$$V = \frac{2}{3}r^3 \times \frac{8}{9}\left(1+\frac{1}{3}\right) = \frac{64}{81}r^3$$

Exercises

Single Correct Answer Type

1. (4) $f(x) = 3x^2 - 2x + 1$

$\therefore\quad f'(x) = 6x - 2$

f is increasing. Thus, $f'(x) \geq 0$, i.e.,

$6x - 2 \geq 0$ or $x \geq \frac{1}{3}$

2. (4) $f(x) = x^{100} + \sin x - 1$

$\therefore\quad f'(x) = 100\,x^{99} + \cos x$

If $0 < x < \frac{\pi}{2}$, then $f'(x) > 0$.

Therefore, $f(x)$ is increasing on $(0, \pi/2)$.

If $0 < x < 1$, then

$100\,x^{99} > 0$ and $\cos x > 0$

$\therefore\quad f'(x) = 100x^{99} + \cos x > 0$

Thus, $f(x)$ is increasing on $(0, 1)$.

If $\frac{\pi}{2} < x < \pi$, then

$100\,x^{99} > 100$ $\quad[\because x > 1 \Rightarrow x^{99} > 1]$

or $100\,x^{99} + \cos x > 0$ $[\because \cos x \geq -1 \Rightarrow 100\,x^{99} + \cos x > 99]$

$f'(x) > 0$ implies $f(x)$ is increasing in $(\pi/2, \pi)$.

3. (1) Here, $f(x) = \tan^{-1}(\sin x + \cos x)$

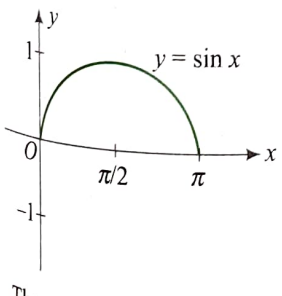
$$\therefore\quad f'(x) = \frac{1}{1+(\sin x + \cos x)^2}(\cos x - \sin x)$$

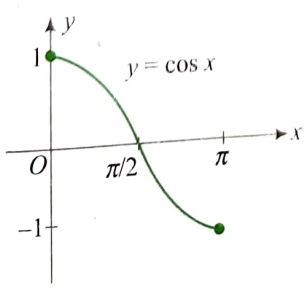
$$= \frac{\cos x - \sin x}{2 + \sin 2x}$$

For $-\frac{\pi}{2} < x < \frac{\pi}{4}$, $\cos x > \sin x$

Hence, $y = f(x)$ is increasing in $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$.

4. (4) From the graph, it is clear that both $\sin x$ and $\cos x$ in the internal $(\pi/2, \pi)$ are the decreasing functions.





Therefore, S is correct.

To disprove R, let us consider the counter example,

$f(x) = \sin x$ in $(0, \pi/2)$

so that $f'(x) = \cos x$.

Again, from the graph, it is clear that $f(x)$ is increasing in $(0, \pi/2)$, but $f'(x)$ is decreasing in $(0, \pi/2)$.

Therefore, R is wrong. Therefore, (4) is the correct option.

5. (1) $3\sin x - 4\sin^3 x = \sin 3x$ which increases for

$3x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ or $x \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$ whose length is $\frac{\pi}{3}$.

6. (3) $f'(x) = x^x[1 + \log x] = x^x \log(ex)$

$f'(x) < 0$

or $\log(ex) < 0$

or $0 < ex < 1$

or $0 < x < 1/e$

7. (1) $f(x) = x\sqrt{4ax - x^2}$ (domain is $[0, 4a]$)

$$\therefore\quad f'(x) = \sqrt{4ax - x^2} + \frac{x(4a - 2x)}{2\sqrt{4ax - x^2}}$$

$$= \frac{2x(3a - x)}{\sqrt{4ax - x^2}}$$

Now, if $f'(x) > 0$, then

$2x(3a - x) > 0$

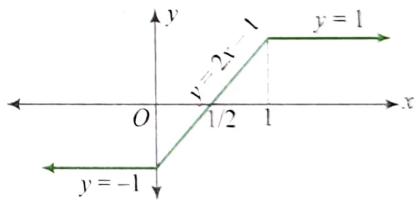
or $2x(x - 3a) < 0$

or $x \in (0, 3a)$

Thus, $f(x)$ increases in $(0, 3a)$ and decreases in $(3a, 4a)$.

8. (4) $f(x) = |x| - |x - 1|$

$$= \begin{cases} -x-(1-x), & x<0 \\ x-(1-x), & 0\le x<1 \\ x-(x-1), & x<0 \end{cases} = \begin{cases} -1, & x<0 \\ 2x-1, & 0\le x<1 \\ 1, & x\ge 1 \end{cases}$$



Graph of the function shows that $f(x)$ clearly increases in $(0, 1)$.

9. (1) $f'(x) = |x| - \{x\} = |x| - (x - [x]) = |x| - x + [x]$

For $x \in (-1/2, 0)$,

$f'(x) = -x - x - 1 = -2x - 1$

Also, for $-\frac{1}{2} < x < 0$ or $0 < -2x < 1$ or $-1 < -2x - 1 < 0$

or $f'(x) < 0$, $f(x)$ decreases in $(-1/2, 0)$.

Similarly, we can check for other given options say for $x \in (-1/2, 2)$,

$$f'(x) = \begin{cases} (-x) - x - 1, & -\frac{1}{2} < x < 0 \\ x - x + 0, & 0 \le x < 1 \\ x - x + 1, & 1 \le x < 2 \\ \vdots & \end{cases}$$

Here, $f(x)$ decreases only in $(-1/2, 0)$; otherwise $f(x)$ in other intervals is constant.

10. (2) $f'(x) = 4 - 2\sec^2 2x = 2(1 - \tan^2 2x)$

For the continuous domain $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, $f'(x) \geq 0$ in $\left[-\frac{\pi}{8}, \frac{\pi}{8}\right]$

and $f'(x) \le 0$ in $\left(-\frac{\pi}{4}, -\frac{\pi}{8}\right] \cup \left[\frac{\pi}{8}, \frac{\pi}{4}\right)$.

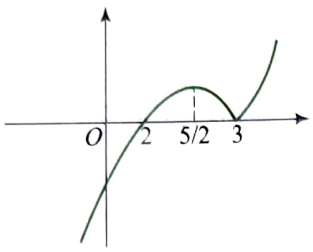
So, the required largest continuous interval is $\left[-\frac{\pi}{8}, \frac{\pi}{8}\right]$ and length is $\frac{\pi}{4}$.

11. (1) $f(x) = (x-2)\,|x-3|$

For $f(x) = (x-2)(x-3) = x^2 - 5x + 6$,

$f'(x) = 2x - 5 = 0$ or $x = 5/2$

Now, the graph of $f(x) = (x-2)\,|x-3|$ is as follows:



Clearly, from the graph, $f(x)$ increases in $(-\infty, 5/2) \cup (3, \infty)$.

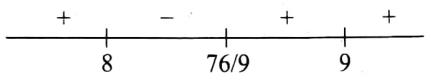
12. (2) $f(x) = (x-8)^4 (x-9)^5,\ 0 \le x \le 10$

$\therefore \quad f'(x) = 4(x-8)^3 (x-9)^5 + 5(x-9)^4 (x-8)^4$

$= (x-8)^3 (x-9)^4 [4(x-9) + 5(x-8)]$

$= 9(x-8)^3 (x-9)^4 \left(x - \frac{76}{9}\right)$

Sign scheme of $f'(x)$ is as follows:

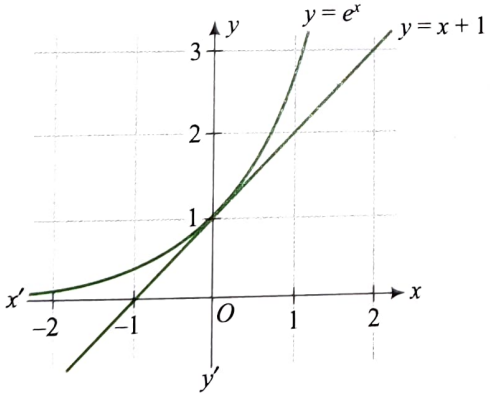


$f'(x) < 0$ if $x \in \left(8, \frac{76}{9}\right)$, i.e., $f(x)$ decreases if $x \in \left(8, \frac{76}{9}\right)$.

13. (2) $y = e^x$ or $\frac{dy}{dx} = e^x$

Then, equation of the tangent at $x = 0$ is

$y - 1 = 1(x - 0)$ or $y = x + 1$



Graph of $y = e^x$ always lies above the graph of $y = 1 + x$.

Hence, $e^x > 1 + x$ or $x > \log_e(1 + x)$. Hence, (b) is true.

Option (3) is wrong as $\sin x < x$ for $x \in (0, 1)$.

Option (4) is wrong as $x > \log_e x$ for $\forall\ x > 0$.

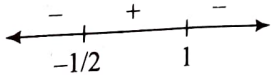
14. (1) $f(x) = xe^{x(1-x)}$

or $\quad f'(x) = e^{x(1-x)} + (1 - 2x)\,xe^{x(1-x)}$

$= -e^{x(1-x)} (2x^2 - x - 1)$

$= -e^{x(1-x)} (2x + 1)(x - 1)$

Sign scheme of $f'(x)$ is as follows:



Thus, $g(x)$ is increasing in $[-1/2, 1]$.

15. (3) $f'(x) = 3kx^2 - 18x + 9 = 3[kx^2 - 6x + 3] \ge 0\ \forall\ x \in R$

or $\quad D = b^2 - 4ac \le 0,\ k > 0$, i.e., $36 - 12k \le 0$

or $\quad k \ge 3$

16. (4) Since $f(x) = \frac{K \sin x + 2\cos x}{\sin x + \cos x}$ is strictly increasing for all x,

$f'(x) > 0$ for all x

or $\quad \frac{K - 2}{(\sin x + \cos x)^2} > 0$ for all x

or $\quad K - 2 > 0$ or $K > 2$

17. (4) $f'(x) = a + 3\cos x - 4\sin x$

$= a + 5\cos(x + \alpha)$, where $\cos\alpha = \frac{3}{5}$

For invertible, $f(x)$ must be monotonic. Thus,

$f'(x) \ge 0\ \forall\ x$ or $f'(x) \le 0\ \forall\ x$

i.e., $a + 5\cos(x + \alpha) \ge 0$ or $a + 5\cos(x + \alpha) \le 0$

i.e., $a \ge -5\cos(x + \alpha)$ or $a \le -5\cos(x + \alpha)$

i.e., $a \ge 5$ or $a \le -5$

18. (4) Clearly if $f(x)$ increases then $f^{-1}(x)$ increases. Draw the graph and verify.

If $f(x)$ increases then $f'(x) > 0$

$\therefore \quad \frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{f^2(x)} < 0$

$\Rightarrow \quad \frac{1}{f(x)}$ decreases

$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$

If f and g are +ve functions and $f' < 0$ and $g' > 0$ then

$\frac{d}{dx}\left(\frac{f}{g}\right) < 0$

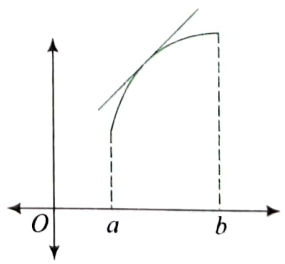
19. (3) $f(x)\,f'(x) < 0\ \forall\ x \in R$

or $\quad \frac{1}{2}\frac{d}{dx}(f^2(x)) < 0$

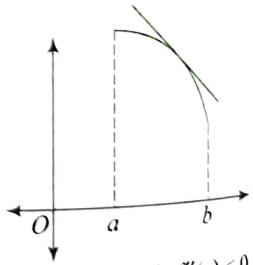
or $\quad \frac{d}{dx}(f^2(x)) < 0$

Thus, $f^2(x)$ is a decreasing function.

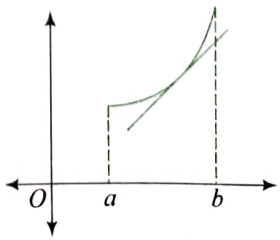
20. (3)



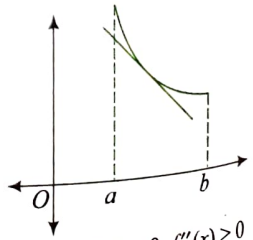
(a) $f'(x) > 0, f''(x) < 0$



(b) $f'(x) < 0, f''(x) < 0$



(c) $f'(x) > 0, f''(x) > 0$



(d) $f'(x) < 0, f''(x) > 0$

Clearly, for $f'(x) > 0$, $f''(x) > 0$ [in Figure (c)], tangent always lies below the graph.

For $f'(x) < 0$, $f''(x) > 0$ [in Figure (d)], tangent always lies below the graph.

21. (2) We must have $\log_{1/3}(\log_3(\sin x + a)) \le 0 \ \forall\, x \in R$

or $\log_3(\sin x + a) >\ge \forall\, x \in R$

or $\sin x + a \ge 3 \ \forall\, x \in R$

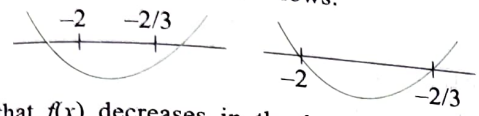
or $a \ge 3 - \sin x \ \forall\, x \in R$

or $a \ge 4$

22. (1) Here, $f'(x) \le 0$

or $3x^2 + 8x + \lambda \le 0 \ \forall\, x \in \left(-2, -\frac{2}{3}\right)$

Then situations for $f'(x)$ are as follows:



Given that $f(x)$ decreases in the largest possible interval $\left(-2, -\frac{2}{3}\right)$. Then $f'(x) = 0$ must have roots -2 and $-2/3$. Thus,

Product of roots $= (-2)\left(-\frac{2}{3}\right) = \frac{\lambda}{3}$ or $\lambda = 4$

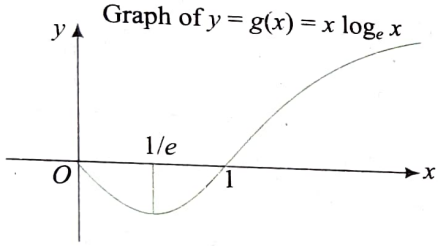
23. (2) $f(x) = |x \log_e x|$

For $g(x) = x \log_e x$,

$$g'(x) = x \cdot \frac{1}{x} + \log_e x = 1 + \log_e x$$

Thus, $g(x)$ increases for $\left(\frac{1}{e}, \infty\right)$ and decreases for $\left(0, \frac{1}{e}\right)$.

Graph of $y = g(x) = x \log_e x$



From the graph, $f(x) = |x \log_e x|$ decreases in $\left(\frac{1}{e}, 1\right)$.

24. (1) $f'(x) = ax^2 + 2(a+2)x + (a-1)$

$f''(x) = 2ax + 2(a+2) = 0$

Thus, $x = -\frac{a+2}{a}$ which is the point of inflection

Given that we must have $\frac{a+2}{a} < 0$ or $a \in (-\infty, -2) \cup (0, \infty)$.

25. (3) $$f'(x) = \frac{0.6(1+x)^{-0.4}(1+x^{0.6}) - 0.6x^{-0.4}(1+x)^{0.6}}{(1+x^{0.6})^2}$$

$$= 0.6\frac{(1+x^{0.6}) - x^{-0.4}(1+x)^1}{(1+x^{0.6})^2(1+x)^{0.4}}$$

$$= 0.6\frac{(1+x^{0.6})x^{0.4} - (1+x)}{(1+x^{0.6})^2(1+x)^{0.4}x^{0.4}}$$

$$= 0.6\frac{x^{0.4} - 1}{(1+x^{0.6})^2(1+x)^{0.4}x^{0.4}} < 0 \ \forall\, x \in (0, 1)$$

Hence, $f(x)$ is decreasing. Thus,

$f(x)_{max} = f(0) = 1$

26. (4) The derivative of a degree 3 polynomial is quadratic. This must have either 0, 1, or 2 roots. If this has precisely one root, then this must be repeated. Hence, we have $f'(x) = m(x-\alpha)^2$, where α is the repeated root and $m \in R$. So, our original function f has a critical point at $x = \alpha$.

Also, $f''(x) = 2m(x-\alpha)$, in which case $f''(\alpha) = 0$. But we are told that the second derivative is nonzero at critical point. Hence, there must be either 0 or 2 critical points.

27. (2) $g'(x) = xf'(2x^2-1) - xf'(1-x^2) = x(f'(2x^2-1) - f'(1-x^2))$

$g'(x) > 0$

If $x > 0$, $2x^2 - 1 > 1 - x^2$ (as f' is an increasing function)

or $3x^2 > 2$ or $x \in \left(-\infty, -\sqrt{\frac{2}{3}}\right) \cup \left(\sqrt{\frac{2}{3}}, \infty\right)$

or $x \in \left(\sqrt{\frac{2}{3}}, \infty\right)$

If $x < 0$, $2x^2 - 1 < 1 - x^2$

or $3x^2 < 2$ or $x \in \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$ or $x \in \left(-\sqrt{\frac{2}{3}}, 0\right)$

28. (1) Given $\phi'(x) - \phi(x) > 0 \ \forall\, x \ge 1$

or $e^{-x}\{\phi'(x) - \phi(x)\} > 0 \ \forall\, x \ge 1$

or $\frac{d}{dx}e^{-x}\phi(x) > 0 \ \forall\, x \ge 1$

Therefore, $e^{-x}\phi(x)$ is an increasing function $\forall\, x \ge 1$.

Since $\phi(x)$ is a polynomial,

$e^{-x}\phi(x) > e^{-1}\phi(1)$ or $e^{-x}\phi(x) > 0$ $[\because \phi(1) = 0]$

or $\phi(x) > 0$

29. (4) $g'(x) = (f'((\tan x - 1)^2 + 3))\, 2(\tan x - 1)\sec^2 x$

Since $f''(x) > 0$, $f'(x)$ is increasing. So,

$\therefore \quad f'((\tan x - 1)^2 + 3) > f'(3) = 0 \ \forall\, x \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

Also, $(\tan x - 1) > 0$ $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

So, $g(x)$ is increasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

30. (3) $u = \sqrt{c+1} - \sqrt{c}$

$$u = \frac{1}{\sqrt{c+1} + \sqrt{c}} \text{ and } v = \frac{1}{\sqrt{c-1} + \sqrt{c}}$$

Clearly, $u < v$.

Also, f is increasing whereas g is decreasing. Thus,

$u < v$

or $f(u) < f(v)$

or $gof(u) > gof(v)$

31. (4) Let $f(x) = x^3 + 2x^2 + 5x + 2\cos x$

$\therefore \quad f'(x) = 3x^2 + 4x + 5 - 2\sin x$

Now, the least value of $3x^2 + 4x + 5$ is

$$-\frac{D}{4a} = -\frac{(4)^2 - 4(3)(5)}{4(3)} = \frac{11}{3}$$

and the greatest value of $2\sin x$ is 2. Therefore

$3x^2 + 4x + 5 > 2\sin x \ \forall\, x \in R$

or $f'(x) = 3x^2 + 4x + 5 - 2\sin x > 0 \ \forall\, x \in R$

Thus, $f(x)$ is strictly an increasing function.

Also, $f(0) = 2$ and $f(2\pi) > 0$.

Thus, for the given interval, $f(x)$ never becomes zero.

Hence, the number of roots is zero.

32. (2) $f'(x) = -\pi \sin \pi x + 10 + 6x + 3x^2$

$= 3(x+1)^2 + 7 - \pi \sin \pi x > 0$ for all x

Thus, $f(x)$ is increasing in $-2 \le x \le 3$.

So, absolute minimum $= f(-2) = 1 - 20 + 12 - 8$

33. (2) Let $g(x) = 4x^3 - 12x^2 + 11x - 3$

$\therefore \quad g'(x) = 12x^2 - 24x + 11$

$= 12(x-1)^2 - 1$

> 0 for $x \in [2, 3]$

Thus, $g(x)$ is increasing in $[2, 3]$.

$f(x)_{max} = f(3) = \log_{10}(4(27) - 12(9) + 11(3) - 3)$

$= \log_{10}(30)$

$= 1 + \log_{10} 3$

34. (3) $f(f(x)) = k(x^5 + x)$ or $f'(f(x))f'(x) = k(5x^4 + 1)$

Thus, $f(x)$ is always increasing or decreasing as $k(5x^4 + 1)$ is either always negative or positive.

35. (4) If $f(x)$ has an extremum at $x = \pi/3$, then $f'(x) = 0$ at $x = \pi/3$.

Now, $f(x) = a \sin x + \frac{1}{3}\sin 3x$

$\therefore \quad f'(x) = a \cos x + \cos 3x$

$f'(\pi/3) = 0$

or $\quad a \cos(\pi/3) + \cos \pi = 0$

or $\quad a = 2$

36. (2) We have $f(x) = a \log |x| + bx^2 + x$

or $f'(x) = \frac{a}{x} + 2bx + 1$

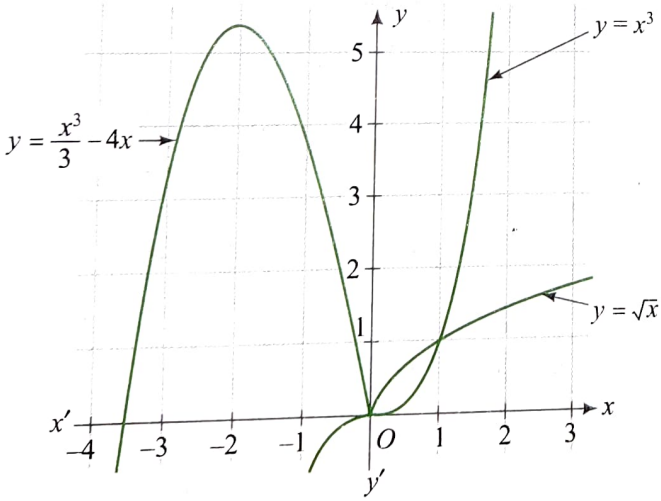
Since $f(x)$ attains its extremum values at $x = -1, 2$,

$f'(-1) = 0$ and $f'(2) = 0$

or $-\ a - 2b + 1 = 0$ and $\frac{a}{2} + 4b + 1 = 0$ or $a = 2$ and $b = -1/2$.

37. (3) Consider functions x^3, $-x^4$, x^4.

38. (3) Function is increasing in $(-\infty, -2) \cup (0, \infty)$ and decreasing in $(-2, 0)$.



$x = -2$ is local maxima and $x = 0$ is local minima.

It is derivable $\forall\ x \in R - \{0, 1\}$ and continuous $\forall\ x \in R$.

39. (2) We have $f(x) = \sin^4 x + \cos^4 x = \frac{3}{4} + \frac{1}{4}\cos 4x$

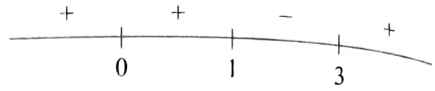
$\therefore \quad f'(x) = -\sin 4x$

Now, $f'(x) > 0$ or $-\sin 4x > 0$

or $\quad \sin 4x < 0$ or $\pi < 4x < 2\pi$

40. (2) $\frac{dy}{dx} = 5x^2(x-1)(x-3) = 0$

$\therefore \quad x = 0, 1, 3$



Clearly, $x = 0$ is neither a point of maxima nor a point of minima as derivative does not change sign at $x = 0$.

$x = 1$ is a point of maxima and $x = 3$ is a point of minima.

41. (3) The given polynomial is

$p(x) = a_0 + a_1x^2 + a_2x^4 + \ldots + a_n x^{2n}, x \in R$

and $\quad 0 < a_0 < a_1 < a_2 < \ldots < a_n$.

Here, we observe that all coefficients of different powers of x, i.e., $a_0, a_1, a_2, \ldots, a_n$, are positive.

Also, only even powers of x are involved.

Therefore, $P(x)$ cannot have any maximum value.

Moreover, $P(x)$ is minimum, when $x = 0$, i.e., a_0.

Therefore, $P(x)$ has only one minimum.

Alternative method:

We have

$P'(x) = 2a_1x + 4a_2x^3 + \ldots + 2na_nx^{2n-1}$

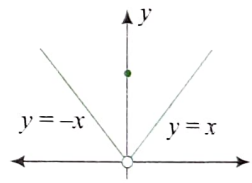
$= x(2a_1 + 4a_2x^2 + \ldots + 2na_nx^{2n-2})$

Clearly, $P'(x) > 0$ for $x > 0$ and $P'(x) < 0$ for $x < 0$.

Thus, $P(x)$ increases for all $x > 0$ and decreases for all $x < 0$.

Therefore, $P'(x)$ has $x = 0$ as the point of manima.

42. (1)



From the graph, $f(0^+) < f(0)$ and $f(0^-) < 0$. Thus, $x = 0$ is the point of maxima.

43. (1) $f(x) = x^3 + bx^2 + cx + d, 0 < b^2 < c$

$f'(x) = 3x^2 + 2bx + c$

Discriminant $= 4b^2 - 12c = 4(b^2 - 3c) < 0$

$\therefore \quad f'(x) > 0\ \forall\ x \in R$

Thus, $f(x)$ is strictly increasing $\forall\ x \in R$.

44. (1) $f(0) = \sin 0 = 0$

$f(0^+) \to 0^+$

$f(0^-) = \lim_{x \to 0^-} \sin(x^2 - 3x) = \lim_{h \to 0} \sin(h^2 + 3h) \to 0^+$

Thus $f(0^+) > f(0)$ and $f(0^-) > f(0)$

Hence $x = 0$ is point of manima.

45. (2) Since $\cos\theta \le 1$ for all θ, $f(x) \le 1$ for all x.

46. (1) $f(x) = (4\sin^2 x - 1)^n (x^2 - x + 1)$

$x^2 - x + 1 > 0\ \forall\ x$

$f\left(\frac{\pi}{6}\right) = 0$

$f\left(\frac{\pi^+}{6}\right) = \lim_{x \to \frac{\pi^+}{6}} (4\sin^2 x - 1)^n (x^2 - x + 1) = \to 0^+$

$$f\left(\frac{\pi}{6}^-\right) = \lim_{x\to\frac{\pi}{6}^-}(4\sin^2 x-1)^n(x^2-x+1)$$

$$= (\to 0^-)^n \quad \text{(A positive value)}$$

Thus, $f\left(\frac{\pi}{6}^-\right) > 0$ if n is an even number.

47. (4) We have $f(x) = x\log_e x - x + 1,\ x > 0$

$\therefore\ f(1) = 0$

$f'(x) = 1 + \log_e x - 1 = \log_e x$

$\therefore\ f'(x) < 0$ if $0 < x < 1$

$f'(x) > 0$ if $x > 1$

Also $\lim_{x\to 0^+}(x\log_e x - x + 1) = 1$

So, in $(0, 1)$, $f(x)$ decreases from 1 to 0 and in $(1, \infty)$ increases from 0 to ∞.

Hence, $f(x) > 0$ for $x \in (0, 1) \cup (1, \infty)$.

48. (2) We have $f(x) = (x+1)^{1/3} - (x-1)^{1/3}$

$$\therefore\ f'(x) = \frac{1}{3}(x+1)^{\frac{-2}{3}} - \frac{1}{3}(x-1)^{\frac{-2}{3}}$$

$$= \frac{(x-1)^{2/3} - (x+1)^{2/3}}{3\left(x^2-1\right)^{2/3}}$$

Clearly, $f'(x)$ does not exist at $x = \pm 1$. Now,

$f'(x) = 0$

or $(x-1)^{2/3} = (x+1)^{2/3}$

or $(x-1)^2 = (x+1)^2$

or $-2x = 2x$ or $4x = 0$ or $x = 0$

Clearly, $f'(x) \neq 0$ for any other values of $x \in [0, 1]$.

The value of $f(x)$ at $x = 0$ is 2.

Hence, the greatest value of $f(x)$ is 2.

49. (2) $$f'(x) = \frac{\log(e+x)\times\frac{1}{\pi+x} - \log(\pi+x)\frac{1}{e+x}}{(\log(e+x))^2}$$

$$= \frac{\log(e+x)\times(e+x) - (\pi+x)\log(\pi+x)}{(\pi+x)(e+x)(\log(e+x))^2}$$

Since log function is an increasing function and $e < \pi$,

$\log(e+x) < \log(\pi+x)$

Thus, $(e+x)\log(e+x) < (e+x)\log(\pi+x) < (\pi+x)\log(\pi+x)$ for all $x > 0$.

Thus, $f'(x) < 0$.

Therefore, $f(x)$ decreases on $(0, \infty)$.

50. (2) $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$

$\therefore\ f'(x) = 6x^2 - 18ax + 12a^2$ and $f''(x) = 12x - 18a$

For maximum/minimum, $6x^2 - 18ax + 12a^2 = 0$

or $x^2 - 3ax + 2a^2 = 0$

or $(x-a)(x-2a) = 0$

i.e., $x = a$ or $x = 2a$

Now, $f''(a) = 12a - 18a = -6a < 0$

and $f''(2a) = 24a - 18a = 6a > 0$

Therefore, $f(x)$ is maximum at $x = a$ and minimum at $x = 2a$.

Thus, $p = a$ and $q = 2a$.

Given that $p^2 = q$ or $a^2 = 2a$ or $a(a-2) = 0$ or $a = 2$.

51. (4) $f(0) > f(0^+)$ and $f(0) < f(0^-)$. Hence, $x = 0$ is neither a maximum nor a minimum.

52. (1) $f(0) = \pi/2,\ f(0^+) = 0,\ f(0^-) = 0$.

Hence, $x = 0$ is the point of maxima.

53. (3) We have $f(x) = \begin{cases}(-1)^{m+n}x^m(x-1)^n, & \text{if } x < 0\\ (-1)^n x^m(x-1)^n, & \text{if } 0 \le x < 1\\ x^m(x-1)^n, & \text{if } x \ge 1\end{cases}$

Let $g(x) = x^m(x-1)^n$. Then,

$g'(x) = mx^{m-1}(x-1)^n + nx^m(x-1)^{n-1}$

$= x^{m-1}(x-1)^{n-1}\{mx - m + nx\}$

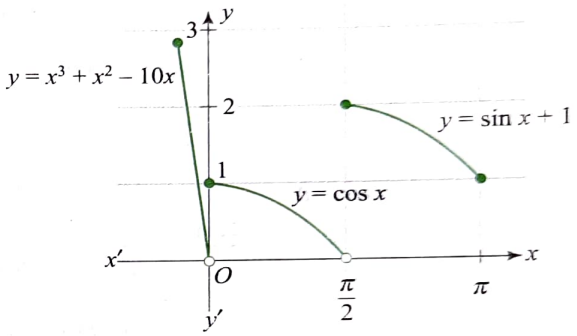
Now, $f'(x) = 0$ or $g'(x) = 0$, i.e., $x = 0, 1$ or $\frac{m}{m+n}$.

$f(0) = 0, f(1) = 0$, and

$$f\left(\frac{m}{m+n}\right) = (-1)^n\frac{m^m n^n(-1)^n}{(m+n)^{m+n}} = \frac{m^m n^n}{(m+n)^{m+n}} > 0$$

$$\therefore\ \text{Maximum value} = \frac{m^m n^n}{(m+n)^{m+n}}$$

54. (3)



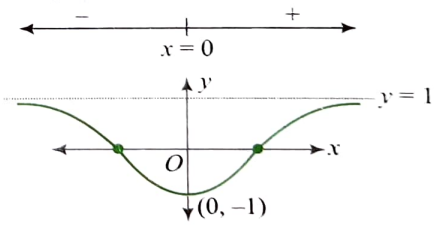
55. (4) We have $f(x) = \frac{x^2-a}{x^2+a} = 1 - \frac{2a}{x^2+a}$

Clearly, range of f is $[-1, 1)$. Now,

$$f'(x) = \frac{4ax}{(x^2+a)^2}$$

and $$f''(x) = \frac{4a}{(x^2+a)^3}(a - 3x^2)$$

Sign scheme of $f'(x)$ is as follows:



Thus, $f(x)$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

Therefore, $f(x)$ has a local minimum at $x = 0$.

56. (1) Here, $f(x) = 4\tan x - \tan^2 x + \tan^3 x$

$\therefore\ f'(x) = 4\sec^2 x - 2\tan x\sec^2 x + 3\tan^2 x\sec^2 x$

$= \sec^2 x(4 - 2\tan x + 3\tan^2 x)$

$$= 3\sec^2 x\left\{\tan^2 x - \frac{2}{3}\tan x + \frac{4}{3}\right\}$$

$$= 3\sec^2 x\left\{\left(\tan x - \frac{1}{3}\right)^2 + \left(\frac{4}{3} - \frac{1}{9}\right)\right\}$$

$$= 3\sec^2 x\left\{\left(\tan x - \frac{1}{3}\right)^2 + \frac{11}{9}\right\} > 0\ \forall\ x$$

Therefore, $f(x)$ is increasing for all $x \in$ domain.

57. (1) $h'(x) = \frac{m}{n} x^{\frac{m-n}{n}} = \frac{m}{n} x^{-\left(\frac{\text{even}}{\text{odd}}\right)}$

As $h'(x)$ is undefined at $x = 0$ and $h'(x)$ does not change its sign in the neighborhood, there are no extremums.

58. (3) $f(x) = \dfrac{(\sin x + \cos x)^2 - 1}{\frac{1}{\sqrt{2}}(\sin x + \cos x)} = \sqrt{2}\,\dfrac{t^2 - 1}{t}$

$= \phi(t) = \sqrt{2}\left(t - \dfrac{1}{t}\right)$

where $t = g(x) = \sin x + \cos x = \sqrt{2}\cos\left(\dfrac{\pi}{4} + x\right), x \in [0, \pi/2]$

$\therefore \quad t \in [1, \sqrt{2}]$

Now, $\phi(t) = \sqrt{2}\left(t - \dfrac{1}{t}\right)$, where $t \in [1, \sqrt{2}]$

$\phi'(t) = \sqrt{2}\left(1 + \dfrac{1}{t^2}\right) = +\text{ve}$

Therefore, $\phi(t)$ is increasing.

Hence $\phi(t)$ is greatest at the endpoint of interval $[1, \sqrt{2}]$, i.e., $t = \sqrt{2}$. Therefore,

$$f(x) = \phi(t) = \sqrt{2}\left[\sqrt{2} - \frac{1}{\sqrt{2}}\right] = 1$$

Alternative method:

$$f(x)\ \frac{\sin 2x}{\sin\left(x + \frac{\pi}{4}\right)} = \frac{2\sin x \cos x}{\frac{1}{\sqrt{2}}(\sin x + \cos x)} = 2\sqrt{2}\,\frac{1}{\sec x + \operatorname{cosec} x}$$

For $x \in (0, \pi/2)$, maximum value of $\sec x + \operatorname{cosec} x$ occurs when $\sec x = \operatorname{cosec} x$ or $x = \pi/4$. Hence,

$$f_{\max} = \frac{2\sqrt{2}}{\sec\frac{\pi}{4} + \operatorname{cosec}\frac{\pi}{4}} = \frac{2\sqrt{2}}{2\sqrt{2}} = 1$$

59. (3) Given $y = e^{(2x^2 - 2x + 1)\sin^2 x} = e^{2\left[\left(x - \frac{1}{2}\right)^2 + \frac{1}{4}\right]\sin^2 x}$

Clearly, the minimum value occurs when $\sin^2 x = 0$ as

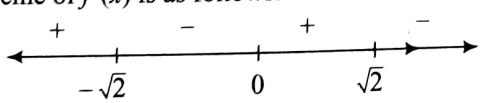
$$\left[\left(x - \frac{1}{2}\right)^2 + \frac{1}{4}\right] \geq \frac{1}{4}.$$

60. (4) $f(x) = x^4 e^{-x^2}$

or $\quad f'(x) = 4x^3 e^{-x^2} + x^4 e^{-x^2}(-2x)$

$= 2x^3 e^{-x^2}(2 - x^2)$

Sign scheme of $f'(x)$ is as follows:



Hence, $f(x)$ is maximum at $x = \pm\sqrt{2}$. Thus, maximum value $= 4e^{-2}$.

61. (3) $a^2x^4 + b^2y^4 = c^6$

or $\quad y = \left(\dfrac{c^6 - a^2x^4}{b^2}\right)^{1/4}$

or $\quad f(x) = xy = x\left(\dfrac{c^6 - a^2x^4}{b^2}\right)^{1/4}$

$= \left(\dfrac{c^6x^4 - a^2x^8}{b^2}\right)^{1/4}$

Differentiate $f(x)$ w.r.t x, we get

$$f'(x) = \frac{1}{4}\left(\frac{c^6x^4 - a^2x^8}{b^2}\right)^{-3/4}\left(\frac{4x^3c^6}{b^2} - \frac{8x^7a^2}{b^2}\right)$$

$$f'(x) = 0 \Rightarrow \frac{4x^3c^6}{b^2} - \frac{8x^7a^2}{b^2} = 0$$

or $\quad x^4 = \dfrac{c^6}{2a^2}$ or $x = \pm\dfrac{c^{3/2}}{2^{1/4}\sqrt{a}}$

At $\quad x = \dfrac{c^{3/2}}{2^{1/4}\sqrt{a}}$, $f(x)$ will be maximum. So,

$$f\left(\frac{c^{3/2}}{2^{1/4}\sqrt{a}}\right) = \left(\frac{c^{12}}{2a^2b^2} - \frac{c^{12}}{4a^2b^2}\right)^{1/4} = \left(\frac{c^{12}}{4a^2b^2}\right)^{1/4}$$

$$= \frac{c^3}{\sqrt{2ab}}$$

Alternative method:

Since A.M. $\geq$ G.M.,

$$\frac{a^2x^4 + b^2y^4}{2} \geq \sqrt{a^2x^4b^2y^4}$$

or $\quad abx^2y^2 \leq \dfrac{c^6}{2}$

or $\quad xy \leq \dfrac{c^3}{\sqrt{2ab}}$

Hence, maximum value of xy is $\dfrac{c^3}{\sqrt{2ab}}$.

62. (2) Let $f(x) = x + ax^{-2} - 2$

Let $\quad f'(x) = 1 - 2ax^{-3} = 0$ or $x = (2a)^{1/3}$

Also, $f''(x) = 6ax^{-4}$ or $f''((2a)^{1/3}) > 0$

Thus, $x = (2a)^{1/3}$ is the point of minima.

For $x + ax^{-2} - 2 > 0\ \forall\ x$, we must have $f((2a)^{1/3}) > 0$

or $\quad (2a)^{1/3} + a(2a)^{-2/3} - 2 > 0$

or $\quad 2a + a - 2(2a)^{2/3} > 0$

or $\quad 3a > 2(2a)^{2/3}$

or $\quad 27a^3 > 32a^2$

or $\quad a > 32/27$

Hence, the least value of a is 2.

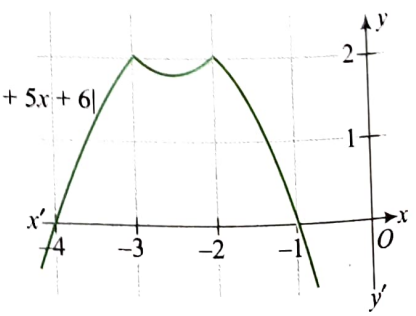
63. (2) Clearly, $f(x)$ in decreasing just before $x = 3$ and increasing after $x = 3$. For $x = 3$ to be the point of local minima, $f(3) \leq f(3^-)$.

or $\quad -15 \leq 12 - 27 + \ln(a^2 - 3a + 3)$

or $\quad a^2 - 3a + 3 \geq 1$

or $\quad a \in (-\infty, 1) \cup (2, \infty)$.

64. (1) $f(x)$ will have maxima at $x = -2$ only if $a^2 + 1 \geq 2$ or or $|a| \geq 1$.



65. (2) Since $f(x)$ has a relative minimum at $x = 0$, $f'(0) = 0$ and $f''(0) > 0$ (assuming $f''(0) \neq 0$).

If the function $y = f(x) + ax + b$ has a relative minimum at $x = 0$, then

$\frac{dy}{dx} = 0$ at $x = 0$ or $f'(x) + a = 0$ for $x = 0$

or $f'(0) + a = 0$ or $0 + a = 0$ $[\because f'(0) = 0]$

or $a = 0$

Now, $\frac{d^2y}{dx^2} = f''(x)$ or $\left(\frac{d^2y}{dx^2}\right)_{x=0} = f''(0) > 0$ $[\because f''(0) > 0]$

Hence, y has a relative minimum at $x = 0$ if $a = 0$ and b can attain any real value.

66.(4) $f'(x) = 12x^2 - 2x - 2$

$= 2(3x + 1)(2x - 1)$

$\therefore \quad g(x) = \begin{cases} f(x), & 0 \le x < 1/2 \\ f(1/2), & 1/2 \le x \le 1 \\ 3 - x, & 1 < x \le 2 \end{cases}$

$\Rightarrow \quad g\left(\frac{1}{4}\right) + g\left(\frac{3}{4}\right) + g\left(\frac{5}{4}\right)$

$= f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + g\left(\frac{5}{4}\right)$

$= \frac{5}{2}$

67.(1) $f(x) + f''(x) = -x\,g(x) f'(x)$

Let $h(x) = f^2(x) + (f'(x))^2$

$\therefore \quad h'(x) = 2f(x)\,f'(x) + 2f'(x)\,f''(x)$

$= 2f'(x)[-x]\,g(x) f'(x)$

$= -2x\,(f'(x))^2\,g(x)$

Thus, $x = 0$ is a point of maxima for $h(x)$.

68.(2) Given $A + B = 60°$ or $B = 60° - A$

$\therefore \quad \tan B = \tan(60° - A) = \frac{\sqrt{3} - \tan A}{1 + \sqrt{3}\tan A}$

Now $z = \tan A \tan B$

or $z = \frac{t(\sqrt{3} - t)}{1 + \sqrt{3}\,t} = \frac{\sqrt{3}\,t - t^2}{1 + \sqrt{3}\,t}$

where $t = \tan A$

$\frac{dz}{dt} = -\frac{(t + \sqrt{3})(\sqrt{3}t - 1)}{(1 + \sqrt{3}\,t)^2} = 0$

or $t = 1/\sqrt{3}$

or $t = \tan A = \tan 30°$

The other value is rejected as both A and B are +ve acute angles.

If $t < \frac{1}{\sqrt{3}}$, $\frac{dz}{dt}$ is positive and if $t > \frac{1}{\sqrt{3}}$, $\frac{dz}{dt}$ is negative.

Hence, maximum when $t = \frac{1}{\sqrt{3}}$ and maximum value $= \frac{1}{3}$.

69.(3) $f(x) = \frac{t + 3x - x^2}{x - 4}$; $f'(x) = \frac{(x-4)(3-2x) - (t + 3x - x^2)}{(x-4)^2}$

For maximum or minimum, $f'(x) = 0$

or $-2x^2 + 11x - 12 - t - 3x + x^2 = 0$

or $-x^2 + 8x - (12 + t) = 0$

For one maxima and minima,

$D > 0$

or $64 - 4(12 + t) > 0$

or $16 - 12 - t > 0$, i.e., $4 > t$ or $t < 4$

70. (1) Since $a = \left(\frac{4}{\sin x} + \frac{1}{1 - \sin x}\right)$, a is least.

$\therefore \quad \frac{da}{dx} = \left[-\frac{4}{\sin^2 x} + \frac{1}{(1 - \sin x)^2}\right]\cos x = 0$

We have to find the values of x in the interval $(0, \pi/2)$.

Thus, $\cos x \neq 0$ and the other factor when equated to zero gives $\sin x = 2/3$. Now,

$$\frac{d^2a}{dx^2} = \left[-\frac{4}{\sin^2 x} + \frac{1}{(1 - \sin x)^2}\right](-\sin x) + \left[\frac{8}{\sin^3 x} + \frac{2}{(1 - \sin x)^3}\right]\cos^2 x$$

Put $\sin x = \frac{2}{3}$ and $\cos^2 x = 1 - \frac{4}{9} = \frac{5}{9}$

$\therefore \quad \frac{d^2a}{dx^2} = 0 + \left[\frac{8}{8/27} + 2 \times 27\right]\frac{5}{9} = 81 \times \frac{5}{9} = 45 > 0$

Thus, a is minimum and its value is

$\frac{4}{2/3} + \frac{1}{1 - (2/3)} = 6 + 3 = 9$

71. (1) $f'(x) = -3x^2 - 6x - 2 < 0 \;\forall\; x > 0$

So, $f(x)$ is decreasing for $x > 0$.

Now $x^2 + 1,\ 2x^2 + 2x + 3 > 0$

$\therefore \quad f(x^2 + 1) > f(2x^2 + 2x + 3)$

$\Rightarrow \quad x^2 + 1 < 2x^2 + 2x + 3$

$\Rightarrow \quad x^2 + 2x + 2 > 0$

$\Rightarrow \quad x \in R$

72. (1) Let $f(x) = \frac{\log_e(2x-1)}{\log_e x}$ for $x > 1$

$$\Rightarrow \quad f'(x) = \frac{\frac{2}{2x-1}\log_e x - \frac{1}{x}\log_e(2x-1)}{\{\log_e(2x-1)\}^2},\ x > 1$$

$$= \frac{2x\log_e x - (2x-1)\log_e(2x-1)}{x(2x-1)\{\log_e(2x-1)\}^2},\ x > 1$$

Let $g(x) = 2x \log_e x - (2x - 1)\log_e(2x - 1)$

$\Rightarrow \quad g'(x) = 2\log_e x + 2 - 2\log_e(2x - 1) - 2$

$= -2\log_e\left(2 - \frac{1}{x}\right) < 0$ for $x > 1$

So, for $x > 1$, we have $g(x) < g(1)$.

$\Rightarrow \quad g(x) < 0$

$\Rightarrow \quad f'(x) < 0$ for $x > 1$

Thus, $f(x)$ is decreasing for $x > 1$.

$\because \quad 5 > 4 > 3 > 2$

$\therefore \quad f(5) > f(4) > f(3) > f(2)$

$\Rightarrow \quad \frac{\log_e 9}{\log_e 5} < \frac{\log_e 7}{\log_e 4} < \frac{\log_e 5}{\log_e 3} < \frac{\log_e 3}{\log_e 2}$

73. (3) Let $f(x) = 4x^3 + 5x + k$

$\therefore \quad f'(x) = 12x^2 + 5 > 0 \;\forall\; x \in R$

$\therefore \quad f(x)$ is strictly increasing on R.

For negative root of the equation $f(x) = 0$, graph of $y = f(x)$ must intersect y-axis at positive side.

74. (2) Equation of the tangent to the ellipse $\frac{x^2}{27}+y^2=1$ at $(3\sqrt{3}\cos\theta, \sin\theta)$, $\theta\in(0,\pi/2)$, is

$$\frac{\sqrt{3}\,x\cos\theta}{9}+y\sin\theta=1$$

$\therefore$ Sum of the intercepts $= S = 3\sqrt{3}\sec\theta+\operatorname{cosec}\theta$

For minimum values of S, $\frac{dS}{d\theta}=0$

or $3\sqrt{3}\sec\theta\tan\theta-\operatorname{cosec}\theta\cot\theta=0$

or $\frac{3\sqrt{3}\sin\theta}{\cos^2\theta}-\frac{\cos\theta}{\sin^2\theta}=0$

or $3\sqrt{3}\sin^3\theta-\cos^3\theta=0$

or $\tan^3\theta=\frac{1}{3\sqrt{3}}=\left(\frac{1}{\sqrt{3}}\right)^3$

or $\tan\theta=\frac{1}{\sqrt{3}}=\tan\pi/6\Rightarrow\theta=\pi/6.$

75. (3) Consider the function $f(x)=\frac{x^2}{(x^3+200)}$ (1)

$\therefore\quad f'(x)=x\frac{(400-x^3)}{(x^3+200)^2}=0$

When $x=(400)^{1/3}$, $(\because x\neq 0)$

$x=(400)^{1/3}-h$ or $f'(x)>0$

$x=(400)^{1/3}+h$ or $f'(x)<0$

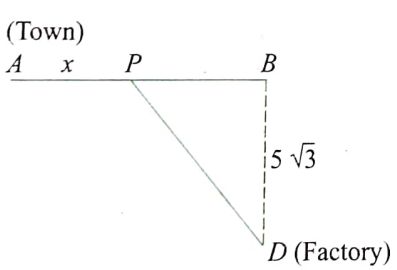
Thus, $f(x)$ has maxima at $x=(400)^{1/3}$.

Since $7<(400)^{1/3}<8$, either a_7 or a_8 is the greatest term of the sequence. Therefore,

$$a_7=\frac{49}{543}\text{ and }a_8=\frac{8}{89}\text{ and }\frac{49}{543}>\frac{8}{89}$$

Thus, $a_7=\frac{49}{543}$ is the greatest term.

76. (1)



Let the charges for railway line be k ₹/km.

Now, total freight charges, $T=kx+2k\sqrt{(20-x)^2+75}$.

Let $\frac{dT}{dx}=0$

or $k+2k\frac{2(x-20)}{2\sqrt{(x-20)^2+75}}=0$

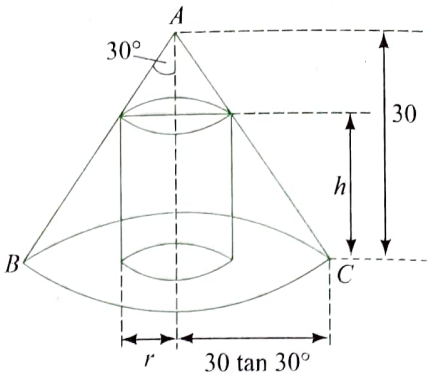
or $4(x-20)^2=75+(x-20)^2$

or $(x-20)^2=25$

or $x=25,\ 15$ or $x=15$ (as $AP<AB$)

or $PB=AB-AP=20-15=5$ km

77. (1)



From geometry, we have $\frac{r}{30\tan 30^\circ}=\frac{30-h}{30}$

or $h=30-\sqrt{3}r$

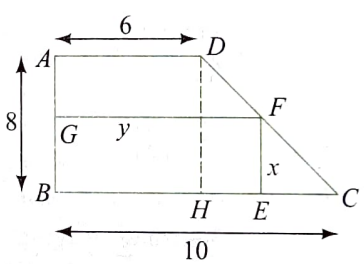
Now, the volume of cylinder, $V=\pi r^2h=\pi r^2(30-\sqrt{3}r)$

Now, let $\frac{dV}{dr}=0$ or $\pi(60r-3\sqrt{3}r^2)=0$ or $r=\frac{20}{\sqrt{3}}$

Hence, $V_{\max}=\pi\left(\frac{20}{\sqrt{3}}\right)^2\left(30-\sqrt{3}\frac{20}{\sqrt{3}}\right)$

$$=\pi\ \frac{400}{3}\times 10=\frac{4000\pi}{3}$$

78. (2)



Let rectangle $BEFG$ is inscribed.

Its area, $A=x\,y$

Now, ΔFEC and ΔDHC are similar, i.e.,

$$\frac{x}{8}=\frac{10-y}{4}\text{ or }y=10-\frac{x}{2}$$

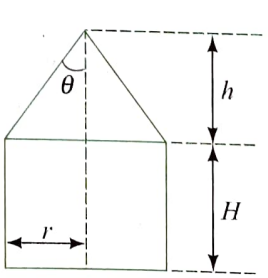
or $A=x\left(10-\frac{x}{2}\right)$ where $x\in(0,8)$

Now, $\frac{dA}{dx}=10-x$. For $x\in(0,8)$, $\frac{dA}{dx}>0$, i.e., A increases.

Hence, $A_{\max}$ occurs when $x=8$.

Hence, max area $=A_{\max}=8\left(10-\frac{8}{2}\right)=48\ \text{cm}^2$.

79. (1)



Given volume and r.

Now, $V=$ volume of cone + volume of cylinder

$$=\frac{\pi}{3}r^2h+\pi r^2H$$

$$=\frac{\pi}{3}r^2(h+3H)$$

or $H = \dfrac{\dfrac{3V}{\pi r^2} - h}{3}$

Now, surface area, $S = \pi rl + 2\pi rH$

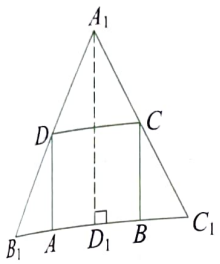
$$= \pi r\sqrt{h^2 + r^2} + 2\pi r \times \left(\frac{\dfrac{3V}{\pi r^2} - h}{3}\right)$$

Let $\dfrac{dS}{dh} = 0$ or $\pi r \dfrac{h}{\sqrt{h^2 + r^2}} - \dfrac{2\pi r}{3} = 0$

or $\dfrac{h}{\sqrt{h^2 + r^2}} = \dfrac{2}{3}$ or $5h^2 = 4r^2$ or $\dfrac{r}{h} = \dfrac{\sqrt{5}}{2} = \tan\theta$

or $\cos\theta = \dfrac{2}{3}$ or $\theta = \cos^{-1}\dfrac{2}{3}$

80. (4)



Let $BD_1 = x$ or $BC_1 = (a - x)$

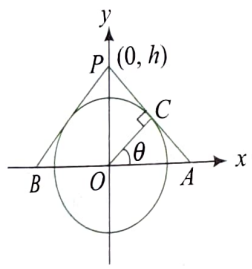
$\therefore$ $BC = (a - x)\tan\dfrac{\pi}{3} = \sqrt{3}(a - x)$.

Now, area of rectangle $ABCD$,

$\Delta = (AB)(BC) = 2\sqrt{3}x(a - x)$

$$\le 2\sqrt{3}\left(\frac{x + a - x}{2}\right)^2 = \frac{\sqrt{3}a^2}{2}$$ (using A.M. $\ge$ G.M.)

81. (4)



Let $\angle COA = \theta$. Then $OA = OC \sec\theta = 4\sec\theta$.

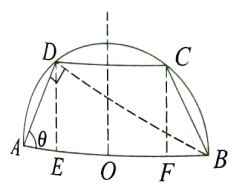
Also, $\angle OPC = \theta$. Then $OP = OC \operatorname{cosec}\theta = 4\operatorname{cosec}\theta$.

Now, $\Delta_{PAB} = OA \cdot OP = \dfrac{32}{\sin 2\theta}$

For Δ_{PAB} to be minimum, $\sin 2\theta = 1$ or $\theta = \dfrac{\pi}{4}$

$\therefore$ $P \equiv (0, 4\sqrt{2})$

82. (1)



$AD = AB\cos\theta = 2R\cos\theta$,

$AE = AD\cos\theta = 2R\cos^2\theta$

or $EF = AB - 2AE = 2R - 4R\cos^2\theta$

$DE = AD\sin\theta = 2R\sin\theta\cos\theta$

Thus, area of trapezium,

$$S = \frac{1}{2}(AB + CD) \times DE$$

$$= \frac{1}{2}(2R + 2R - 4R\cos^2\theta) \times 2R\sin\theta\cos\theta$$

$$= 4R^2\sin^3\theta\cos\theta$$

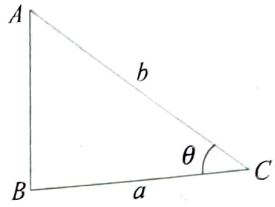
$$\frac{dS}{d\theta} = 12R^2\sin^2\theta\cos^2\theta - 4R^2\sin^4\theta$$

$$= 4R^2\sin^2\theta(3\cos^2\theta - \sin^2\theta)$$

For maximum area, $\dfrac{dS}{d\theta} = 0$ or $\tan^2\theta = 3$

or $\tan\theta = \sqrt{3}$ (θ is acute) or $S_{max} = \dfrac{3\sqrt{3}}{4}R^2$

83. (3) $b\cos\theta = a$ or $b\cos\theta + b = 4$



or $b = \dfrac{4}{1 + \cos\theta}$

$\therefore$ $a = \dfrac{4\cos\theta}{1 + \cos\theta}$

or Area $= \Delta = \dfrac{1}{2}ba\sin\theta$

$$= \frac{1}{2}\frac{4}{1 + \cos\theta}\frac{4\cos\theta}{1 + \cos\theta} \times \sin\theta = \frac{4\sin 2\theta}{(1 + \cos\theta)^2}$$

$\therefore$ $\dfrac{d\Delta}{d\theta} = 4\dfrac{2\cos 2\theta(1 + \cos\theta)^2 + 2\sin 2\theta(1 + \cos\theta)\sin\theta}{(1 + \cos\theta)^4}$

$\dfrac{d\Delta}{d\theta} = 0 \Rightarrow \cos 2\theta\ (1 + \cos\theta) + \sin 2\theta\sin\theta = 0$

or $\cos 2\theta + \cos\theta = 0$ or $\cos 2\theta = -\cos\theta = \cos(\pi - \theta)$

or $\theta = \dfrac{\pi}{3}$

Therefore, Δ is maximum when $\theta = \dfrac{\pi}{3}$.

84. (2) $\tan\theta = \tan(\theta_2 - \theta_1)$

$\Rightarrow$ $\tan\theta = \dfrac{3x - x}{1 + 3x.x} = \dfrac{2x}{1 + 3x^2}$

Let $y = \dfrac{2x}{1 + 3x^2}$

$\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{2(1 - 3x^2)}{(1 + 3x^2)^2}$

$\dfrac{dy}{dx} = 0 \Rightarrow x = \dfrac{1}{\sqrt{3}}$

and $\dfrac{d^2y}{dx^2} = \dfrac{-24x}{(1 + 3x^2)^3} < 0$ for $x = 1/\sqrt{3}$

$\therefore$ $\theta_{max.} = \pi/6$

85. (2) Fuel charges $\propto v^2$. Let F represents fuel charges. Then $F \propto v^2$ or $F = kv^2$ (1)

Given that $F = ₹48/h$, $v = 16$ km/h. Thus,

$$48 = k(16)^2 \text{ or } k = \frac{3}{16}$$

From (1), $F = \dfrac{3v^2}{16}$

Let the train covers λ km in t hours. Then

$$\lambda = vt \text{ or } t = \frac{\lambda}{v}$$

$\therefore$ Fuel charges in time $t = \frac{3}{16}v^2 \times \frac{\lambda}{v} = \frac{3v\lambda}{16}$

Thus, total cost for running the train,

$$C = \frac{3v\lambda}{16} + 300 \times \frac{\lambda}{v}$$

$\therefore$ $\frac{dC}{dv} = \frac{3\lambda}{16} - \frac{300\lambda}{v^2}$ and $\frac{d^2C}{dv^2} = \frac{600\lambda}{v^3}$

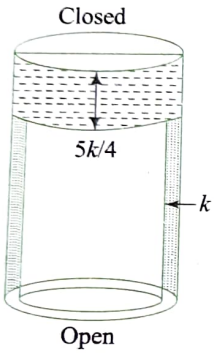
For the maximum or minimum values of C,

$$\frac{dC}{dv} = 0 \text{ or } v = 40 \text{ km/h}$$

Also, $\left.\frac{d^2C}{dv^2}\right|_{v=40} = \frac{60\lambda}{(40)^3} > 0$ $(\because \lambda > 0)$

Thus, C is minimum when $v = 40$ km/h.

86. (3)



Let x be the radius and y the height of the cylindrical gas container. Also, let k be the thickness of the plates forming the cylindrical sides. Therefore, the thickness of the plate forming the top will be $5k/4$.

Capacity of the vessel = volume of cylinder

$= \pi x^2 y = V$ (Given)

$\therefore$ $y = V/(\pi x^2)$ (1)

Now, the volume V_1 of the iron plate used for construction of the container is given by

$$V_1 = \pi (x + k)^2 (y + 5k/4) - \pi x^2 y$$

$\therefore$ $\frac{dV_1}{dx} = 2Vk(x+k) \times \left(\frac{5\pi}{4V} - \frac{1}{x^3}\right)$

For maximum or minimum of V_1, $dV_1/dx = 0$ or $x = [4V/(5\pi)]^{1/3}$.

For this value of x, d^2V_1/dx^2 is +ve.

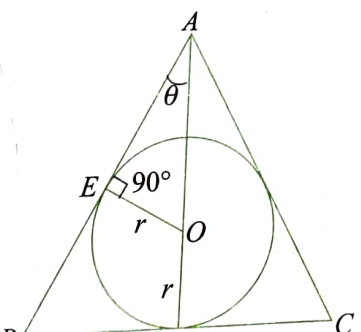
Hence, V_1 is minimum when $x = [4V/(5\pi)]^{1/3}$.

Now, $x = [4V/(5\pi)]^{1/3}$

or $5\pi x^3 = 4V = 4\pi x^2 y$ or $x/y = 4/5$

Hence, the required ratio is 4 : 5.

87. (3)



Let ABC be an isosceles triangle in which a circle of radius r is inscribed.

Let $\angle BAD = \theta$ (semi-vertical angle).

In ΔOAE, $OA = OE$ cosec $\theta = r$ cosec θ, $AE = r$ cot θ. Thus,

$AD = OA + OD = r$ (cosec $\theta + 1$)

In ΔABD, $BD = AD \tan\theta = r$ (cosec $\theta + 1$) tan θ

$AB = AD$ sec $\theta = r$ (cosec $\theta + 1$) sec θ

Now, perimeter of the ΔABC,

$S = AB + AC + BC = 2AB + 2BD$ $(\because AC = AB)$

$= 2r$ (cosec $\theta + 1$) (sec θ + tan θ) $= \frac{4r(1+\sin\theta)^2}{\sin 2\theta}$

$\therefore$ $\frac{dS}{d\theta} = 4r[2(1 + \sin\theta)\cos\theta \sin 2\theta - (1 + \sin\theta)^2 2\cos 2\theta]/(\sin 2\theta)^2$

$= 8r(1 + \sin\theta)[\sin 2\theta \cos\theta - \cos 2\theta \sin\theta - \cos 2\theta]/(\sin 2\theta)^2$

$= 8r(1 + \sin\theta)(\sin\theta - 1 + 2\sin^2\theta)/(\sin 2\theta)^2$

$= 16r(1 + \sin\theta)^2(\sin\theta - 1/2)/(\sin 2\theta)^2$

For maximum or minimum of S, $dS/d\theta = 0$ or $\sin\theta = 1/2$

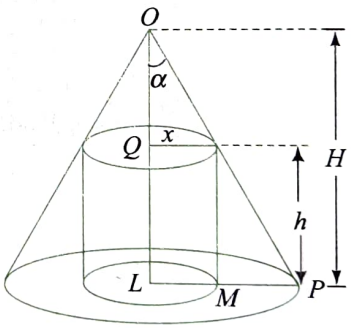
$\therefore$ $\theta = \pi/6$ $(\because \sin\theta \neq -1$ as θ is an acute angle)

Now, if θ is little less and little greater than $\pi/6$, then sign of $dS/d\theta$ changes from –ve to +ve. Hence, S is minimum when $\theta = \pi/6$, which is the point of minima.

Hence, the least perimeter of the triangle is

$\Delta = 4r[1 + \sin(\pi/6)]^2/\sin(\pi/3) = 6\sqrt{3}r$

88. (1)



Let H be the height of the cone and α be its semi-vertical angle. Suppose that x is the radius of the inscribed cylinder and h is its height.

$h = QL = OL - OQ = H - x \cot\alpha$

V = Volume of the cylinder $= \pi x^2(H - x\cot\alpha)$

Also, $p = \frac{1}{3}\pi(H\tan\alpha)^2 H$ (1)

$$\frac{dV}{dx} = \pi(2Hx - 3x^2\cot\alpha)$$

So, $\frac{dV}{dx} = 0$, i.e., $x = 0$ or $x = \frac{2}{3}H\tan\alpha$

$$\left.\frac{d^2V}{dx^2}\right|_{x=\frac{2}{3}H\tan\alpha} = -2\pi H < 0$$

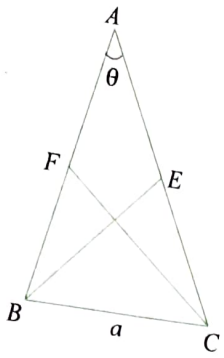
So, V is maximum when $x = \frac{2}{3}H\tan\alpha$.

$$q = V_{max} = \pi\frac{4}{9}H^2\tan^2\alpha\frac{1}{3}H$$

$$= \frac{4}{27}\frac{3\pi p\tan^2\alpha}{\pi\tan^2\alpha} = \frac{4}{9}p$$ [From (1)]

Hence, $p : q = 9 : 4$

89. (4) Triangle ABC is isosceles with $AB = AC$.
BE and CF are medians.



Area of triangle,

$$\Delta = \frac{1}{2} bc \sin\theta$$

$$= \frac{1}{2} c^2 \sin\theta \text{ (as } b = c)$$

In ΔABE, using cosine rule,

$$l^2 = c^2 + \frac{b^2}{4} - bc\cos\theta$$

$$= \frac{5c^2}{4} - c^2\cos\theta$$

$$\therefore \quad c^2 = \frac{4l^2}{5 - 4\cos\theta}$$

$$\therefore \quad \Delta = 2l^2 \frac{\sin\theta}{5 - 4\cos\theta}$$

$$\therefore \quad \frac{d\Delta}{d\theta} = \frac{2l^2((5 - 4\cos\theta)\cos\theta - \sin\theta\,(4\sin\theta))}{(5 - 4\sin\theta)^2}$$

$$= \frac{2l^2(5\cos\theta - 4)}{(5 - 4\sin\theta)^2}$$

$\therefore$ For Δ to be maximum, $\cos\theta = 0.8$

90. (3) The dimensions of the box after cutting equal squares of side x on the corner will be $21 - 2x$, $16 - 2x$, and height x.

$$V = x(21 - 2x)(16 - 2x)$$

$$= x(336 - 74x + 4x^2) = 4x^3 + 336x - 74x^2$$

$$\therefore \quad \frac{dV}{dx} = 12x^2 + 336 - 148x$$

$\frac{dV}{dx} = 0$ gives $x = 3$ for which $\frac{d^2V}{dx^2}$ is –ve and, hence, maximum.

91. (1) $f(x) = \frac{\sin^3 x\cos x}{2}$

$$\therefore \quad f'(x) = \frac{3\sin^2 x\cos^2 x - \sin^4 x}{2}$$

$f'(x) = 0 \Rightarrow 3\sin^2 x\cos^2 x - \sin^4 x = 0$

or $3\cos^2 x - \sin^2 x = 0$

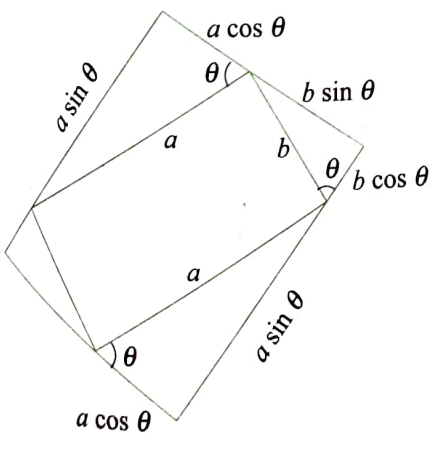
or $4\cos^2 x - 1 = 0$

or $\cos x = \frac{1}{2}$

or $x = \frac{\pi}{3}$, which is the point of maxima

$$\therefore \quad f_{max.} = \frac{(\sqrt{3}/2)^3(1/2)}{2} = \frac{3\sqrt{3}}{32}$$

92. (2)



Area, $A = (a\sin\theta + b\cos\theta)(a\cos\theta + b\sin\theta)$

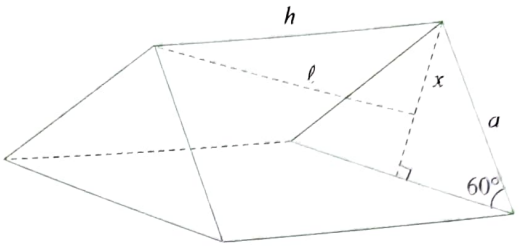
$$= ab(\sin^2\theta + \cos^2\theta) + (a^2 + b^2)\sin\theta\cos\theta$$

$$= ab + \frac{(a^2 + b^2)}{2}\sin 2\theta$$

A is maximum when $\sin 2\theta$ is maximum. Therefore,

$$A_{max} = ab + \frac{(a^2 + b^2)}{2} = \frac{1}{2}(a + b)^2$$

93. (2)



From the figure, $\ell^2 = h^2 + x^2$

Area of base (equilateral triangle) is $\frac{\sqrt{3}}{4}a^2$.

Also, $\frac{3x}{2} = a\sin 60° = \frac{\sqrt{3}}{2}a$

$\therefore \quad a = \sqrt{3}x$

Now, Volume (V) = Height × Area of base.

$$= h\frac{\sqrt{3}}{4}a^2 = h \cdot \frac{\sqrt{3}}{4} \cdot 3 \cdot x^2$$

$$= \frac{3\sqrt{3}}{4} h(\ell^2 - h^2)$$

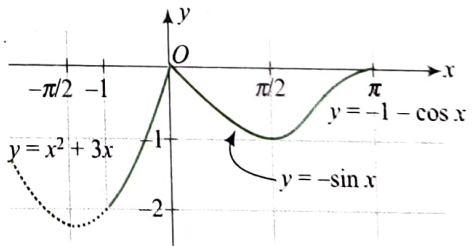
$$= \frac{3\sqrt{3}}{4} (h\ell^2 - h^3)$$

$$\therefore \quad \frac{dV}{dh} = \frac{3\sqrt{3}}{4}(\ell^2 - 3h^2)$$

$\frac{dV}{dh} = 0 \Rightarrow h = \frac{\ell}{\sqrt{3}}$, which is point of maxima

Multiple Correct Answers Type

1. **(1), (2), (3), (4)**



From the graph, global minimum value is $f(-1) = -2$ and global maximum value is $f(0) = f(\pi) = 0$.

2. **(1), (3)**

$f'(x) = 4(x^3 - 3x^2 + 3x - 1) = 4(x - 1)^3 > 0$ for $x > 1$

Hence, f increases in $[1, \infty)$. Moreover, $f'(x) < 0$ for $x < 1$.

Hence, f has a minimum at $x = 1$.

3. **(1), (2), (3), (4)**

$f(x) = 2x - \sin x$ or $f'(x) = 2 - \cos x > 0\ \forall\ x$. Hence, $f(x)$ is strictly increasing. Hence, the function is one-one and onto.

$$g(x) = x^{1/3} \text{ or } g'(x) = \frac{1}{3}x^{-2/3} > 0\ \forall\ x$$

Hence, $g(x)$ is strictly increasing and, hence, one-one and onto.

Also, gof is one-one.

$gof(x) = (2x - \sin x)^{1/3}$ has range R as the range of $2x - \sin x$ is R.

4. (1), (4)

We have

$$f(x) = 2x + \cot^{-1} x + \log(\sqrt{1+x^2} - x)$$

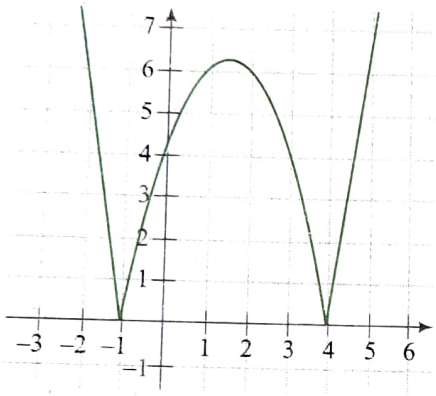
$$\therefore \quad f'(x) = 2 - \frac{1}{1+x^2} + \frac{1}{\sqrt{1+x^2} - x}\left(\frac{x}{\sqrt{1+x^2}} - 1\right)$$

$$= \frac{1+2x^2}{1+x^2} - \frac{1}{\sqrt{1+x^2}} = \frac{1+2x^2}{1+x^2} - \frac{\sqrt{(1+x^2)}}{1+x^2}$$

$$= \frac{x^2 + \sqrt{1+x^2}(\sqrt{1+x^2} - 1)}{1+x^2} > 0 \text{ for all } x$$

Hence, $f(x)$ is an increasing function in $(-\infty, \infty)$ and, in particular, in $(0, \infty)$.

5. (1), (2), (3), (4)



Refer the graph for the answers.

6. (2), (4)

$$f'(x) = \frac{\sin x}{x}$$

For $f'(x) = 0$, $\frac{\sin x}{x} = 0$ or $x = n\pi$, $(n \in I, n \neq 0)$

$$f''(x) = \frac{x\cos x - \sin x}{x^2}$$

$$f''(n\pi) = \frac{\cos n\pi}{n\pi} < 0 \text{ if } n = 2k-1 \text{ and } > 0 \text{ if } n = 2k,\ k \in I^+$$

Hence, $f(x)$ has local maxima at $x = n\pi$, where $n = 2k - 1$, and local minima at $x = n\pi$, $n = 2k$, where $k \in I^+$.

7. (1), (2)

Given that $\frac{x^2+x+2}{x^2+5x+6} < 0$ or $x \in (-3, -2)$

We have to find the extrema for the function

$$f(x) = 1 + a^2x - x^3$$

For maximum or minimum, $f'(x) = 0$

or $\quad a^2 - 3x^2 = 0$ or $x = \pm\frac{a}{\sqrt{3}}$

and $\quad f''(x) = -6x$ is +ve when x is negative.

If a is positive, then the point of minima is $-\frac{a}{\sqrt{3}}$, i.e.,

$$-3 < -\frac{a}{\sqrt{3}} < -2 \text{ or } 2\sqrt{3} < a < 3\sqrt{3}$$

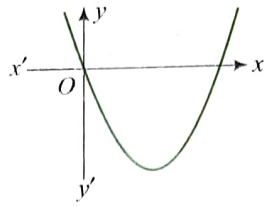
If a is negative, then the point of minima is $\frac{a}{\sqrt{3}}$, i.e.,

$$-3 < \frac{a}{\sqrt{3}} < -2 \text{ or } -3\sqrt{3} < a < -2\sqrt{3}$$

Then, $a \in (-3\sqrt{3}, -2\sqrt{3}) \cup (2\sqrt{3}, 3\sqrt{3})$.

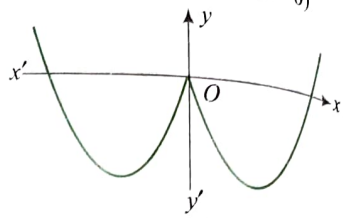
8. (1), (3)

$y = ax^2 - bx$ $(a > 0, b > 0)$



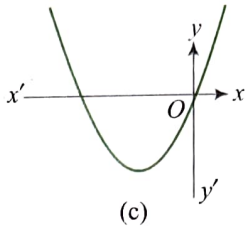
(a)

$y = ax^2 - b|x|$ $(a > 0, b > 0)$



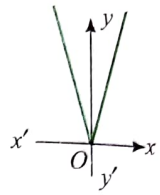
(b)

$y = ax^2 - bx$ $(a > 0, b < 0)$



(c)

$y = ax^2 - b|x|$ $(a > 0, b < 0)$



(d)

9. (1), (2), (4)

$$y = \frac{2x-1}{x-2}$$

$$\frac{dy}{dx} = \frac{2(x-2) - (2x-1)}{(x-2)^2} = \frac{-3}{(x-2)^2} < 0 \ \forall\ x \neq 2$$

Therefore, y is decreasing in $(-\infty, 2)$ as well as in $(2, \infty)$. Thus,

$$y = \frac{2x-1}{x-2} \text{ or } x = \frac{2y-1}{y-2}$$

$$\therefore \quad f^{-1}(x) = \frac{2x-1}{x-2}$$

Thus, $f(x)$ is its own inverse.

10. (2), (3)

Since $g(x)$ is increasing and $f(x)$ is decreasing,

$g(x+1) > g(x-1)$ and $f(x+1) < f(x-1)$

or $\quad f\{g(x+1)\} < f\{g(x-1)\}$

and $\quad g\{f(x+1)\} < g\{f(x-1)\}$

11. (2), (3)

$$f(x) = x^3 - x^2 + 100x + 2002$$

$f'(x) = 3x^2 - 2x + 100 > 0\ \forall\ x \in R$. Thus,

Therefore, $f(x)$ is increasing (strictly). Thus,

$$f\left(\frac{1}{2000}\right) > f\left(\frac{1}{2001}\right)$$

Also, $f(x-1) > f(x-2)$ as $x - 1 > x - 2\ \forall\ x$.

12. (1), (4)

Since $g(a) \neq 0$, either $g(a) > 0$ or $g(a) < 0$.

Let $g(a) > 0$. Since $g(x)$ is continuous at $x = a$, there exists a neighborhood of a in which $g(x) > 0$.

Thus, $f'(x) > 0$. Therefore, $f(x)$ is increasing in the neighborhood of a.

Let $g(a) < 0$. Since $g(x)$ is continuous at $x = a$, there exists a neighborhood of a in which $g(x) < 0$.

Thus, $f'(x) < 0$. Therefore, $f(x)$ is decreasing in the neighborhood of a.

13. **(1), (4)**

We have $f(x) = (4a-3)(x + \log 5) + 2(a-7)\cot\frac{x}{2}\sin^2\frac{x}{2}$

$= (4a-3)(x + \log 5) + (a-7)\sin x$

or $f'(x) = (4a-3) + (a-7)\cos x$

If $f(x)$ does not have critical points, then $f'(x) = 0$ does not have any solution in R. Now,

$$f'(x) = 0 \text{ or } \cos x = \frac{4a-3}{7-a}$$

or $$\left|\frac{4a-3}{7-a}\right| \le 1 \qquad [\because |\cos x| \le 1]$$

or $$-1 \le \frac{4a-3}{7-a} \le 1$$

or $a - 7 \le 4a - 3 \le 7 - a$

or $a \ge -4/3$ and $a \le 2$

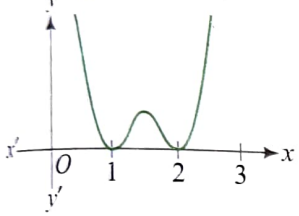
Thus, $f'(x) = 0$ has solutions in R if $-4/3 \le a \le 2$.

So, $f'(x) = 0$ is not solvable in R if $a < -4/3$ or $a > 2$, i.e., $a \in (-\infty, -4/3) \cup (2, \infty)$.

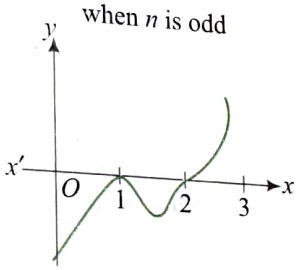
14. **(1), (3), (4)**

Graph of $f(x)$

when n is even

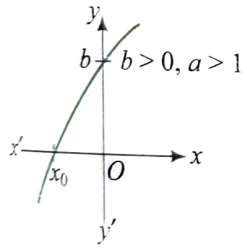


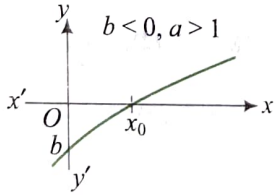
when n is odd

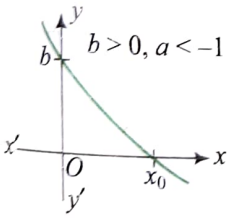


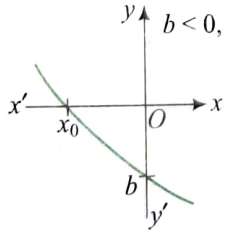
15. **(1), (2), (3)**

$f'(x) = \cos x + a$









If $a > 1$, then $f'(x) > 0$ or $f(x)$ is an increasing function. Then $f(x) = 0$ has +ve root if $b < 0$ and –ve root if $b > 0$.

$f'(x) = \cos x + a$

If $a < -1$, then $f'(x) < 0$ or $f(x)$ is a decreasing function. Then $f(x) = 0$ has negative root if $b < 0$.

16. **(1), (2), (3)**

$$f(x) = \frac{\sin(x+a)}{\sin(x+b)}$$

$$f'(x) = \frac{\sin(x+b)\cos(x+a) - \sin(x+a)\cos(x+b)}{\sin^2(x+b)}$$

$$= \frac{\sin(b-a)}{\sin^2(x+b)}$$

If $\sin(b-a) = 0$, then $f'(x) = 0$ or $f(x)$ will be a constant, i.e., $b - a = n\pi$ or $b - a = (2n+1)\pi$ or $b - a = 2n\pi$.

Then $f(x)$ has no minima.

17. **(3), (4)**

$f(x) = ax^4 + cx^2 + dx + e$

$\Rightarrow f''(x) = 12ax^2 + 2c = 0$

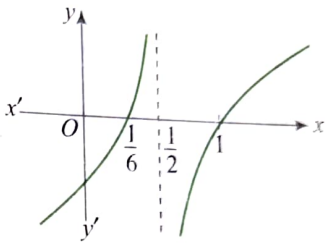
$\Rightarrow x^2 = -\frac{c}{6a}$

Since the function has no point of inflection, above equation has no real roots.

$\Rightarrow c/a > 0$

18. **(1), (2), (4)**

$$f'(x) = \frac{12x^2 - 12x + 5}{(2x-1)^2} > 0 \ \forall\ x \in R$$



Hence, f is increasing $\forall\ x \in R$.

$x = 1/2$ is the point of inflection as concavity changes at $x = 1/2$.

19. **(1), (2), (4)**

At the point of inflection, concavity of the curve changes irrespective of any other factor.

20. **(2), (3), (4)**

Since f is defined on $(0, \infty)$, $2a^2 - a - 1 > 0$ which is true as $D < 0$.

Also, $3a^2 - 4a + 1 > 0$

$(3a-1)(a-1) > 0$, i.e., $a < 1/3$ or $a > 1$ (1)

As f is increasing,

$f(2a^2 + a + 1) > f(3a^2 - 4a + 1)$

or $2a^2 + a + 1 > 3a^2 - 4a + 1$

or $0 > a^2 - 5a$

or $a(a-5) < 0$ or $a \in (0, 5)$ (2)

From (1) and (2), we get

$a \in (0, 1/3) \cup (1, 5)$.

Therefore, possible integers are $\{2, 3, 4\}$.

21. **(1), (4)**

$f(x) = (\sin^2 x - 1)^n$

$$f\left(\frac{\pi}{2}\right) = 0$$

$$f\left(\frac{\pi^+}{2}\right) = (\to 0^+)^n \text{ and } f\left(\frac{\pi^-}{2}\right) = (\to 0^-)^n$$

If n is even, $f\left(\frac{\pi^+}{2}\right)$ and $f\left(\frac{\pi^-}{2}\right) > 0$. Then $x = \frac{\pi}{2}$ is the point of minima.

If n is odd, $f\left(\frac{\pi^+}{2}\right)$ and $f\left(\frac{\pi^-}{2}\right) < 0$. Then $x = \frac{\pi}{2}$ is the point of maxima.

22. (1), (2), (4)

$f(x) = 2x^3 + 9x^2 + 12x + 1$

$\therefore \quad f'(x) = 6[x^2 + 3x + 2]$

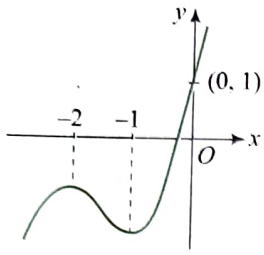
$= 6(x + 2)(x + 1)$

$f'(x) < 0$ for $x \in (-2, -1)$, where $f(x)$ decreases.

$f'(x) > 0$ for $x \in (-\infty, -2) \cup (-1, \infty)$, where $f(x)$ increases.

$f''(x) = 6(2x + 3) = 0$

Thus, $x = -3/2$ is the point of inflection.



From the graph, f is many-one. Hence, it is not bijective.

23. (1), (2), (3)

$f(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x$

Thus, $f(x) = 0$ has one root, $x = 0$.

Also, given that $f(x) = 0$ has positive root a_0.

Thus, the equation must have at least three real roots (as complex root occurs in conjugate pair). Thus, $f'(x) = 0$ has at least two real roots as between two roots of $f(x) = 0$, there lies at least one root of $f'(x) = 0$.

Similarly, we can say that $f''(x) = 0$ has at least one real root.

Further, $f'(x) = 0$ has one root between roots $x = 0$ and $x = a_0$ of $f(x) = 0$.

24. (1), (2), (3)

Let $y = f(x)^{g(x)}$

$$\therefore \quad \frac{dy}{dx} = f(x)^{g(x)}\left[g(x)\frac{f'(x)}{f(x)} + g'(x)\log f(x)\right]$$

$f(x)^{g(x)}$, $g(x)$, $f(x)$, $f'(x)$ and $g'(x)$, are positive, but $\log f(x)$ can be negative, which can cause $\frac{dy}{dx} < 0$. Hence, statement (a) is false.

If $f(x) < 1$, then $\log f(x) < 0$, which does not necessarily make $\frac{dy}{dx} < 0$. Hence, statement (b) is false.

$f(x) < 0$ can also cause $\frac{dy}{dx} > 0$. Hence statement (c) is false.

But reverse of (c) is true.

25. (1), (3)

$$f(x) = \frac{2-x}{\pi}\cos\pi(x+3) + \frac{1}{\pi^2}\sin\pi(x+3)$$

$$f'(x) = -\frac{1}{\pi}\cos\pi(x+3) - (2-x)\sin\pi(x+3) + \frac{1}{\pi}\cos\pi(x+3)$$

$$= (x-2)\sin\pi(x+3) = 0 \text{ (Let)}$$

$\therefore \quad x = 2, 1, 3$

$f''(x) = \sin\pi(x+3) + \pi(x-2)\cos\pi(x+3)$

$f''(1) = -\pi < 0, f''(2) = 0, f''(3) = \pi > 0$

Therefore, $x = 1$ is a maximum and $x = 3$ is a minimum. Hence, $x = 2$ is the point of inflection.

26. (1), (2), (3), (4)

$y = f(x) = x^4(12\log_e x - 7); x > 0$

$$\therefore \quad \frac{dy}{dx} = 16x^3(3\log_e x - 1) \text{ and } \frac{d^2y}{dx^2} = 16x^2(9\log_e x)$$

$$\frac{dy}{dx} = 0 \Rightarrow x = e^{1/3}$$

At $x = e^{1/3}$, $\frac{d^2y}{dx^2} > 0$. Hence, $x = e^{1/3}$ is point of minima.

Also, for $0 < x < 1$, $\frac{d^2y}{dx^2} < 0$ and for $x > 1$, $\frac{d^2y}{dx^2} > 0$.

Hence $x = 1$ is point of inflection, and for $0 < x < 1$, graph is concave downward and for $x > 1$, graph is concave upward.

27. (1), (2), (3)

$$f(x) = \log(2x - x^2) + \sin\frac{\pi x}{2}$$

$$= \log(1 - (x-1)^2) + \sin\frac{\pi x}{2}$$

$$f(1-x) = \log(1 - (1-x-1)^2) + \sin\frac{\pi(1-x)}{2}$$

$$= \log(1 - x^2) + \cos\frac{\pi x}{2}$$

$$\text{Also, } f(1+x) = \log(1 - (x+1-1)^2) + \sin\frac{\pi(1+x)}{2}$$

$$= \log(1 - x^2) + \cos\frac{\pi x}{2}$$

Hence, function is symmetrical about line $x = 1$.

Also, $f(1) = 1$.

Also, for domain of the function $2x - x^2 > 0$ or $x \in (0, 2)$.

For $x > 1$, $f(x)$ decreases. Hence, $x = 1$ is point of maxima.

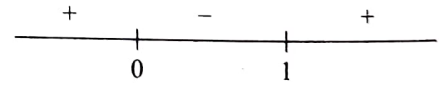
Also, maximum value of the function is 1.

Further, $f(x) \to \infty$, when $x \to 2$. Hence, absolute minimum value of f does not exist.

28. (1), (2), (3), (4)

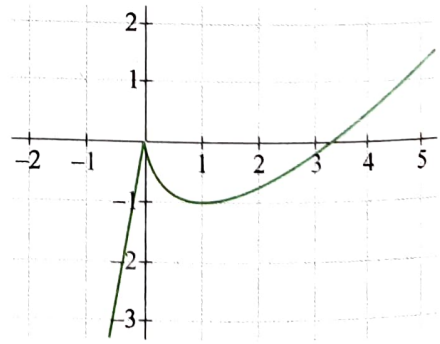
$$f'(x) = 2 - 2x^{-1/3} = 2\left(1 - \frac{1}{x^{1/3}}\right) = 2\left(\frac{x^{1/3} - 1}{x^{1/3}}\right)$$

Sign scheme of derivative is as follows:



$f(x)$ has point of maxima at $x = 0$ and point of minima at $x = 1$.

Also, $f(x)$ is non-differentiable at $x = 0$.



29. (1), (2), (3)

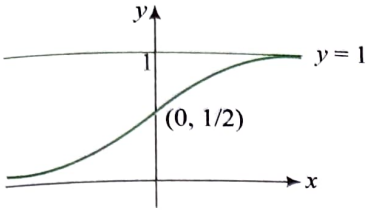
$$f(x) = \frac{e^x}{1+e^x}$$

$$\therefore \quad f'(x) = \frac{e^x(1+e^x) - e^xe^x}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2} > 0 \ \forall x \in R$$

Thus, $f(x)$ is an increasing function.

Also, $\lim_{x\to-\infty}\frac{e^x}{1+e^x}=0$ and $\lim_{x\to\infty}\frac{e^x}{1+e^x}=\lim_{x\to\infty}\frac{1}{1+\frac{1}{e^x}}=1$

Hence, the graph of $f(x)=\frac{e^x}{1+e^x}$ is as shown.



Also, let $f''(x)=\frac{e^x(1+e^x)^2-2(1+e^x)e^xe^x}{(1+e^x)^4}=0$

or $(1+e^x)-2e^x=0$

or $e^x=1$

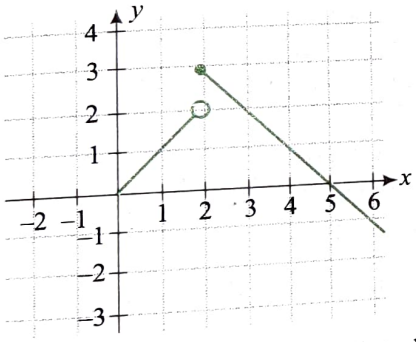
or $x=0$, which is point of inflection

Thus, $x=0$ is the inflection point and f is bounded in $(0, 1)$.

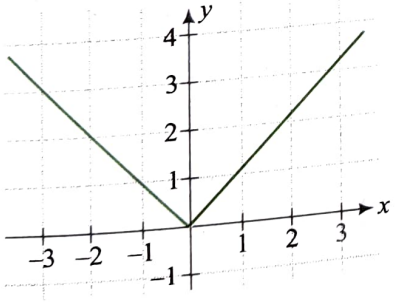
There is no maxima and f has two asymptotes.

30. **(1), (2), (3)**

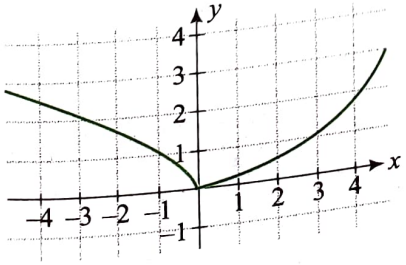
The following function is discontinuous at $x=2$, but has point of maxima.



$f(x)=|x|$ has point of minima at $x=0$, though it is non-differentiable at $x=0$.

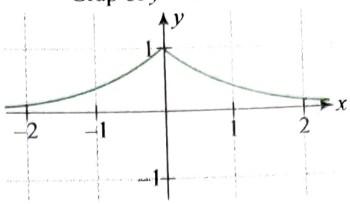


In the following figure, $y=f(x)$ has point of inflection at $x=0$, as curve changes its concavity at $x=0$. However, $x=0$ is point of minima for the function.

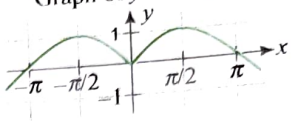


31. **(1), (2), (4)**

Grap of $y=e^{|x|}$

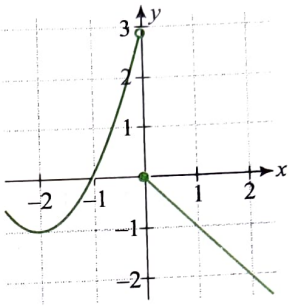


Graph of $y=\sin|x|$



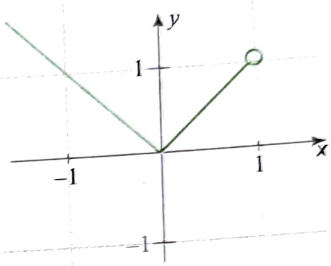
Graph of

$$f(x)=\begin{cases}x^2+4x+3, & x<0\\ -x, & x\geq 0\end{cases}$$



Graph of

$$f(x)=\begin{cases}|x|, & x<0\\ \{x\}, & x\geq 0\end{cases}$$



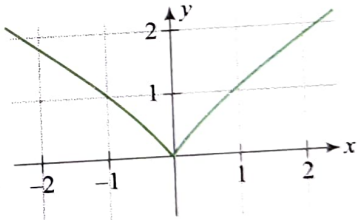
32. **(3), (4)**

$f(x)=x^{6/7}$

or $f''(x)=-\frac{6}{7}\frac{13}{7}x^{\frac{-8}{7}}$,

Here, $f''(x)$ does not change sign. Hence, it has no point of inflection.

Graph of $f(x)=x^{6/7}$



For $f(x)=x^6, f''(x)=30x^4$, but $f'(x)$ does not change sign in the neighborhood of $x=0$.

$f(x)=\cos x+2x$

or $f''(x)=-\cos x$

or $f''(x)=0$ for $x=(2n+1)\pi/2, n\in Z$

Also, sign of $f''(x)$ changes sign in the neighborhood of $(2n+1)\pi/2$. Hence, function has infinite points of inflection.

$$f(x)=x|x|=\begin{cases}-x^2, & x<0\\ x^2, & x\geq 0\end{cases}$$

or $$f''(x)=\begin{cases}-2, & x<0\\ 2, & x>0\end{cases},$$

Here, $f'(x)$ changes sign in the neighborhood of $x=0$. Hence, has point of inflection.

33. **(1), (3), (4)**

$$f'(x)=2x-\frac{\lambda}{x^2}, f'(x)=0$$

$$\Rightarrow\quad x=\left(\frac{\lambda}{2}\right)^{1/3}$$

If $\lambda=16, x=2.$

Now, $f''(x) = 2 + \frac{2\lambda}{x^3}$

Thus, if $\lambda = 16$, $f''(x) > 0$, i.e., $f(x)$ has a minimum at $x = 2$.

Also, $f''\left\{\left(\frac{\lambda}{2}\right)^{1/3}\right\} = 2 + \frac{2\lambda}{\lambda/2} = 2 + 4 > 0$

Hence, $f(x)$ has maximum for no real value of λ.

When $\lambda = -1$, $f'(x) = 0$ if $x = 1$. So, $f(x)$ has a point of inflection at $x = 1$.

34. (1), (2), (3), (4)

$$f(x) = x^{1/3}(x - 1)$$

or $$\frac{df(x)}{dx} = \frac{4}{3}x^{1/3} - \frac{1}{3}\cdot\frac{1}{x^{2/3}} = \frac{1}{3x^{2/3}}[4x - 1]$$

Thus, $f'(x)$ changes sign from –ve to +ve, at $x = 1/4$, which is point of minima.

Also, $f'(x)$ does not exist at $x = 0$ as $f(x)$ has vertical tangent at $x = 0$.

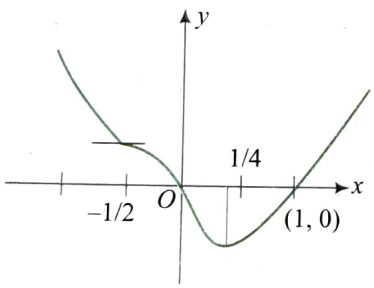
$$f''(x) = \frac{4}{9}\cdot\frac{1}{x^{2/3}} + \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{1}{x^{5/3}} = \frac{2}{9x^{2/3}}\left[2 + \frac{1}{x}\right]$$

$$= \frac{2}{9x^{2/3}}\left[\frac{2x+1}{x}\right]$$

Thus, $f''(x) = 0$ at $x = -\frac{1}{2}$ which is the point of inflection.

At $x = 0$, $f''(x)$ does not exist but $f''(x)$ changes sign. Hence, $x = 0$ is also the point of inflection.

From the above information, the graph of $y = f(x)$ is as shown.



Also, minimum value of $f(x)$ is at $x = 1/4$ which is $-3 \times 2^{-8/3}$.

Hence, range is $[-3 \times 2^{-8/3}, \infty)$.

35. (1), (4)

$$f(x) = \cos x - \left(1 - \frac{x^2}{2}\right)$$

$\therefore$ $f'(x) = -\sin x + x$

$\sin x < x$ if $x > 0$ and $\sin x > x$ if $x < 0$.

36. (1), (2), (3)

$f(-x) = -x\cos(-x) - \sin(-x) = -x\cos x + \sin x = -f(x)$.

$\therefore$ $f(x)$ is odd function.

$f'(x) = -x\sin x + \cos x - \cos x = -x\sin x$

$f'(x) = 0 \Rightarrow x = n\pi, n \in Z$.

$f''(x) = -x\cos x - \sin x$

$f''(0) = 0$

$\therefore$ $x = 0$ is point of inflection.

$$\left.\begin{aligned} f'(0^-) &= (-)(-)(-) < 0 \\ f'(0^+) &= (-)(+)(+) < 0 \end{aligned}\right\}$$

Thus, at $x = 0$ $f(x)$ decreases.

$f''(\pi) = \pi > 0$

$x = \pi$ is point of minima.

37. (1), (4)

$$f(x) = \frac{x^2}{2 - 2\cos x} = \frac{x^2}{4}\operatorname{cosec}^2\frac{x}{2}$$

$$\Rightarrow \quad f'(x) = \frac{x}{2}\operatorname{cosec}^2\frac{x}{2}\cot\frac{x}{2}\left(\tan\frac{x}{2} - \frac{x}{2}\right)$$

For $0 < x < 1$, $\tan\frac{x}{2} > \frac{x}{2}$

$\Rightarrow$ $f'(x) > 0$

So, $f(x)$ is increasing.

$$g(x) = \frac{x^2}{6x - 6\sin x}$$

$$\Rightarrow \quad g'(x) = \frac{1}{6}\frac{2x(x - \sin x) - x^2(1 - \cos x)}{(x - \sin x)^2}$$

$$= \frac{x}{6}\frac{x - 2\sin x + x\cos x}{(x - \sin x)^2}$$

Let $h(x) = x - 2\sin x + x\cos x$

$h'(x) = 1 - 2\cos x + \cos x - x\sin x$

$= 1 - \cos x - x\sin x$

$= 2\sin^2\frac{x}{2} - 2x\sin\frac{x}{2}\cos\frac{x}{2}$

$= 2\sin\frac{x}{2}\cos\frac{x}{2}\left(\tan\frac{x}{2} - x\right)$

For $0 < x < 1$, $\tan\frac{x}{2} < x$

$\therefore$ $h'(x) < 0$

So, $h(x)$ is decreasing.

For $x > 0$, $h(x) < h(0)$

$\therefore$ $h(x) < 0$

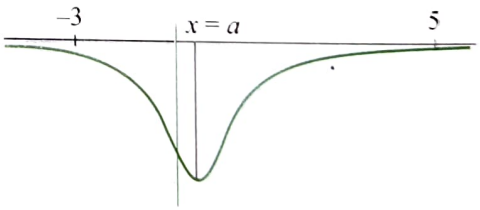
$\therefore$ $g'(x) < 0$

So, $g(x)$ is decreasing.

38. (1), (2), (3), (4)

$$f(x) = \frac{1}{2ax - x^2 - 5a^2} = \frac{1}{-4a^2 - (x - a)^2}$$

Clearly $f(x)$ is continuous $\forall\ x \in R$.



The graph is symmetrical about the line $x = a$.

If $a = 1$ (mid point of $x = -3$ and $x = 5$), greatest value is $f(5) = f(-3)$

If $a < 1$, $f_{max}(x) = f(5) = \frac{-1}{5(a^2 - 2a + 5)}$

and if $a > 1$, $f_{max}(x) = f(-3) = \frac{-1}{5a^2 + 6a + 9}$

39. (1), (2)

Let $f(x) = \frac{\sin x}{x}$

$$\Rightarrow \quad f'(x) = \frac{x\cos x - \sin x}{x^2} = \frac{\cos x(x - \tan x)}{x^2} < 0, x \in \left(0, \frac{\pi}{2}\right);$$

(as $\tan x > x$)

$$\Rightarrow \quad f''(x) = \frac{-x^2\sin x - 2x\cos x + 2\sin x}{x^3}$$

Let $g(x) = -x^2\sin x - 2x\cos x + 2\sin x$

$\Rightarrow$ $g'(x) = -x^2\cos x < 0 \ \forall\ x \in (0, \pi/2)$

$\Rightarrow$ for $x > 0$, we have $g(x) < g(0)$, i.e., $g(x) < 0$

$\Rightarrow$ $f'(x) < 0$ and $f''(x) < 0, x \in \left(0, \dfrac{\pi}{2}\right)$

$\Rightarrow$ $f\left(\dfrac{A+B+C}{3}\right) > \dfrac{f(A)+f(B)+f(C)}{3}$

$\Rightarrow$ $\dfrac{\sin\left(\dfrac{A+B+C}{3}\right)}{\dfrac{A+B+C}{3}} \geq \dfrac{\dfrac{\sin A}{A}+\dfrac{\sin B}{B}+\dfrac{\sin C}{C}}{3}$

$\Rightarrow$ $\dfrac{\sin A}{A}+\dfrac{\sin B}{B}+\dfrac{\sin C}{C} \leq \dfrac{9\sqrt{3}}{2\pi} \cong 2.5$

40. (2), (3)

Let $G(x) = e^x \cos x\, f(x)$

$\therefore$ $G'(x) = e^x\cos x\, f'(x) + e^x(\cos x - \sin x)\, f(x)$

$= e^x(\cos x\, f'(x) + (\cos x - \sin x)\, f(x))$

≤ 0 (as given that $\cos x \cdot f'(x) \leq (\sin x - \cos x)\, f(x)$)

$\therefore$ $G(x)$ is a non increasing function.

$\therefore$ $G(6) = e^6 \cos 6\, f(6) \leq 0$

$\therefore$ $f(6) \leq 0$

But given that $f(x)$ is non-negative

$\therefore$ $f(6) = 0$

With similar reasons $f(5), f(7) = 0$

Thus $x^2 - 3x + 2 + f(7) = 0$

or $x^2 - 3x + 2 = 0$ has 2 distinct solutions

$$\lim_{x\to 6} \frac{f(x) - \sin(\pi x)}{(x-6)}$$

$$= \lim_{x\to 6} \frac{f'(x) - \pi\cos(\pi x)}{1}$$ (Applying L'Hospital Rule)

$= f'(6) - \pi < 0$ (as $f'(6) \leq 0$)

Linked Comprehension Type

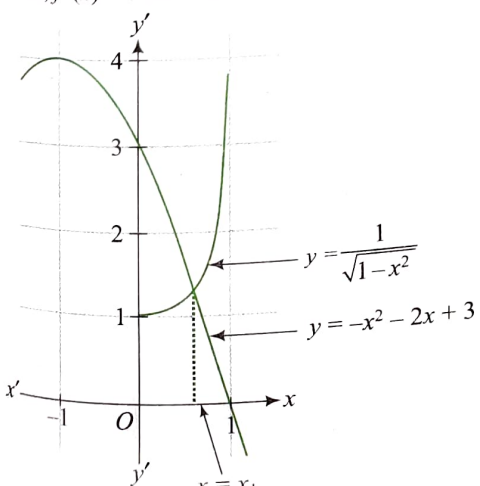
For Problems 1 and 2

1. (2), 2. (1)

Let $f(x) = \sin^{-1} x + x^2 - 3x + \dfrac{x^3}{3}$

$\therefore$ $f'(x) = \dfrac{1}{\sqrt{1-x^2}} + 2x - 3 + x^2$

Thus, $f'(x) = 0$ for some $x = x_1 \in (0, 1)$.



$f''(x) = \dfrac{x}{(1-x^2)^{3/2}} + 2 + 2x > 0 \ \forall\ x \in (0, 1)$

Thus, $x = x_1$ is the point of minimum.

$f(x)$ is continuous $\forall\ x \in [0, 1]$.

Hence, the global maxima exists at $x = 0$ or $x = 1$.

$f(0) = 0, f(1) = \pi/2 - 5/3 < 0$

$f(0)$ is global maxima $\forall\ x \in [0, 1]$. Thus,

$f(x) \leq f(0), x \in [0, 1]$ or $\sin^{-1} x + x^2 - 3x + x^3/3 \leq 0$

or $\sin^{-1} x + x^2 \leq \dfrac{x(9-x^2)}{3} \ \forall\ x \in [0, 1]$

For Problems 3 and 4

3. (1), 4. (4)

$g'(x) = f'(\sin x)\cos x - f'(\cos x)\sin x$

or $g''(x) = -f'(\sin x)\sin x + \cos^2 x\, f''(\sin x)$

$+ f''(\cos x)\sin^2 x - f'(\cos x)\cos x > 0 \ \forall\ x \in (0, \pi/2)$

[as it is given $f'(\sin x) = f'(\cos x\,(\pi/2 - x)) < 0$

and $f''(\sin x) = f''(\cos x(\pi/2 - x)) > 0$]

Thus, $g'(x)$ is increasing in $(0, \pi/2)$. Also, $g'(\pi/4) = 0$

or $g'(x) > 0 \ \forall\ x \in \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

and $g'(x) < 0 \ \forall\ x \in (0, \pi/4)$.

Thus, $g(x)$ is decreasing in $(0, \pi/4)$.

For Problems 5–8

5. (4), 6. (1), 7. (4), 8. (3)

If $f(x)$ is continuous, then $f(3^-) = f(3^-)$

or $-9 + 12 + a = 3a + b$ or $2a + b = 3$ (1)

or $-b + 6$ or $2a + b = 3$ (2)

Also, $f(4^-) = f(4^+)$ or $4a + b =$

Thus, $f(x)$ is continuous for infinite values of a and b. Also,

$$f'(x) = \begin{cases} -2x + 4, & x < 3 \\ a, & 3 < x < 4 \\ -\dfrac{b}{4}, & x > 4 \end{cases}.$$

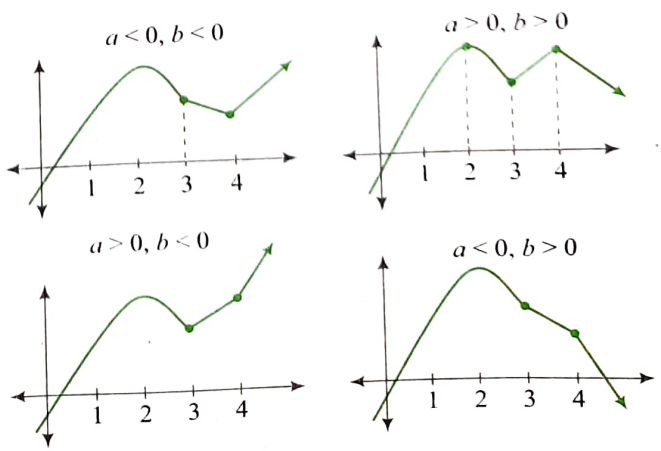
For $f(x)$ to be differentiable,

$f'(3^-) = f(3^+)$

or $a = -2$ and $-\dfrac{b}{4} = a = -2$ or $b = 8$

But these values do not satisfy equation (1).

Hence, $f(x)$ cannot be differentiable.



For Problems 9 and 10

9. (1) $\dfrac{dP(x)}{dx} > P(x)$

or $e^{-x}\dfrac{dP(x)}{dx} - e^{-x}P(x) > 0$

or $\dfrac{d}{dx}\left(P(x)e^{-x}\right) > 0$

Thus, $P(x)\,e^{-x}$ is an increasing function, i.e.,

$P(x)\,e^{-x} > P(1)\,e^{-1}\ \forall\ x \geq 1$

or $P(x)\,e^{-x} > 0\ \forall\ x > 1$ or $P(x) > 0\ \forall\ x > 1$

10. (2) Given that $\dfrac{d}{dx}H(x) > 2cxH(x)$

or $e^{-cx^2}\dfrac{d}{dx}H(x) - e^{-cx^2}\,2cxH(x) > 0$

or $\dfrac{d}{dx}(H(x)e^{-cx^2}) > 0$

Thus, $H(x)\,e^{-cx^2}$ is an increasing function.

But $H(x_0) = 0$ and e^{-cx^2} is always positive.

Thus, $H(x) > 0$ for all $x > x_0$

Hence, $H(x)$ cannot be zero for any $x > x_0$.

For Problems 11–13

11. (1) $h(x) = f(x) - a(f(x))^2 + a(f(x))^3$

or $h'(x) = f'(x) - 2af(x)f'(x) + 3a(f(x))^2 f'(x)$

$= f'(x)[3a(f(x))^2 - 2af(x) + 1]$

Now, $h(x)$ increases if $f(x)$ increases and

$3a(f(x))^2 - 2af(x) + 1 > 0$ for all $x \in R$

or $3a > 0$ and $4a^2 - 12a \leq 0$

or $a > 0$ and $a \in [0, 3]$

or $a \in [0, 3]$

12. (4) $h(x)$ increases as $f(x)$ decreases for all real values of x if $3a(f(x))^2 - 2af(x) + 1 \leq 0$ for all $x \in R$

or $3a < 0$ and $4a^2 - 12a \leq 0$

or $a < 0$ and $a \in [0, 3]$

So, no such a is possible.

13. (4) $h(x)$ is non-monotonic function if $3a(f(x))^2 - 2af(x) + 1$ changes sign for which $D > 0$ or $4a^2 - 12a > 0$, i.e.,

$a \in (-\infty, 0) \cup (3, \infty)$

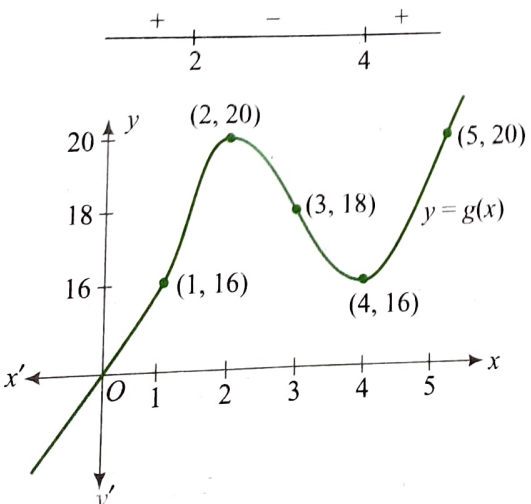
For Problems 14–16

14. (1), 15. (3), 16. (2)

Let $g(x) = x^3 - 9x^2 + 24x = x(x^2 - 9x + 24)$

$\therefore\quad g'(x) = 3(x-2)(x-4)$

Sign scheme of $g(x)$:



For three real roots of $f(x) = x^3 - 9x^2 + 24x + c = 0$, c must lie in the interval $(-20, -16)$.

$f(0) = c < 0$

$f(1) = 1 - 9 + 24 + c = c + 16 < 0$ for $\forall\ c \in (-20, -16)$

$f(2) = 8 - 36 + 48 + c = c + 20 > 0$

$\alpha \in (1, 2)$ or $[\alpha] = 1$

$f(3) = 27 - 81 + 72 + c = 18 + c$

$\therefore\quad f(3) < 0$ if $c \in (-20, -18)$ or $f(3) > 0$ if $c \in (-18, -16)$

or $\beta \in (2, 3)$ if $c \in (-20, -18)$

and $\beta \in (3, 4)$ if $c \in (-18, -16)$

Now, $f(4) = 64 - 144 + 96 + c = 16 + c < 0\ \forall\ c \in (-20, -16)$

$f(5) = 125 - 225 + 120 + c = c + 20 > 0\ \forall\ c \in (-20, -16)$

or $\gamma \in (4, 5)$ or $[\gamma] = 4$

Thus, $[\alpha] + [\beta] + [\gamma] = \begin{cases} 1+2+4, & -20 < c < -18 \\ 1+3+4, & -18 < c < -16 \end{cases}$

Now, if $c \in (-20, -18)$,

$\alpha \in (1, 2)$, $\beta \in (2, 3)$, $\gamma \in (4, 5)$

or $[\alpha] + [\beta] + [\gamma] = 7$

If $c \in (-18, -16)$, $\alpha \in (1, 2)$, $\beta \in (3, 4)$, $\gamma \in (4, 5)$, then

$[\alpha] + [\beta] + [\gamma] = 8$

For Problems 17–21

17. (2) $f'(x) \leq 0\ \forall\ x \in [a, b]$. So, $f(x)$ is a decreasing function and $f(c) = 0$. Thus, $f(x)$ cuts x-axis once when $x = c$.

18. (4) We note that $f(c) = 0$, $f'(c) = 0$. Also, tangent to $f'(x)$ at $x = c$ is $y = 0$. So, $f''(c) = 0$.

Thus, $x = c$ is the repeated root of third order. That is, the equation $f(x) = 0$ has at least three repeated roots.

19. (2) We have $f''(c) = 0$. So, the graph of $y = f(x)$ has one point of inflection at $x = c$.

20. (4) As $f(x)$ is a decreasing function for all $x \in (a, b)$, $f(x)$ has no local maxima or minima.

21. (2) $f''(c) = 0$. Thus, $x = c$ is a root of $f''(x) = 0$.

For Problems 22–24

22. (2), 23. (4), 24. (1)

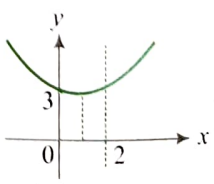
$f(x) = 4x^2 - 4ax + a^2 - 2a + 2$.

Vertex of this parabola is $\left(\dfrac{a}{2}, 2 - 2a\right)$.

Case I: $0 < \dfrac{a}{2} < 2$

In this case, $f(x)$ will attain the minimum value at $x = \dfrac{a}{2}$. Thus,

$f\left(\dfrac{a}{2}\right) = 3$



or $3 = -2a + 2$ or $a = -\dfrac{1}{2}$ (Rejected)

Case II: $\dfrac{a}{2} \geq 2$

In this, $f(x)$ attains the global minimum value at $x = 2$. Thus, $f(2) = 3$

or $3 = 16 - 8a + a^2 - 2a + 2$ or $a = 5 \pm \sqrt{10}$

Thus, $a = 5 + \sqrt{10}$.

Case III: $\frac{a}{2} \le 0$

In this case, $f(x)$ attains the global minimum value at $x = 0$. Thus, $f(0) = 3$.

$\therefore \quad 3 = a^2 - 2a + 2$ or $a = 1 \pm \sqrt{2}$.

Thus, $a = 1 - \sqrt{2}$.

Hence, the permissible values of a are $1 - \sqrt{2}$ and $5 + \sqrt{10}$.

$f(x) = 4x^2 - 49x + a^2 - 2a + 2$ is monotonic in $[0, 2]$.

Hence, the point of minima of function should not lie in $[0, 2]$.

Now, $f'(x) = 0$ or $8x - 4a = 0$ or $x = a/2$.

$\frac{a}{2} \in [0, 2]$ or $a \in [0, 4]$.

For $f(x)$ to be monotonic in $[0, 2]$, $a \notin [0, 4]$, i.e., $a \le 0$ or $a \ge 4$.

For Problems 25–27

25. (1), 26. (2), 27. (2)

$f(x) = x^3 - 3(7 - a)x^2 - 3(9 - a^2)x + 2$

or $f'(x) = 3x^2 - 6(7 - a)x - 3(9 - a^2)$

For real root, $D \ge 0$

or $49 + a^2 - 14a + 9 - a^2 \ge 0$ or $a \le \frac{58}{14}$ (1)

When point of minima is negative, point of maxima is also negative.

Hence, equation $f'(x) = 3x^2 - 6(7 - a)x - 3(9 - a^2) = 0$ has both roots negative.

Sum of roots $= 2(7 - a) < 0$ or $a > 7$, which is not possible as from (1), $a \le \frac{58}{14}$.

When point of maxima is positive, point of minima is also positive.

Hence, equation $f'(x) = 3x^2 - 6(7 - a)x - 3(9 - a^2) = 0$ has both roots positive.

Sum of roots $= 2(7 - a) > 0$ or $a < 7$ (2)

Also, product of roots is positive or $-(9 - a^2) > 0$ or $a^2 > 9$ or $a \in (-\infty, -3) \cup (3, \infty)$. (3)

From (1), (2), and (3), $a \in (-\infty, -3) \cup (3, 58/14)$.

For points of extrema of opposite sign, equation (1) has roots of opposite sign.

Thus, $a \in (-3, 3)$.

For Problems 28–30

28. (3), 29. (4), 30. (4)

$$f(x) = \left(1 + \frac{1}{x}\right)^x,$$

$f(x)$ is defined if $1 + \frac{1}{x} > 0$ or $\frac{x+1}{x} > 0$

or $(-\infty, -1) \cup (0, \infty)$

$$\text{Now, } f'(x) = \left(1 + \frac{1}{x}\right)^x \left[\ln\left(1 + \frac{1}{x}\right) + \frac{x}{1 + \frac{1}{x}} \frac{-1}{x^2}\right]$$

$$= \left(1 + \frac{1}{x}\right)^x \left[\ln\left(1 + \frac{1}{x}\right) - \frac{1}{x+1}\right]$$

Now, $\left(1 + \frac{1}{x}\right)^x$ is always positive. Hence, the sign of $f'(x)$ depends on the sign of $\ln\left(1 + \frac{1}{x}\right) - \frac{1}{x+1}$.

Let $g(x) = \ln\left(1 + \frac{1}{x}\right) - \frac{1}{x+1}$

$$g'(x) = \frac{1}{1 + \frac{1}{x}} \frac{-1}{x^2} + \frac{1}{(x+1)^2} = \frac{-1}{x(x+1)^2}$$

(i) For $x \in (0, \infty)$, $g'(x) < 0$

Thus, $g(x)$ is monotonically decreasing for $x \in (0, \infty)$

or $g(x) > \lim_{x \to \infty} g(x)$

or $g(x) > 0$, So, $f'(x) > 0$

(ii) for $x \in (-\infty, -1)$, $g'(x) > 0$

Thus, $g(x)$ is monotonically increasing for $x \in (-\infty, -1)$

or $g(x) > \lim_{x \to -\infty} g(x) > 0$

$\therefore \quad f'(x) > 0$

Hence, from (i) and (ii), we get

$f'(x) > 0 \ \forall \ x \in (-\infty, -1) \cup (0, \infty)$.

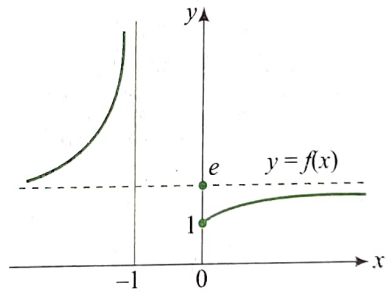
Thus, $f(x)$ is monotonically increasing in its domain.

Also,

$$\lim_{x \to \pm\infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$\lim_{x \to 0^+} \left(1 + \frac{1}{x}\right)^x = 1 \text{ and } \lim_{x \to -1} \left(1 + \frac{1}{x}\right)^x = \infty$$

The graph of $f(x)$ is shown in Figure.



Range is $y \in (1, \infty) - \{e\}$.

For Problems 31–33

31. (4), 32. (4), 33. (2)

$f(x) = x + \cos x - a$ or $f'(x) = 1 - \sin x \ge 0 \ \forall \ x \in R$.

Thus, $f(x)$ is increasing in $(-\infty, \infty)$, as for $f'(x) = 0$, x is not forming an interval. Also,

$f''(x) = -\cos x = 0$

or $x = (2n + 1)\frac{\pi}{2}, n \in Z$

Hence, there are infinite points of infletion.

Now, $f(0) = 1 - a$.

For positive root, $1 - a < 0$ or $a > 1$. For negative root, $1 - a > 0$ or $a < 1$.

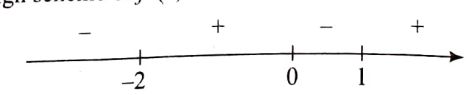
For Problems 34–36

34. (3), 35. (2), 36. (4)

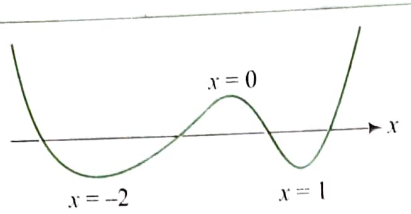
$f(x) = 3x^4 + 4x^3 - 12x^2$

$\therefore \quad f'(x) = 12(x^3 + x^2 - 2x) = 12x(x - 1)(x + 2)$

The sign scheme of $f'(x)$ is as follows.



The graph of the function is as follows.



Thus, we have

$f(-2) = -32$ and $f(1) = -5$

Hence, range of the function is $[-32, \infty)$.

Also, $f(x) = a$ has no real roots if $a < -32$.

For Problems 37–39

37. (4), 38. (4), 39. (2)

$$f(x) = \frac{x^2 - 6x + 4}{x^2 + 2x + 4} = 1 - \frac{8x}{x^2 + 2x + 4}$$

$$f'(x) = -8\left[\frac{(x^2 + 2x + 4) - x(2x + 2)}{(x^2 + 2x + 4)^2}\right]$$

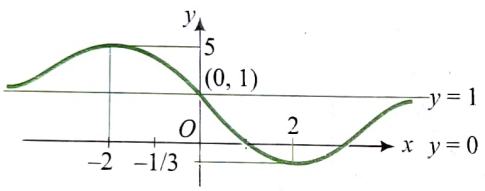
$$= -8\left[\frac{-x^2 + 4}{(x^2 + 2x + 4)^2}\right] = \frac{8(x^2 - 4)}{(x^2 + 2x + 4)^2}$$

$f'(x) = 0 \Rightarrow x = 2$ or -2

$$f(2) = \frac{4 - 12 + 4}{4 + 4 + 4} = \frac{-4}{12} = \frac{-1}{3}$$

$$f(-2) = \frac{4 + 12 + 4}{4 - 4 + 4} = 5$$

The graph of $y = f(x)$ is as shown.



Hence, $-\frac{1}{3} \le f(x) \le 5$

For Problems 40–42

40. (3), 41. (4), 42. (3)

Since two points of inflection occur at $x = 1$ and $x = 0$,

$P''(1) = P''(0) = 0$

$\therefore \quad P''(x) = a(x^2 - x)$

or $\quad P'(x) = a\left(\frac{x^3}{3} - \frac{x^2}{2}\right) + b$

Also, given $\left(\frac{dy}{dx}\right)_{x=0} = \sec^{-1}\sqrt{2} = \tan^{-1}1$

Hence, $P'(0) = 1$. So, $b = 1$. Thus,

$$P'(x) = a\left(\frac{x^3}{3} - \frac{x^2}{2}\right) + 1$$

$$\therefore \quad P(x) = a\left(\frac{x^4}{12} - \frac{x^3}{6}\right) + x + c$$

As $P(-1) = 1$, we have

$$a\left(\frac{1}{12} + \frac{1}{6}\right) - 1 + c = 1 \text{ or } \frac{a}{4} + c = 2 \qquad (1)$$

As $P(1) = 2$, we have

$$a\left(\frac{1}{12} - \frac{1}{6}\right) + 1 + c = 1$$

or $\quad -\frac{a}{12} + c = 0 \qquad (2)$

Solving (1) and (2), we have $a = 6$ and $c = \frac{1}{2}$. Thus,

$$P(x) = 6\left(\frac{x^4}{12} - \frac{x^3}{6}\right) + x + \frac{1}{2}$$

$$P(2) = \frac{5}{2} \text{ and } P(0) = \frac{1}{2}$$

$$P'(x) = 6\left(\frac{x^3}{3} - \frac{x^2}{2}\right) + 1 = (x - 1)^2(2x + 1)$$

For Problems 43–45

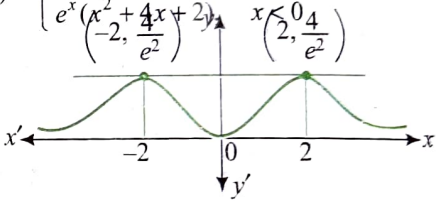
43. (3), 44. (4), 45. (4)

We have $f(x) = x^2 e^{-|x|} = \begin{cases} x^2 e^{-x}, & x \ge 0 \\ x^2 e^{x}, & x < 0 \end{cases}$

$$\therefore \quad f'(x) = \begin{cases} e^{-x}(2x - x^2), & x \ge 0 \\ e^{x}(x^2 + 2x), & x < 0 \end{cases}$$

$f(x)$ increases in $(-\infty, -2) \cup (0, 2)$ and $f(x)$ decreases in $(-2, 0) \cup (2, \infty)$. Thus,

$$f''(x) = \begin{cases} e^{-x}(x^2 - 4x + 2), & x \ge 0 \\ e^{x}(x^2 + 4x + 2), & x < 0 \end{cases}$$



$f''(x) = 0$ has four roots. Hence, there are four points of inflection.

For Problems 46 and 47

46. (3), 47. (1)

$p(x)$ is polynomial of degree '3'

Given $p'(x)$ has minima at $x = 1$

$\therefore \quad p''(x) = a(x - 1)$

$$\therefore \quad p'(x) = a\left(\frac{x^2}{2} - x\right) + b$$

Given $p(x)$ has maxima at $x = -1$

$$\therefore \quad p'(-1) = a\left(\frac{1}{2} + 1\right) + b = 0$$

$$\therefore \quad b = -\frac{3}{2}a$$

$$\therefore \quad p'(x) = a\left(\frac{x^2}{2} - x - \frac{3}{2}\right)$$

$$\therefore \quad p(x) = a\left(\frac{x^3}{6} - \frac{x^2}{2} - \frac{3}{2}x\right) + c$$

Given $p(-1) = 10$

$$\therefore \quad a\left(-\frac{1}{6} - \frac{1}{2} + \frac{3}{2}\right) + c = 10$$

$\therefore \quad 5a + 6c = 60 \qquad (1)$

Given $p(1) = -6$

$$\therefore \quad p(1) = a\left(\frac{1}{6} - \frac{1}{2} - \frac{3}{2}\right) + c = -6$$

$\therefore \quad -11a + 6c = -36 \quad (2)$

Solving (1) and (2), we get

$a = 6$ and $c = 5$

$\therefore \quad p(x) = x^3 - 3x^2 - 9x + 5$

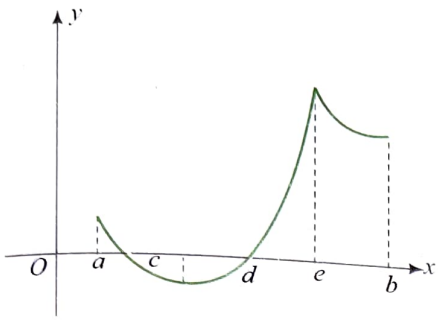
$\therefore \quad p(2) = 8 - 12 - 18 + 5 = -17$

$p'(x) = 3x^2 - 6x - 9$

$\therefore \quad p'(1) = 3 - 6 - 9 = -12$

For Problems 48–50

48. (3)



$f'(x)$ changes sign from positive to negative at $x = c$.

So $x = c$ is point maxima.

$f'(x)$ changes sign from negative to positive at $x = d$

So $x = d$ is point minima

49. (3) Clearly one point of inflection lies in the interval (a, d) where derivative of $f'(x)$ is zero.

Also $f''(x)$ does not exist at $x = e$, but $f''(x)$ changes sign in the neighbourhood of e, so, $x = e$ is another point of inflection.

50. (3) The graph of $y = f'(x)$ is always concave upward, so $f''(x) > 0$ at every point where it exists so graph of $y = f''(x)$ is always increasing whenever defined. Clearly $y = f''(x)$ is discontinuous at $x = e$.

Matrix Match Type

1. **a → r; b → s; c → q; d → p.**

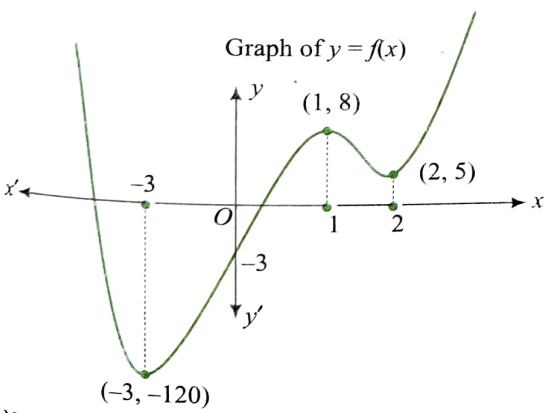
$f'(x) = 4x^3 - 28x + 24$

$= 4(x^3 - 7x + 6)$

$= 4(x^3 - x^2 + x^2 - x - 6x + 6)$

$= 4(x - 1)(x^2 + x - 6)$

$= 4(x - 1)(x + 3)(x - 2)$



Now, nature of roots of $f(x) + p = 0$ can be obtained by shifting the graph of $y = f(x)$ by p units upward or downward depending on whether p is positive or negative.

2. **a → p, s; b → p, s; c → q, r; d → q.**

a. $f(x) = x^2 \log x$

For $f'(x) = x(2 \log x + 1) = 0$, $x = \frac{1}{\sqrt{e}}$, which is the point of minima as derivative changes sign from negative to positive.

Also, the function decreases in $\left(0, \frac{1}{\sqrt{e}}\right)$.

b. $y = x \log x$

$\therefore \quad \frac{dy}{dx} = x \times \frac{1}{x} + \log x \times 1$

$= 1 + \log x$ and $\frac{d^2y}{dx^2} = \frac{1}{x}$

For $\frac{dy}{dx} = 0$, $\log x = -1$ or $x = \frac{1}{e}$

$\frac{d^2y}{dx^2} = \frac{1}{1/e} = e > 0$ at $x = \frac{1}{e}$

Thus, y is minimum for $x = \frac{1}{e}$

c. $f(x) = \frac{\log x}{x}$

For $f'(x) = \frac{1 - \log x}{x^2} = 0$, $x = e$. Also, derivative changes sign from positive to negative at $x = e$. Hence, it is the point of maxima.

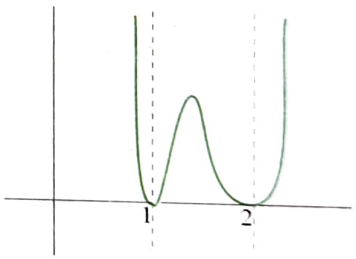
d. $f(x) = x^{-x}$

$f'(x) = -x^{-x}(1 + \log x) = 0$ or $x = 1/e$,

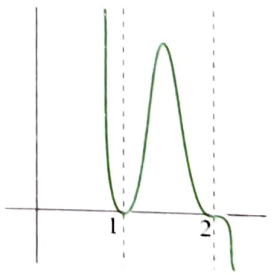
which is clearly point of maxima.

3. **a → p, r; b → p, s; c → q, r; d → q, s.**

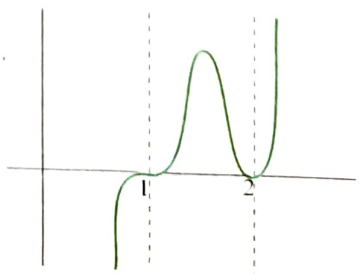
a. Both m and n are even



b. m is even and n is odd

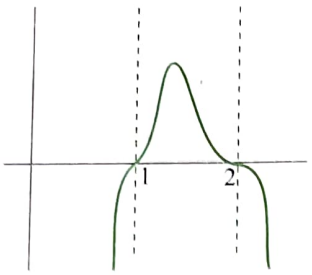


c. m is odd n is even



Both m and n are odd

d.



4. **a → s; b → p; c → q; d → r.**

Since $f(x)$ is minimum at $x = -2$ and maximum at $x = 2$, let $g(x) = ax^3 + bx^2 + cx + d$.

Thus, $g(x)$ is also minimum at $x = -2$ and maximum at $x = 2$.

Thus, $a < 0$

Since a is a root of $x^2 - x - 6 = 0$, i.e., $x = 3, -2$, we get $a = -2$

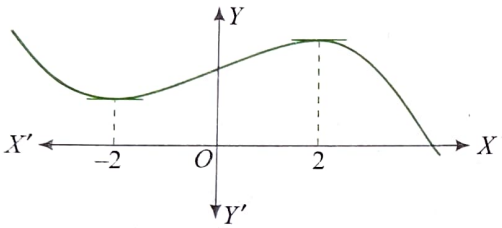
Then, $g(x) = -2x^3 + bx^2 + cx + d$

$\therefore g'(x) = -6x^2 + 2bx + c = -6(x + 2)(x - 2)$

[$\because g(x)$ is minimum at $x = -2$ and maximum at $x = 2$]

On comparing, we get

$b = 0$ and $c = 24$



Since minimum and maximum values are positive,

$g(-2) > 0 \Rightarrow 16 - 48 + d > 0 \Rightarrow d > 32$

and $g(2) > 0 \Rightarrow -16 + 48 + d > 0 \Rightarrow d > -32$

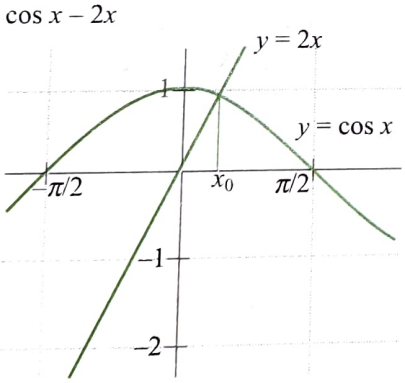
It is clear that $d > 32$.

Hence, $a = -2$, $b = 0$, $c = 24$, $d > 32$.

5. **a → q; b → p; c → s; d → r.**

$f(x) = \sin x - x^2 + 1$

$f'(x) = \cos x - 2x$



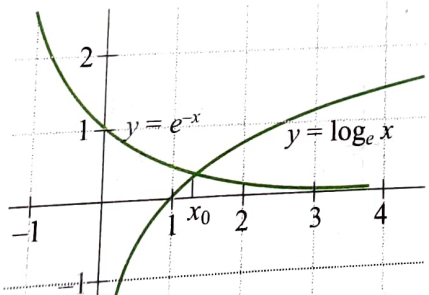
or $f'(x) < 0$ for $x > x_0$

and $f'(x) > 0$ for $x < x_0$

Hence, $x = x_0$ is the point of maxima.

b. p. $f(x) = x\log_e x - x + e^{-x}$

$f'(x) = \log_e x + 1 - 1 - e^{-x} = \log_e x - e^{-x}$



From the graph, for $x < x_0$, $e^{-x} > \log_e x$ or $f'(x) < 0$.

For $x > x_0$, $e^{-x} < \log_e x$ or $f'(x) > 0$.

Hence, $x = x_0$ is point of minima.

c. s $f(x) = -x^3 + 2x^2 - 3x + 1$

$f'(x) = -3x^2 + 4x - 3$

Now, $D = 16 - 4(-3)(-3) = -20 < 0$

Hence, $f'(x) < 0$, for all real x.

Thus, $f(x)$ is always decreasing.

d. r. $f(x) = \cos \pi x + 10x + 3x^2 + x^3$

or $f'(x) = -\pi \sin \pi x + 10 + 6x + 3x^2$

$= 3(x^2 + 2x + 10/3) - \pi \sin \pi x$

$= 3((x + 1)^2 + 7/3) - \pi \sin \pi x$

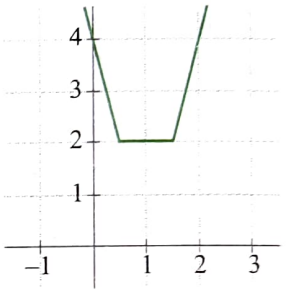
Now minimum value of $3((x + 1)^2 + 7/3)$ is 7 but maximum value of $\pi \sin \pi x$ is π.

Hence, $f'(x) > 0$ for all real x.

Hence, $f(x)$ is always increasing.

6. **a → s; b → s; c → r; d → q.**

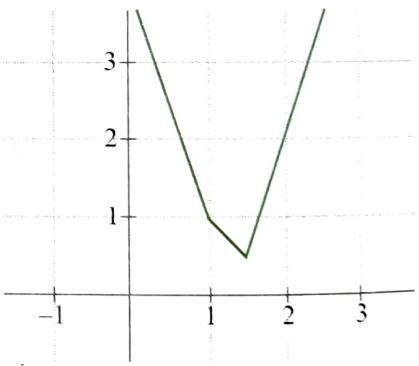
a. s. Graph of $f(x) = |2x - 1| + |2x - 3|$



From the graph, $f(x)$ has infinite points of minima.

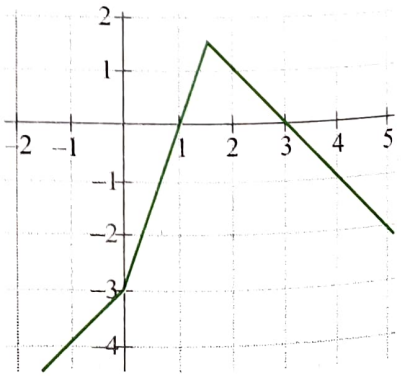
b. s. $f(x) = 2\sin x - x$. Thus, for $f'(x) = 2\cos x - 1 = 0$, we have $\cos x = 1/2$, which has infinite points of extrema.

c. r. Graph of $f(x) = |x - 1| + |2x - 3|$



From the graph, $f(x)$ has one point of minima.

d. q. Graph of $f(x) = |x| - |2x - 3|$



From the graph, $f(x)$ has one point of maxima.

7. **a → q, r; b → r, s; c → p, r; d → r, s.**

a. **q, r.**

$f(x) = (x-1)^3(x+2)^5$

or $f'(x) = 3(x-1)^2(x+2)^5 + 5(x-1)^3(x+2)^4$

$= (x-1)^2(x+2)^4[3(x+2) + 5(x-1)]$

$= (x-1)^2(x+2)^4[8x+1]$

Sign of derivative does not change at $x = 1$ and $x = -2$.

Sign of derivative changes at $x = -1/8$ from –ve to +ve.

Hence, function has point of minima.

Also, $f''(x) = 0$ for $x = 1$ and $x = -2$.

Hence, function has two points of inflection.

b. **r, s.**

$f(x) = 3\sin x + 4\cos x - 5x$

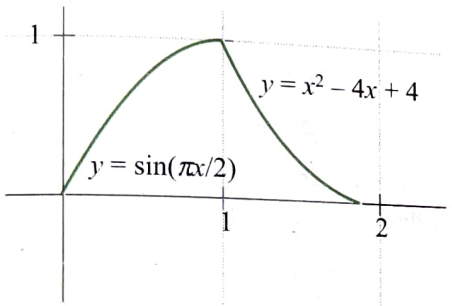
or $f'(x) = 3\cos x - 4\sin x - 5 \le 0$

Hence, $f(x)$ is decreasing function.

Also, $f''(x) = -3\sin x - 4\cos x = 0$ for infinite values of x.

Hence, function has infinite points of inflection.

c. **p, r**



From the graph, $x = 1$ is point of maxima as well as point of inflection.

d. **r, s.**

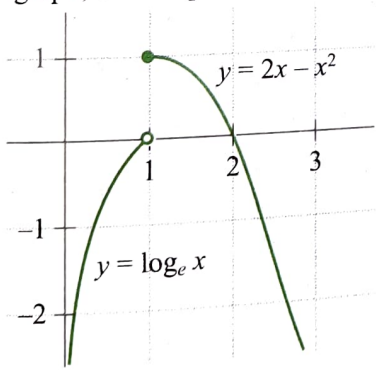
$f(x) = (x-1)^{3/5}$ or $f'(x) = \frac{3}{5}(x-1)^{-\frac{2}{5}} \ge 0$ for all real x.

Also, $f''(x) = -\frac{3}{5}\frac{2}{5}(x-1)^{-\frac{7}{5}}$ which changes sign at $x = 1$.

Hence, $x = 1$ is point of inflection.

8. **a → r; b → s; c → p; d → q.**

a. r. From the graph, $x = 1$ is point of maxima.



b. s. $f(x) = \begin{cases} x-1, & x<2 \\ 0, & x=2 \\ \sin x, & x>2 \end{cases}$

$f(2) = 0, f(2^+) = \sin(2^+) > 0$, and $f(2^-) > 0$

Hence, $x = 2$ is point of minima.

c. p. $f(x) = \begin{cases} 2x+3 & x<0 \\ 5, & x=0 \\ x^2+7, & x>0 \end{cases}$

$f(0^-) = 3, f(0) = 5, f(0^+) = 7$

Hence, $f(0^-) < f(0) < f(0^+)$.

Thus, $f(x)$ is increasing at $x = 0$.

d. q. $f(x) = \begin{cases} e^{-x} & x<0 \\ 0, & x=0 \\ -\cos x, & x>0 \end{cases}$

$f(0) = 0, f(0^+) = -1, f(0^-) = 1$

Thus, $f(0^-) > f(0) > f(0^+)$.

Hence, $f(x)$ is decreases at $x = 0$.

9. (2)

10. (4) Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6$

Given $\lim_{x\to 0}\left(1 + \frac{f(x)}{x^3}\right)^{1/x} = e^2$

$\therefore \quad \lim_{x\to 0}\frac{f(x)}{x^3} = 0$

or $a_0 = a_1 = a_2 = a_3 = 0$

$\therefore \quad \lim_{x\to 0} e^{(a_4 + a_5x + a_6x^2)} = e^2$

or $a_4 = 2$

$\therefore \quad f(x) = 2x^4 + a_5x^5 + a_6x^6$

$\therefore \quad f'(x) = x^3(8 + 5a_5x + 6a_6x^2)$

$x = 1$ and $x = 2$ are points of local maxima and local minima. Thus,

$f'(1) = 0$ and $f'(2) = 0$

$\therefore \quad 8 + 5a_5 + 6a_6 = 0$

and $4 + 5a_5 + 12a_6 = 0$

Solving, we get

$a_5 = -\frac{12}{5}, \quad a_6 = \frac{2}{3}$

$\therefore \quad f(x) = 2x^4 - \frac{12}{5}x^5 + \frac{2}{3}x^6$

Numerical Value Type

1. (9) Let $y = 2x\tan^{-1}x - \ln(1+x^2)$

$y' = 2\tan^{-1}x + \frac{2x}{1+x^2} - \frac{2x}{1+x^2}$

$\therefore \quad y' > 0 \ \forall\, x \in R^+, y' < 0 \ \forall\, x \in R^-$

or $y \ge 0 \ \forall\, x \in R$

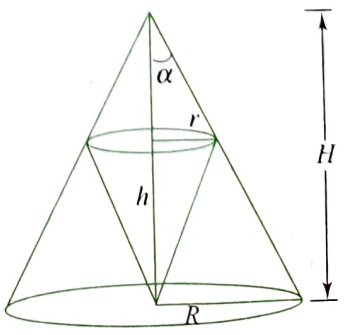
Therefore, $4 - |[x]|$ takes the values 0, 1, 2, 3, 4

$(\because |\alpha| \le 4 - |[x]|)$

$|\alpha| \le 4 - |[x]|$ is satisfied by $\alpha = 0, \pm1, \pm2, \pm3, \pm4$.

Therefore, number of values of α is 9.

2. (3)



$\frac{r}{R} = \frac{H-h}{H}$

$$r = \frac{R(H-h)}{H}$$

$$\text{Volume } (V) = \frac{1}{3}\pi\frac{R^2(H-h)^2}{H^2}\cdot h = \frac{\pi R^2}{3H^2}(H-h)^2 h$$

$$\therefore \quad \frac{dV}{dh} = \frac{\pi R^2}{3H^2}\left[(H-h)^2 - 2h(H-h)\right]$$

$$= \frac{\pi R^2}{3H^2}(H-h)(H-h-2h)$$

Therefore, $\frac{dV}{dh} = 0$ if $h = \frac{H}{3}$.

Also, $h = \frac{H}{3}$ is a point of maximum. Thus, $\frac{H}{h} = 3$.

3. (1) $f(x) = \begin{cases} |x^3 + x^2 + 3x + \sin x|\left(3 + \sin\left(\frac{1}{x}\right)\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Let $g(x) = x^3 + x^2 + 3x + \sin x$

$\therefore \quad g'(x) = 3x^2 + 2x + 3 + \cos x$

$$= 3\left(x^2 + \frac{2x}{3} + 1\right) + \cos x$$

$$= 3\left\{\left(x + \frac{1}{3}\right)^2 + \frac{8}{9}\right\} + \cos x > 0$$

and $2 < 3 + \sin\left(\frac{1}{x}\right) < 4$

Hence, minimum value of $f(x)$ is 0 at $x = 0$.

Hence, number of points = 1.

4. (32) Let $f''(x) = 6a(x-1)$, $(a > 0)$

or $\quad f'(x) = 6a\left(\frac{x^2}{2} - x\right) + b = 3a(x^2 - 2x) + b$

Given $f'(-1) = 0$

or $\quad 9a + b = 0 \quad$ or $\quad b = -9a$

$\therefore \quad f'(x) = 3a(x^2 - 2x - 3) = 0$

or $\quad x = -1$ and 3

So, $y = f(-1)$ and $y = f(3)$ are two horizontal tangents. Thus,

Distance between these tangents $= |f(3) - f(-1)| = |-22 - 10|$.

5. (2.4) Given $\lim_{x\to 0}\left(\frac{P(x)}{x^3} - 2\right) = 4$

$\therefore \quad \lim_{x\to 0}\frac{P(x)}{x^3} = 6$

Consider $P(x) = ax^5 + bx^4 + 6x^3$

$\therefore \quad P'(x) = 5ax^4 + 4bx^3 + 18x^2$

Now, $P'(-1) = 0 \Rightarrow 5a - 4b = -18$

and $P'(1) = 0 \Rightarrow 5a + 4b = -18$

Therefore, on solving, we get $a = \frac{-18}{5}$, $b = 0$.

Hence, $P(x) = \frac{-18}{5}x^5 + 6x^3$

or $\quad P(1) = \frac{12}{5}$

6. (3) We have $f(x, y) = x^2 + y^2 - 4x + 6y$

Let $(x, y) \equiv (\cos\theta, \sin\theta)$. Then $\theta \in [0, \pi/2]$ and

$f(x, y) = f(\theta) = \cos^2\theta + \sin^2\theta - 4\cos\theta + 6\sin\theta$

$f'(\theta) = 6\cos\theta + 4\sin\theta > 0 \ \forall\ \theta \in [0, \pi/2]$

Therefore, $f'(\theta)$ is strictly increasing in $[0, \pi/2]$. Thus,

$f(\theta)_{\min} = f(0) = 1 - 4 + 0 = -3$

7. (–15) $f''(x) = 4x$

$f'(x) = 2x^2 + C$

Given $f'(-2) = 1$ or $C = -7$

$\therefore \quad f'(x) = 2x^2 - 7$

$f(x) = \frac{2}{3}x^3 - 7x + C$, $f(-2) = 0$

$0 = -\frac{16}{3} + 14 + C$ or $C = -\frac{26}{3}$

$\therefore \quad f(x) = \frac{2}{3}x^3 - 7x - \frac{26}{3} = \frac{1}{3}(2x^3 - 21x - 26)$

$\therefore \quad f(1) = -15$

8. (1) $f(x) = \frac{x^3}{3} - x - b$

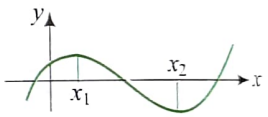
$\therefore \quad f'(x) = x^2 - 1 = 0$

$\therefore \quad x = 1$ or -1

For three distinct roots, $f(x_1)\cdot f(x_2) < 0$, where x_1 and x_2 are the roots of $f'(x) = 0$. Thus,

$$\left(\frac{1}{3} - 1 - b\right)\left(-\frac{1}{3} + 1 - b\right) < 0$$

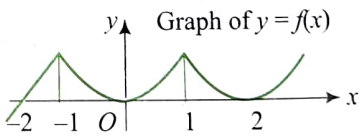
or $\quad \left(b + \frac{2}{3}\right)\left(b - \frac{2}{3}\right) < 0$



or $\quad b \in \left(-\frac{2}{3}, \frac{2}{3}\right)$

9. (4) $f'(x) = \begin{cases} 1, & x < -1 \\ 2x, & -1 < x < 1 \\ 2(x-2), & x > 1 \end{cases}$

$f'(x)$ changes sign at $x = -1, 0, 1, 2$.



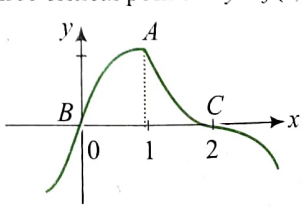
10. (4) $f''(x) = 12x^2 + 6ax + 3 \geq 0 \ \forall\ x \in R$

or $\quad 36a^2 - 144 \leq 0$

or $\quad a \in [-2, 2]$

Thus, number of nonzero integral values of a is 4.

11. (3) A, B, C are the three critical points of $y = f(x)$.



At B, it has vertical tangent. Hence, it is non-differentiable.

At A, it is non-differentiable.

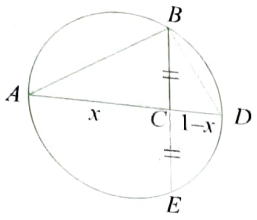
At C, $\frac{dy}{dx} = 0$.

12. (9) $BC \times CE = AC \times CD$

$\therefore \quad (BC)(CE) = x(1-x)$

But $BC = CE$

$\therefore \quad BC = \sqrt{x(1-x)}$



$\therefore \quad$ Area $\Delta = \dfrac{x\sqrt{x-x^2}}{2}$

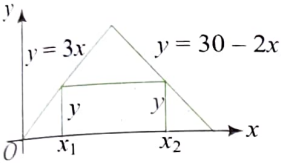
or $\quad \Delta^2 = \dfrac{x^3 - x^4}{4}$

or $\quad \dfrac{d\Delta^2}{dx} = \dfrac{3x^2 - 4x^3}{4}$

If $\dfrac{d\Delta^2}{dx} = 0$, then $x = \dfrac{3}{4}$, which is the point of maxima.

Hence, maximum area is $\dfrac{3\sqrt{3}}{32}$.

13. (67.5)



Area, $A = (x_2 - x_1)y$

$y = 3x_1$ and $y = 30 - 2x_2$

$\therefore \quad A(y) = \left(\dfrac{30-y}{2} - \dfrac{y}{3}\right)y$

or $\quad 6A(y) = (90 - 3y - 2y)y = 90y - 5y^2$

So, $\quad 6A'(y) = 90 - 10y = 0$

or $\quad y = 9; \quad A''(10) = -10 < 0$

$\therefore \quad x_1 = 3; x_2 = \dfrac{21}{2}$

$\therefore \quad A_{max} = \left(\dfrac{21}{2} - 3\right)9 = \dfrac{15 \times 9}{2} = \dfrac{135}{2}$

14. (4) $x^2 - 2x - 3 > 0$

or $\quad (x-3)(x+1) > 0$

i.e., $\quad x < -1$ or $x > 3$ (1)

Now, $f(x) = \log_{1/2}(x^2 - 2x - 3)$

$$= \frac{\log_e(x^2 - 2x - 3)}{\log_e(1/2)}$$

$$f'(x) = \frac{2x-2}{(\log_e(1/2))(x^2 - 2x - 3)}$$

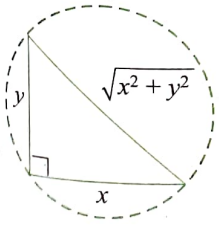
For $f(x)$ to be decreasing, $f'(x) < 0$

or $\quad \dfrac{x-1}{(\log_e(1/2))(x-3)(x+1)} < 0$ (2)

or $\quad x > 1$

From (1) and (2), $x > 3$.

15. (9)



$$\frac{9}{\pi} = S = \frac{xy}{2} = \text{constant}$$

Area of the circles $A(x) = \pi r^2 = \dfrac{\pi(x^2+y^2)}{4}, (x^2 + y^2 = 4r^2)$

$$= \frac{\pi}{4}\left[x^2 + \left(\frac{2S}{x}\right)^2\right]$$

$$A'(x) = \frac{\pi x}{2} - \frac{2\pi S^2}{x^3} = 0$$

or $\quad x^4 = 4S^2$

or $\quad x^2 = 2S$

or $\quad S^2 = \dfrac{x^2 y^2}{4} = \dfrac{2Sy^2}{4}$

or $\quad y^2 = 2S$

Therefore, least area of circle $= \pi r^2 = \dfrac{\pi}{4}(x^2 + y^2)$

$= \pi S = 9$ sq. units.

16. (– 2.25)

$f\left(\dfrac{3}{2}\right) = 0. \lim\limits_{x \to \frac{3}{2}} |x^2 - 3x| + a \le 0$ or $a \le -\dfrac{9}{4}$

Hence, greatest value of a is –2.25.

17. (2) $f(x) = 30 - 2x - x^3$

$f'(x) = -2 - 3x^2 < 0$

$\Rightarrow \quad f(x)$ is decreasing function

Hence $f(f(f(x))) > f(f(-x))$

$\Rightarrow \quad f(f(x)) < f(-x)$

$\Rightarrow \quad f(x) > -x$

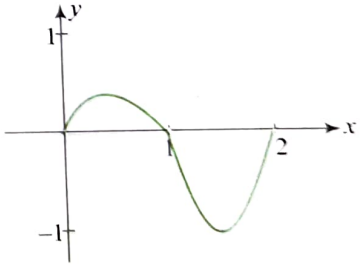
$\Rightarrow \quad 30 - 2x - x^3 > -x$

$\Rightarrow \quad x^3 + x - 30 < 0$

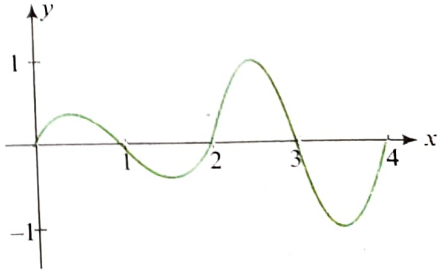
$\Rightarrow \quad (x-3)(x^2 + 3x + 10) < 0$

$\Rightarrow \quad x < 3$

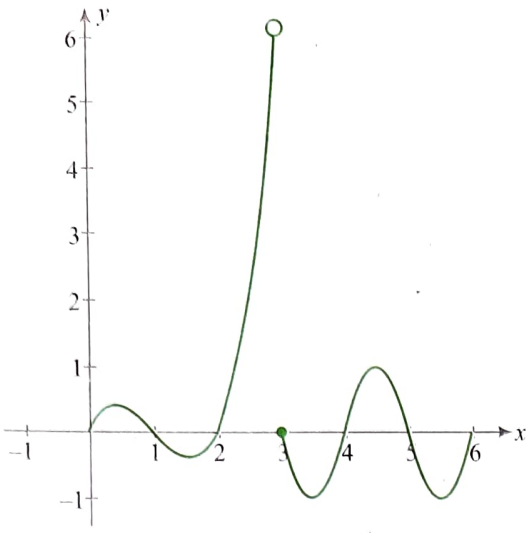
18. (3) For $n = 1$, graph of $f(x)$



For $n = 2$, graph of $f(x)$

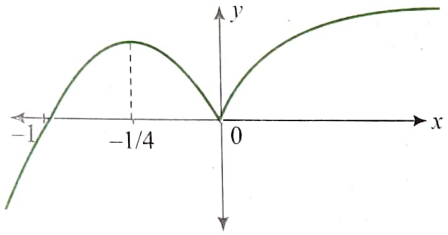


For $n = 3$, graph of $f(x)$



19. (2) $f(x) = x + \sqrt{|x|}$

$$= \begin{cases} x + \sqrt{x} & \text{if } x \geq 0 \\ x + \sqrt{-x} & \text{if } x < 0 \end{cases}$$



The graph of $f(x)$ is shown with $f'(x) = 0$ as $x = -1/4$. Also derivative fails at $x = 0$. Hence there are two critical points.

20. (8) $f(x) = \dfrac{1}{1+|x|} + \dfrac{1}{1+|x-1|}$

Clearly $f(x)$ is non-differentiable at $x = 0, 1$

Also $f(1/2 + x) = f(1/2 - x)$

Hence graph of $f(x)$ is symmetrical about the line $x = 1/2$, where derivative is zero.

$$\text{Local minimum} = f\left(\frac{1}{2}\right) = \frac{4}{3}$$

$$\text{Global maximum} = f(0) = f(1) = \frac{3}{2}$$

$$\therefore \quad k = \frac{4}{3} + \frac{3}{2} = \frac{17}{6}$$

Archives

JEE Main

Single Correct Answer Type

1. (2) $P(x) = x^4 + ax^3 + bx^2 + cx + d$

$$P'(x) = 4x^3 + 3ax^2 + 2bx + c$$

Since $x = 0$ is a solution to $P'(x) = 0$, we get $c = 0$. Therefore,

$$P(x) = x^4 + ax^3 + bx^2 + d \qquad \text{(i)}$$

Also, we have

$$P(-1) < P(1)$$

$$\Rightarrow \quad 1 - a + b + d < 1 + a + b + d \Rightarrow a > 0$$

Since $P'(x) = 0$ only when $x = 0$ and $P(x)$ is differentiable in $(-1, 1)$, we should have the maximum and minimum at the points $x = -1, 0$, and 1 only.

Also, we have $P(-1) < P(1)$

Maximum of $P(x)$ = maximum $\{P(0), P(1)\}$ and

Minimum of $P(x)$ = minimum $\{P(-1), P(0)\}$

In the interval $[0, 1]$,

$$P'(x) = 4x^3 + 3ax^2 + 2bx = x(4x^2 + 3ax + 2b)$$

Since $P'(x)$ has only one root $x = 0$,

$4x^2 + 3ax + 2b = 0$ has no real roots. Therefore,

$$(3a)^2 - 32b < 0$$

$$\Rightarrow \quad \frac{9a^2}{32} < b$$

$$\Rightarrow \quad b > 0$$

Thus, we have $a > 0$ and $b > 0$. Thus,

$$P'(x) = 4x^3 + 3ax^2 + 2bx > 0, \forall\, x \in (0, 1)$$

Hence, $P(x)$ is increasing in $[0, 1]$.

The maximum of $P(x)$ is $P(1)$.

Similarly, $P(x)$ is decreasing in $[-1, 0]$.

Therefore, the minimum of $P(x)$ does not occur at $x = -1$.

2. (1) $f(x) = \begin{cases} k - 2x, & \text{if } x \leq -1 \\ 2x + 3, & \text{if } x > -1 \end{cases}$

For minima at $x = -1$, we must have

$$\lim_{x \to -1^-} f(x) \leq \lim_{x \to -1^+} f(x)$$

$$\Rightarrow \quad k + 2 \leq 1$$

$$\Rightarrow \quad k \leq -1$$

$$\Rightarrow \quad k = -1$$

3. (2) $f(x) = \dfrac{1}{e^x + 2e^{-x}} = \dfrac{e^x}{e^{2x} + 2}$

$$\therefore \quad f'(x) = \frac{(e^{2x} + 2)e^x - 2e^{2x}e^x}{(e^{2x} + 2)^2}$$

$$f'(x) = 0$$

$$\Rightarrow \quad e^{2x} + 2 = 2e^{2x}$$

$$\Rightarrow \quad e^{2x} = 2 \Rightarrow e^x = \sqrt{2}$$

So, the maximum of $f(x)$ is

$$\frac{\sqrt{2}}{4} = \frac{1}{2\sqrt{2}}$$

$$0 < f(x) \leq \frac{1}{2\sqrt{2}} \; \forall\, x \in R$$

$$\because \quad 0 < \frac{1}{3} < \frac{1}{2\sqrt{2}},$$

for some $c \in R$, we get

$$f(c) = \frac{1}{3}$$

4. (2) $f'(x) = \dfrac{1}{x} + 2bx + a$

at $x = -1$, we have

$$-1 - 2b + a = 0$$

$$\Rightarrow \quad a - 2b = 1 \qquad \text{(i)}$$

at $x = 2$, we have

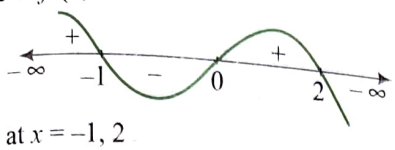
$$\frac{1}{2} + 4b + a = 0$$

$$\Rightarrow \quad a + 4b = -\frac{1}{2} \qquad \text{(ii)}$$

On solving (i) and (ii), $a = \frac{1}{2}, b = -\frac{1}{4}$

$$f'(x) = \frac{1}{x} - \frac{x}{2} + \frac{1}{2} = \frac{2 - x^2 + x}{2x} = \frac{-(x+1)(x-2)}{2x}$$

Sign scheme of $f'(x)$



So, maxima at $x = -1, 2$

5. (4) $f(x) = 2x^3 + 3x + k$

$f'(x) = 6x^2 + 3 > 0 \quad \forall x$

$\therefore$ $f(x)$ is increasing

$\therefore$ $f(x) = 0$ has only one real root

6. (3) $f'(x) = \frac{\alpha}{x} + 2\beta x + 1$

$\Rightarrow$ $2\beta x^2 + x + \alpha = 0$ has roots -1 and 2

$\Rightarrow$ $\alpha = 2, \beta = \frac{-1}{2}$

7. (3) $f'(x) = 0$ at $x = 1$ and $x = 2$

or $f'(1) = 0$ and $f'(2) = 0$

Also, $\lim_{x \to 0} \frac{f(x)}{x^2} = 2$ (Given)

$\therefore$ $f(x) = ax^4 + bx^3 + 2x^2$

$\Rightarrow$ $f'(x) = 4ax^3 + 3bx^2 + 4x$

$\therefore$ $f'(1) = 4a + 3b + 4$

and, $f'(2) = 32a + 12b + 8$

Thus, we have

$8a + 3b + 2 = 0$ (1)

$4a + 3b + 4 = 0$ (2)

on solving (1) and (2), we get

$4a = 2 \Rightarrow a = \frac{1}{2} \quad \therefore b = -2$

$\therefore$ $f(x) = \frac{x^4}{2} - 2x^3 + 2x^2$

Hence, $f(2) = 8 - 16 + 8 = 0$ ...(1)

8. (2) We have $4x + 2\pi r = 2$

Now sum of areas of square and circle is $x^2 + \pi r^2$

$\Rightarrow$ $f(r) = \left(\frac{1 - \pi r}{2}\right)^2 + \pi r^2$

$\Rightarrow$ $\frac{df(r)}{dr} = 2\left(\frac{\pi r - 1}{2}\right)\left(\frac{\pi}{2}\right) + 2\pi r$

$\frac{df(r)}{dr} = 0$

$\Rightarrow$ $r = \frac{1}{\pi + 4}$

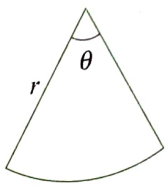
From (1), $2x = \frac{4}{\pi + 4}$

$\Rightarrow$ $x = 2r$

9. (4) Total length, $r + r + r\theta = 20$

$\Rightarrow$ $\theta = \frac{20 - 2r}{r}$

Therefore, area of flower-bed,



$$A = \frac{1}{2}r^2\theta = \frac{1}{2}r^2\left(\frac{20 - 2r}{r}\right)$$

$\Rightarrow$ $A = 10r - r^2 = 25 - (r - 5)^2$

$\therefore$ $A_{max.} = 25$ (when $r = 5$)

10. (1) $h(x) = \frac{x^2 + \frac{1}{x^2}}{x - \frac{1}{x}} = \frac{\left(x - \frac{1}{x}\right)^2 + 2}{\left(x - \frac{1}{x}\right)} = \left(x - \frac{1}{x}\right) + \frac{2}{\left(x - \frac{1}{x}\right)}$

Let $x - \frac{1}{x} = z$.

For local minimum value of $h(x)$, let $z > 0$.

$\therefore$ $h(x) = z + \frac{2}{z} = \left(\sqrt{z} - \sqrt{\frac{2}{z}}\right)^2 + 2\sqrt{2} \geq 2\sqrt{2}$

So, local minimum value of $h(x)$ is $2\sqrt{2}$.

If $z < 0$, then

$$h(x) = -\left[-z - \frac{2}{z}\right] = -\left[\sqrt{-z} - \sqrt{\frac{2}{-z}}\right]^2 - 2\sqrt{2} \leq -2\sqrt{2}$$

So, $-2\sqrt{2}$ is local maximum value of $h(x)$.

JEE Advanced

Single Correct Answer Type

1. (2) $f(x) = 2x^3 - 15x^2 + 36x + 1$

$f'(x) = 6x^2 - 30x + 36$

$= 6(x - 2)(x - 3)$

Thus, $f(x)$ is increasing in $[0, 2]$ and decreasing in $[2, 3]$.

Therefore, $f(x)$ is many-one.

$f(0) = 1$

$f(2) = 29$

$f(3) = 28$

Range is $[1, 29]$.

Hence, $f(x)$ is many-one-onto.

2. (3) Let $f(x) = x^2 - x\sin x - \cos x$

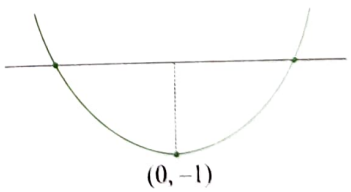
$\therefore$ $f'(x) = 2x - x\cos x$

$f'(x) = 0 \Rightarrow x(2 - \cos x) = 0$ or $x = 0$

($\because 2 - \cos x > 0$ for all real x)

Also, $x = 0$ is point of minima.

$f(0) = -1 < 0$ and $\lim_{x \to \infty} f(x) \to \infty$, $\lim_{x \to -\infty} f(x) \to \infty$.



Hence, it meets x-axis at two points and, hence, two solutions.

3. (3) Using LMVT on $f(x)$ for $x \in \left[\frac{1}{2}, 1\right]$

$$\frac{f(1) - f\left(\frac{1}{2}\right)}{1 - \frac{1}{2}} = f'(c), \text{ where } c \in \left(\frac{1}{2}, 1\right)$$

$$\Rightarrow \quad \frac{1-\frac{1}{2}}{\frac{1}{2}} = f'(c)$$

$$\Rightarrow \quad f'(c) = 1, \text{ where } c \in \left(\frac{1}{2}, 1\right)$$

Now, $f''(x) > 0$, so $f'(x)$ is an increasing function $\forall\, x \in R$.

$\therefore \quad f'(1) > 1$

Multiple Correct Answers Type

1. (1), (3)

Let the sides of rectangle be $15k$ and $8k$ and side of square be x. Then $(15k - 2x)(8k - 2x)x$ is volume.

$$v = 2(2x^3 - 23kx^2 + 60k^2x)$$

$$\left.\frac{dv}{dx}\right|_{x=5} = 0 \text{ or } 6x^2 - 46kx + 60k^2|_{x=5} = 0$$

or $\quad 6k^2 - 23k + 15 = 0$

or $\quad k = 3, k = \frac{5}{6}$

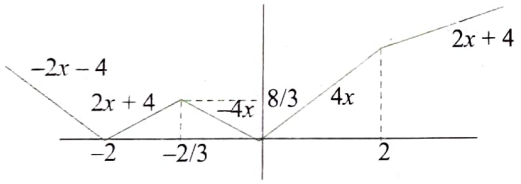
Only $k = 3$ is permissible.

So, the sides are 45 and 24.

2. (1), (2)

As $\quad \frac{f(x) + g(x) - |f(x) - g(x)|}{2} = \text{Min}(f(x), g(x))$

$$\Rightarrow \quad \frac{2|x| + |x+2| - ||x+2| - 2|x||}{2} = \text{Min}(|2x|, |x+2|)$$



From the figure shown, points of local minima/maxima are $x = -2, -2/3, 0$

3. (1), (4)

Let $f(x)$ and $g(x)$ assume their maximum value at x_1 and x_2 respectively, where $x_1 < x_2$.

$\therefore \quad f(x_1) = g(x_2) = \lambda$

Now, let $h(x) = f(x) - g(x)$

$\therefore \quad h(x_1) = f(x_1) - g(x_1)$

$= \lambda - g(x_1)$

> 0

And, $h(x_2) = f(x_2) - g(x_2)$

$= f(x_2) - \lambda$

< 0

If $x_1 > x_2$ then $h(x_1) < 0$ and $h(x_2) > 0$.

So, by intermediate value theorem, for some $c \in [0, 1]$

$h(c) = 0 \qquad (1)$

From $(f(c))^2 + 3f(c) = (g(c))^2 + 3g(c)$

$(f(c) - g(c))(f(c) + g(c) + 3) = 0$

So, there exist a 'c' such that $f(c) - g(c) = 0 \qquad$ (from (1))

Hence, (a) is correct.

Similarly, $(f(c))^2 = (g(c))^2$

$\therefore \quad (f(c) - g(c))(f(c) + g(c)) = 0$

Thus, (4) is correct.

(2) and (3) are wrong by counter example.

If $f(x) = g(x) = \lambda \neq 0$, then

$\lambda^2 + \lambda = \lambda^2 + 3\lambda$, which is not possible.

and $\quad \lambda^2 + 3\lambda = \lambda^2 + \lambda$, which is not possible.

4. (2), (4)

Let $\quad y = f(x) = x^5 - 5x$

$\Rightarrow \quad f'(x) = 5x^4 - 5$

$= 5(x - 1)(x + 1)(x^2 + 1)$

$f'(x) = 0, \therefore x = -1, 1$

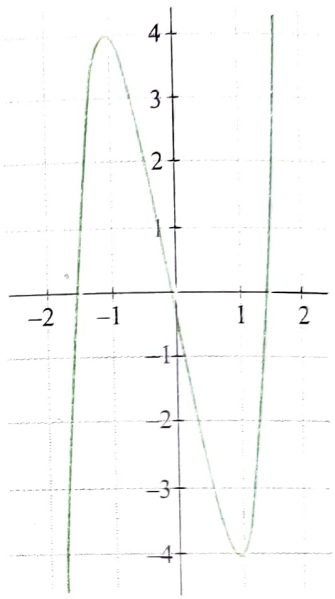
$f''(x) = 20x^3$

$f''(1) = 20$ and $f''(-1) = -20$

$\therefore \quad x = 1$ is point of minima and $x = -1$ is point of maxima

Also $f(1) = -4$ and $f(-1) = 4$

Graph of $y = f(x)$ is as shown in the following figure.



From the graph $x^5 - 5x = -a$ has one real root if $-a < -4$ or $-a > 4$.

i.e., $\quad a > 4$ or $a < -4$

$x^5 - 5x = -a$ has three real roots if $-4 < -a < 4$.

i.e., $\quad -4 < a < 4$

5. (1), (4)

$$\lim_{x \to 2} \frac{f(x)\, g(x)}{f'(x)\, g'(x)} = 1$$

$$\Rightarrow \quad \lim_{x \to 2} \frac{f'(x)\, g(x) + g'(x)\, f(x)}{f''(x)\, g'(x) + f'(x)\, g''(x)} = 1$$

$$\frac{f'(2)\, g(2) + g'(2)\, f(2)}{f''(2)\, g'(2) + f'(2)\, g''(2)} = 1$$

$$\Rightarrow \quad \frac{g'(2).f(2)}{f''(2)g'(2)} = 1$$

$\Rightarrow \quad f''(2) = f(2)$

Hence option (4) is correct

As $f''(2) = f(2)$ and $f(2) \in (0, \infty)$

$\Rightarrow \quad f''(2) > 0$

$\Rightarrow \quad f$ has local min. at $x = 2$

Hence option (1) is correct.

6. (1), (3)

Given that, $f'(x) > 2f(x)\ \forall\, x \in R$

$\Rightarrow \quad f'(x) - 2f(x) > 0\ \forall\, x \in R$

$\therefore \quad e^{-2x}(f'(x) - 2f(x)) > 0\ \forall\, x \in R$

$\Rightarrow \quad \dfrac{d}{dx}(e^{-2x} f(x)) > 0 \quad \forall\, x \in R$

Let $g(x) = e^{-2x} f(x)$.

Now, $g'(x) > 0 \ \forall\, x \in R$

i.e., $g(x)$ is strictly increasing $\forall\, x \in R$

Also, $g(0) = 1$

$\Rightarrow \quad g(x) > g(0) = 1$

$\therefore \quad e^{-2x} f(x) > 1 \ \forall\, x \in (0, \infty)$

$\Rightarrow \quad f(x) > e^{2x} \ \forall\, x \in (0, \infty)$

So, option (1) is correct.

As, $f'(x) > 2f(x) > 2e^{2x} > 2 \ \forall\, x \in (0, \infty)$

i.e., $f(x)$ is strictly increasing on $x \in (0, \infty)$

So, option (3) is correct.

Clearly, option (4) is incorrect.

7. **(2), (4)**

$$f(x) = \begin{vmatrix} \cos(2x) & \cos(2x) & \sin(2x) \\ -\cos x & \cos x & -\sin x \\ \sin x & \sin x & \cos x \end{vmatrix}$$

$$= \cos 4x + \cos 2x$$

Now, $f'(x) = -2\sin 2x - 4\sin 4x = 0$

$\Rightarrow \quad 2\sin 2x(1 + 4\cos 2x) = 0$

$\Rightarrow \quad \sin 2x = 0$ or $\cos 2x = -\dfrac{1}{4}$

For $\sin 2x = 0$; $x = 0, \pi/2, -\pi/2$

For $\cos 2x = -\dfrac{1}{4}$, there are four solutions.

$f'(x) = 0$ has more than three solutions.

8. **(2), (3), (4)**

$$\lim_{t \to x} \frac{f(x)\sin t - f(t)\sin x}{t - x} = \sin^2 x$$

$$\Rightarrow \quad \lim_{t \to x} \frac{f(x)\cos t - f'(t)\sin x}{1} = \sin^2 x$$

(using L'Hospital's rule)

$$\Rightarrow \quad \frac{f(x)\cos x - f'(x)\sin x}{\sin^2 x} = 1$$

$$\Rightarrow \quad -d\left(\frac{f(x)}{\sin x}\right) = 1$$

$$\Rightarrow \quad \frac{f(x)}{\sin x} = -x + c$$

Since $f\left(\dfrac{\pi}{6}\right) = -\dfrac{\pi}{12}$, $c = 0$.

$\Rightarrow \quad f(x) = -x \sin x$

(1) $f\left(\dfrac{\pi}{4}\right) = -\dfrac{\pi}{4\sqrt{2}}$

(2) Let $g(x) = -x\sin x + x^2 - \dfrac{x^4}{6}$

$\Rightarrow \quad g'(x) = -x\cos x - \sin x + 2x - \dfrac{2x^3}{3}$

$\Rightarrow \quad g''(x) = -2\cos x + x\sin x + 2 - 2x^2$

$\Rightarrow \quad g'''(x) = 3\sin x + x\cos x - 4x$

$= 3(\sin x - x) + x(\cos x - 1)$

$\Rightarrow \quad g'''(x) < 0$

Hence, $g''(x)$ is decreasing.

So, for $x > 0$, $g''(x) < g''(0) \quad \Rightarrow g''(x) < 0$

Hence, $g'(x)$ is decreasing.

So, for $x > 0$, $g'(x) < g'(0) \quad \Rightarrow g'(x) < 0$

Hence, $g(x)$ is decreasing.

So, for $x > 0$, $g(x) < g(0)$

$\Rightarrow \quad g(x) < 0$

Hence, $f(x) < \dfrac{x^4}{6} - x^2 \ \forall\, x \in (0, \pi)$.

(3) $f(x)$ is continuous and differentiable and $f(0) = f(\pi) = 0$.

Using Rolle's theorem, $f'(\alpha) = 0$ for some $\alpha \in (0, \pi)$.

(4) $f(x) + f'(x) = -2\cos x$

$\therefore \quad f\left(\dfrac{\pi}{2}\right) + f''\left(\dfrac{\pi}{2}\right) = 0$

Linked Comprehension Type

1. **(4)** Given $f''(x) - 2f'(x) + f(x) \geq e^x$

The terms to the L.H.S. suggest that we have to consider function $g(x) = e^{-x} f(x)$.

Now, $g'(x) = f'(x)e^{-x} - e^{-x} f(x)$

and $g''(x) = f''(x)e^{-x} - f'(x)e^{-x} - e^{-x} f'(x) + e^{-x} f(x)$

$= e^{-x}(f''(x) - 2f'(x) + f(x))$

Given that $f''(x) - 2f'(x) + f(x) \geq e^x$

or $g''(x) = e^{-x}(f''(x) - 2f'(x) + f(x)) \geq 1 > 0$

So, $g(x)$ is concave upward and $g(0) = g(1) = 0$.

Hence, $g(x) < 0 \ \forall\, x \in (0, 1)$ or $e^{-x} f(x) < 0$ or

$f(x) < 0 \ \forall x \in (0, 1)$

2. **(3)** Let $g(x) = e^{-x} f(x)$.

As $g''(x) > 0$, $g'(x)$ is increasing.

So, for $x < \dfrac{1}{4}$, $g'(x) < g'\left(\dfrac{1}{4}\right) = 0$ or $(f'(x) - f(x))e^{-x} < 0$

or $f'(x) < f(x)$ in $(0, 1/4)$

Matrix Match Type

1. **d → s**

c. $f(x) = \dfrac{5}{4} - \left(\sin x - \dfrac{1}{2}\right)^2$

Maximum value occurs when $\sin x = \dfrac{1}{2}$.

d. $f'(x) > 0$ if $\cos x > \sin x$

Note: Solutions of the remaining parts are given in their respective chapters.

2. **(4)**, 3. **(4)**, 4. **(4)**

$f(x) = x + \log_e x - x\log_e x$

$$f'(x) = 1 + \frac{1}{x} - \log_e x - x\left(\frac{1}{x}\right) = \frac{1}{x} - \log_e x$$

$$f''(x) = -\frac{1}{x^2} - \frac{1}{x} < 0 \ \forall x \in (0, \infty)$$

Thus, $f'(x)$ is strictly decreasing function for $x \in (0, \infty)$.

$$\left.\begin{aligned} \lim_{x \to \infty}\left(\frac{1}{x} - \log_e x\right) &= \lim_{x \to \infty} f'(x) = -\infty \\ \lim_{x \to 0^+}\left(\frac{1}{x} - \log_e x\right) &= \lim_{x \to 0^+} f'(x) = \infty \end{aligned}\right\}$$

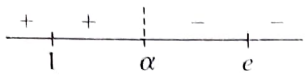
i.e., $f'(x) = 0$ has only one real root in $(0, \infty)$

$f'(1) = 1 > 0$

$$f'(e) = \frac{1}{e} - 1 < 0$$

So, $f'(x) = 0$ has one root in $(1, e)$.

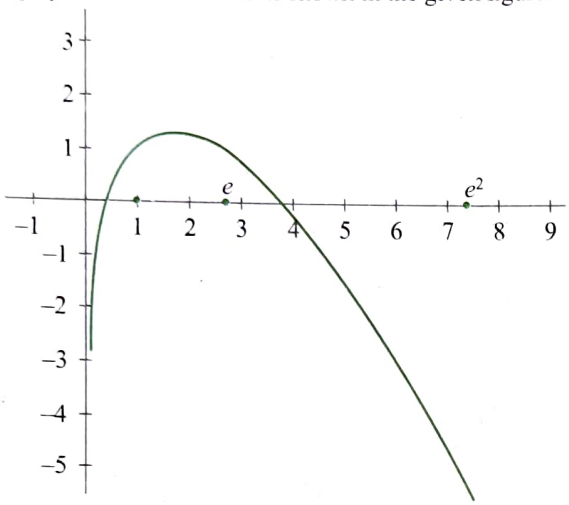
Let $f'(\alpha) = 0$, where $\alpha \in (1, e)$



Therefore, $f(x)$ is increasing in $(0, \alpha)$ and decreasing in (α, ∞).

$f(1) = 1$ and $f(e^2) = e^2 + 2 - 2e^2 = 2 - e^2 < 0$

The graph of the function is as shown in the given figure.



So, $f(x) = 0$ has one root in $(1, e^2)$

From Column I: I and II are correct.

From Column II: ii, iii, and iv are correct

From Column III: P, Q, S are correct

Numerical Value Type

1. (1) $f(x) = \ln\{g(x)\}$

$\therefore \quad g(x) = e^{f(x)}$

$\therefore \quad g'(x) = e^{f(x)} \cdot f'(x)$

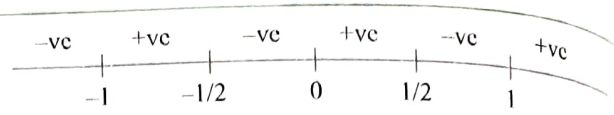
$g'(x) = 0 \Rightarrow f'(x) = 0$ as $e^{f(x)} \neq 0$

or $\quad 2010(x - 2009)(x - 2010)^2 (x - 2011)^3 (x - 2012)^4 = 0$

So, there is only one point of local maxima.

2. (5) $f(x) = |x| + |x^2 - 1|$

$$f'(x) = \frac{|x|}{x} + \frac{|x^2 - 1|}{x^2 - 1} \cdot (2x) = \begin{cases} 2x - 1, & x < -1 \\ -(2x + 1), & -1 < x < 0 \\ 1 - 2x, & 0 < x < 1 \\ 2x + 1, & x > 1 \end{cases}$$



So, $f'(x)$ changes sign at points $x = -1, -1/2, 0, 1/2, 1$.

So, total number of points of local maximum or minimum is 5.

3. (9) Let $p'(x) = k(x - 1)(x - 3)$

$$\therefore \quad p(x) = k\left(\frac{x^3}{3} - 2x^2 + 3x\right) + c$$

Now, $p(1) = 6 \Rightarrow \frac{4}{3}k + c = 6$

Also, $p(3) = 2 \Rightarrow c = 2$

So, $k = 3$. Thus, $p'(0) = 3k = 9$.

4. (4) Let the inner radius and inner length be r and h, respectively.

$\therefore \quad V = \pi r^2 h$

Let the volume of material be M.

$$\Rightarrow \quad M = \pi(r + 2)^2 2 + \pi(r + 2)^2 h - \pi r^2 h$$

$$= 2\pi(r + 2)^2 + 4\pi h(r + 1)$$

$$= 2\pi\left((r + 2)^2 + \frac{2(r + 1)V}{\pi r^2}\right)$$

$$\Rightarrow \quad \frac{dM}{dr} = 2\pi\left(2(r + 2) + \frac{2V}{\pi}\left(\frac{-1}{r^2} - \frac{2}{r^3}\right)\right)$$

Given volume is maximum when $r = 10$

$$\therefore \quad \frac{dM}{dr} = 0, \text{ when } r = 10$$

$$\Rightarrow \quad 24 + \frac{2V}{\pi}\left(\frac{-10 - 2}{10^3}\right) = 0$$

$$\Rightarrow \quad \frac{24V}{10^3\pi} = 24$$

$$\Rightarrow \quad V = 10^3\pi$$

$$\Rightarrow \quad \frac{V}{250\pi} = 4$$

Chapter 7

Concept Application Exercises

Exercise 7.1

1. $\int (\sec x+\tan x)^2\,dx = \int (\sec^2 x+\tan^2 x+2\sec x\tan x)dx$

$= \int \left(2\sec^2 x-1+2\sec x\tan x\right)dx$

$= 2(\sec x+\tan x)-x+C$

2. $\int (1-\cos x)\operatorname{cosec}^2 x\,dx = \int \operatorname{cosec}^2 x\,dx - \int \operatorname{cosec} x\cot x\,dx$

$= -\cot x+\operatorname{cosec} x+C$

$= \dfrac{1-\cos x}{\sin x}+C$

$= \dfrac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}+C$

$= \tan\dfrac{x}{2}+C$

3. $I = \int a^{mx}b^{nx}\,dx$

$= \int \left(a^m b^n\right)^x dx$

$= \dfrac{\left(a^m b^n\right)^x}{\log\left(a^m b^n\right)}+C$

4. $\int \dfrac{\tan x}{(\sec x+\tan x)}dx = \int \dfrac{\tan x(\sec x-\tan x)}{(\sec x+\tan x)(\sec x-\tan x)}dx$

$= \int \dfrac{\tan x(\sec x-\tan x)}{\left(\sec^2 x-\tan^2 x\right)}dx$

$= \int \left(\sec x\tan x-\tan^2 x\right)dx$

$= \int \sec x\tan x\,dx - \int \left(\sec^2 x-1\right)dx$

$= \int \sec x\tan x\,dx - \int \sec^2 x\,dx + \int 1\,dx$

$= \sec x-\tan x+x+C$

5. $\int \dfrac{x^4dx}{x+x^5} = \int \dfrac{(x^4+1)\,dx}{x+x^5} - \int \dfrac{dx}{x+x^5}$

$= \int \dfrac{(x^4+1)\,dx}{x(1+x^4)} - \int \dfrac{dx}{x+x^5}$

$= \int \dfrac{dx}{x} - \int \dfrac{dx}{x+x^5}$

$= \log x - f(x)+C$

6. $\int \dfrac{(x^3+8)(x-1)}{x^2-2x+4}dx = \int \dfrac{(x^3+2^3)(x-1)}{x^2-2x+4}dx$

$= \int \dfrac{(x+2)(x^2-2x+4)(x-1)}{x^2-2x+4}dx$

$= \int (x+2)(x-1)dx = \int (x^2+x-2)dx$

$= \int x^2dx + \int x\,dx - 2\int x^0dx$

$= \dfrac{x^3}{3}+\dfrac{x^2}{2}-2x+C$

7. $\int \dfrac{\sin^3 x+\cos^3 x}{\sin^2 x\cos^2 x}dx = \int \dfrac{\sin^3 x}{\sin^2 x\cos^2 x}dx + \int \dfrac{\cos^3 x}{\sin^2 x\cos^2 x}dx$

$= \int \dfrac{\sin x}{\cos^2 x}dx + \int \dfrac{\cos x}{\sin^2 x}dx$

$= \int \tan x\sec x\,dx + \int \cot x\operatorname{cosec} x\,dx$

$= \sec x-\operatorname{cosec} x+C$

8. $\int \tan^{-1}(\sec x+\tan x)dx$

$= \int \tan^{-1}\left(\dfrac{1+\sin x}{\cos x}\right)dx$

$= \int \tan^{-1}\left\{\dfrac{1-\cos\left(\frac{\pi}{2}+x\right)}{\sin\left(\frac{\pi}{2}+x\right)}\right\}dx$

$= \int \tan^{-1}\left\{\dfrac{2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right\}dx$

$= \int \tan^{-1}\left\{\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)\right\}dx$

$= \int \dfrac{\pi}{4}+\dfrac{x}{2}dx = \dfrac{\pi}{4}\int 1\cdot dx + \dfrac{1}{2}\int x\,dx = \dfrac{\pi}{4}x+\dfrac{x^2}{4}+C$

9. $\int \dfrac{\cos 2x-\cos 2\theta}{\cos x-\cos\theta}dx = \int \dfrac{\cos^2 x-\cos^2\theta}{\cos x-\cos\theta}dx$

$= \int (\cos x+\cos\theta)dx$

$= \sin x+x\cos\theta+C$

Exercise 7.2

1. $\int \dfrac{dx}{\sqrt{x}+\sqrt{x-2}} = \int \dfrac{(\sqrt{x}-\sqrt{x-2})dx}{x-(x-2)}$ (rationalizing)

$= \dfrac{1}{2}\int \left(\sqrt{x}-\sqrt{x-2}\right)dx$

$= \dfrac{1}{3}\left\{x^{3/2}-(x-2)^{3/2}\right\}+C$

2. $\int \left(1+2x+3x^2+4x^3+\cdots\right)dx = \int (1-x)^{-2}dx$

$= (1-x)^{-1}+C$

3. $\dfrac{x^3}{x+1} = \dfrac{x^3+1-1}{x+1} = \dfrac{-1}{(x+1)}+(x^2+1-x)$

Thus, the given integral is

$\int \left(x^2+1-x-\dfrac{1}{1+x}\right)dx = \dfrac{x^3}{3}+x-\dfrac{x^2}{2}-\ln|x+1|+C$

4. $I=\int \dfrac{e^{3x}+e^{5x}}{e^{x}+e^{-x}}$

$=\int \dfrac{e^{4x}+e^{6x}}{e^{2x}+1}dx$

$=\int \dfrac{e^{4x}(e^{2x}+1)}{e^{2x}+1}dx$

$=\int e^{4x}dx=\dfrac{e^{4x}}{4}+C$

5. $\int \tan^2 x\sin^2 x\,dx=\int \dfrac{\sin^4 x}{\cos^2 x}dx$

$=\int \dfrac{\sin^2 x(1-\cos^2 x)}{\cos^2 x}dx$

$=\int\left(\tan^2 x-\sin^2 x\right)dx$

$=\int\left(\sec^2 x-1-\dfrac{1-\cos 2x}{2}\right)dx$

$=\tan x-\dfrac{3}{2}x+\dfrac{\sin 2x}{4}+C$

6. $\int \dfrac{\cos x-\sin x}{\cos x+\sin x}\,(2+2\sin 2x)\,dx$

$=2\int \dfrac{(\cos x-\sin x)\,(\cos x+\sin x)^2}{\cos x+\sin x}\,dx$

$=2\int (\cos x-\sin x)\,(\cos x+\sin x)\,dx$

$=2\int (\cos^2 x-\sin^2 x)\,dx$

$=2\int \cos 2x\,dx$

$=\sin 2x+C$

7. $I=\int \sin x\cos x\cos 2x\cos 4x\cos 8x\,dx$

$=\dfrac{1}{2}\int \sin 2x\cos 2x\cos 4x\cos 8x\,dx$

$=\dfrac{1}{4}\int \sin 4x\cos 4x\cos 8x\,dx$

$=\dfrac{1}{8}\int \sin 8x\cos 8x\,dx$

$=\dfrac{1}{16}\int \sin 16x\,dx=\dfrac{-1}{256}\cos 16x+C$

Exercise 7.3

1. $\int \dfrac{dx}{x+x\log x}=\int \dfrac{\frac{1}{x}dx}{(1+\log x)}$

$=\int \dfrac{(1+\log x)'dx}{(1+\log x)}$

$=\log|1+\log x|+C$

2. $\int \dfrac{x+1}{x(x+\log_e x)}dx=\int \dfrac{1+\frac{1}{x}}{x+\log_e x}dx$

$=\int \dfrac{(x+\log_e x)'}{x+\log_e x}dx$

$=\log_e|x+\log_e x|+C$

3. $I=\int \dfrac{\sin 2x}{\sin^2 x-\sin^2\frac{\pi}{6}}dx=\int \dfrac{\left(\sin^2 x-\sin^2\frac{\pi}{6}\right)'}{\sin^2 x-\sin^2\frac{\pi}{6}}dx$

$=\log_e\left|\sin^2 x-\dfrac{1}{4}\right|+C$

$=\log_e|4\sin^2 x-1|+C$

4. $\int \dfrac{1+x^2\log_e x}{x(1+x\log_e x)}dx=\int\left(\dfrac{1+x}{x}-\dfrac{1+\log_e x}{1+x\log_e x}\right)dx$

$=\int\left(\dfrac{1}{x}+1\right)dx+\int \dfrac{(1+x\log_e x)'}{1+x\log_e x}dx$

$=\log_e x+x-\log_e|1+x\log_e x|+C$

5. $\int \dfrac{(1+\ln x)^5}{x}dx=\int (1+\ln x)^5\,d(1+\ln x)$

$=\dfrac{1}{6}(1+\ln x)^6+C$

6. $\int \log_e(\log_e x)\cdot\dfrac{1}{x\log_e x}dx$

$=\int \log_e(\log_e x)\cdot(\log_e(\log_e x))'dx$

$=\dfrac{(\log_e(\log_e x))^2}{2}+C$

7. $\int (x+\log x)^2\left(\dfrac{x+1}{x}\right)dx$

$=\int (x+\log x)^2\left(1+\dfrac{1}{x}\right)dx$

$=\int (x+\log x)^2\dfrac{d}{dx}(x+\log x)dx=\dfrac{(x+\log x)^3}{3}+C$

8. $\int \dfrac{\sqrt{\tan x}}{\sin x\cos x}dx=\int \dfrac{\sqrt{\tan x}\,\frac{1}{\cos^2 x}}{\frac{\sin x\cos x}{\cos^2 x}}dx$

$=\int \dfrac{\sqrt{\tan x}\sec^2 x}{\tan x}dx=\int (\tan x)^{-1/2}\sec^2 x\,dx$

$=\dfrac{(\tan x)^{1/2}}{1/2}+C=2\sqrt{\tan x}+C$

9. $I=\int \operatorname{cosec}^4 x\,dx=\int \operatorname{cosec}^2 x\operatorname{cosec}^2 x\,dx$

$=\int \operatorname{cosec}^2 x\,(1+\cot^2 x)\,dx$

$=\int \operatorname{cosec}^2 x\,dx+\int \cot^2 x\operatorname{cosec}^2 x\,dx$

$=-\cot x-\dfrac{\cot^3 x}{3}+C$

10. $\int \dfrac{\sin^6 x}{\cos^8 x}dx=\int \dfrac{\sin^6 x}{\cos^6 x}\times\dfrac{1}{\cos^2 x}dx$

$=\int \tan^6 x\sec^2 x\,dx$

$=\dfrac{\tan^7 x}{7}+C$

11. $\int \frac{dx}{(1+\sin x)^{1/2}} = \int \frac{dx}{\cos\frac{x}{2}+\sin\frac{x}{2}}$

$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)}$

$= \frac{1}{\sqrt{2}}\int \operatorname{cosec}\left(\frac{x}{2}+\frac{\pi}{4}\right)dx$

$= \frac{1}{\sqrt{2}} \frac{\log\left|\tan\left(\frac{x}{4}+\frac{\pi}{8}\right)\right|}{\frac{1}{2}} + C$

$= \sqrt{2}\log\left|\left(\tan\left(\frac{x}{4}+\frac{\pi}{x}\right)\right)\right| + C$

12. $\int \frac{\sin x}{\sin(x-a)}dx = \int \frac{\sin(x-a+a)}{\sin(x-a)}dx$

$= \int \frac{\sin(x-a)\cos a + \cos(x-a)\sin a}{\sin(x-a)}dx$

$= \cos a \int dx + \sin a \int \frac{\cos(x-a)}{\sin(x-a)}dx$

$= (\cos a)\,x + \sin a \,|\log \sin(x-a)| + C$

$= (x-a)\cos a + \sin a \log|\sin(x-a)| + C$

13. $I = \int \tan^3 x\,dx$

$= \int \tan^2 x \tan x\,dx$

$= \int (\sec^2 x - 1)\tan x\,dx$

$= \int \tan x \sec^2 x\,dx - \int \tan x\,dx$

$= I_1 - \log|\sec x| + C$, where $I_1 = \int \tan x \sec^2 x\,dx$

Putting $\tan x = t$ and $\sec^2 x\,dx = dt$ in I_1, we get

$$I_1 = \int t\,dt = \frac{t^2}{2} = \frac{1}{2}\tan^2 x + C$$

Hence, $I = \frac{1}{2}\tan^2 x - \log|\sec x| + C$.

14. $\int \frac{dx}{\cot^2 x - 1} = \int \frac{\sin^2 x dx}{\cos 2x}$

$= \frac{1}{2}\int \frac{(1-\cos 2x)dx}{\cos 2x}$

$= \frac{1}{2}\left[\int \sec 2x dx - \int dx\right]$

$= \frac{1}{2}\left[\frac{1}{2}\log_e|\sec 2x + \tan 2x| - x\right] + C$

$= \frac{1}{4}\log_e|\sec 2x + \tan 2x| - \frac{x}{2} + C$

15. Put $x = \tan\theta$

$\Rightarrow \quad dx = \sec^2\theta d\theta$

$\therefore \quad \int \frac{1}{x^2\sqrt{1+x^2}}dx = \int \frac{\sec^2\theta\,d\theta}{\tan^2\theta\sec\theta}$

$= \int \operatorname{cosec}\theta\cot\theta\,d\theta$

$= -\operatorname{cosec}\theta + C$

$= \frac{-\sqrt{x^2+1}}{x} + C$

16. $2ax + x^2 = (x+a)^2 - a^2$.

Now, put $x + a = a\sec\theta$

$\therefore \quad dx = a\sec\theta\tan\theta\,d\theta$

$\therefore \quad \int \frac{dx}{(2ax+x^2)^{3/2}} = \int \frac{a\sec\theta\tan\theta}{a^3\tan^3\theta}d\theta$

$= \frac{1}{a^2}\int \frac{\cos\theta}{\sin^2\theta}d\theta$

$= -\frac{1}{a^2\sin\theta} + C$

$= -\frac{1}{a^2\sqrt{1-\frac{a^2}{(x+a)^2}}} + C$

$= -\frac{x+a}{a^2\sqrt{2ax+x^2}} + C$

Exercise 7.4

1. Let $I = \int \frac{\sin 2x}{(a+b\cos x)^2}dx = \int \frac{2\sin x\cos x}{(a+b\cos x)^2}dx$.

Putting $a + b\cos x = t$ and $-b\sin x\,dx = dt$, we get

$$I = -\frac{2}{b}\int \frac{1}{t^2}\left(\frac{t-a}{b}\right)dt \left[\because a + b\cos x = t,\ \cos x = \frac{t-a}{b}\right]$$

$$= -\frac{2}{b^2}\int\left(\frac{1}{t} - \frac{a}{t^2}\right)dt = -\frac{2}{b^2}\left[\log|t| + \frac{a}{t}\right] + C$$

$$= -\frac{2}{b^2}\left[\log|a+b\cos x| + \frac{a}{a+b\cos x}\right] + C$$

2. $I = \int \tan^{-1}x^3 \cdot \frac{x^2}{1+x^6}dx$

Let $\tan^{-1}x^3 = t$

$\therefore \quad dt = \frac{3x^2}{1+x^6}dx$

$\therefore \quad I = \frac{1}{3}\int t\,dt = \frac{t^2}{6} + C = \frac{(\tan^{-1}x^3)^2}{6} + C$

3. Put $x = t^2$ or $dx = 2t\,dt$. Then,

$$\int \frac{\sqrt{x}\,dx}{1+x} = 2\int \frac{t^2 dt}{1+t^2}$$

$$= 2\int\left(1 - \frac{1}{1+t^2}\right)dt$$

$$= 2(t - \tan^{-1}t) + C$$

$$= 2\sqrt{x} - 2\tan^{-1}\sqrt{x} + C$$

4. $I = \int \frac{dx}{x+\sqrt{x}} = \int \frac{dx}{\sqrt{x}\left(\sqrt{x}+1\right)}$.

Put $\sqrt{x} = z$

$\therefore \quad \frac{1}{2\sqrt{x}}dx = dz$

$\therefore \quad I = \int \frac{2\,dz}{z+1}$

$= 2\log|z+1| + C$

$= 2\log(\sqrt{x}+1) + C$

5. $\int \frac{e^{2x}-2e^x}{e^{2x}+1}\,dx = \frac{1}{2}\int \frac{2e^{2x}}{e^{2x}+1}\,dx - 2\int \frac{e^x\,dx}{(e^x)^2+1}$

$= \frac{1}{2}\log(e^{2x}+1) - 2\int \frac{dz}{z^2+1}$, where $z = e^x$

$= \frac{1}{2}\log(e^{2x}+1) - 2\tan^{-1}(e^x) + C$

6. $I = \int \frac{dx}{e^x+e^{-x}} = \int \frac{e^x dx}{e^{2x}+1}$

Let $e^x = t$.

$\therefore \quad e^x dx = dt$

$\therefore \quad I = \int \frac{dt}{t^2+1} = \tan^{-1}t + C = \tan^{-1}(e^x) + C$

7. Let $\log\left(1+\frac{1}{x}\right) = t$

$\therefore \quad \frac{0-\frac{1}{x^2}}{1+\frac{1}{x}}dx = dt$

$\Rightarrow \quad \frac{-dx}{x(x+1)} = dt$

$\Rightarrow \quad \int \frac{\log\left(1+\frac{1}{x}\right)}{x(1+x)}dx = -\int t\,dt$

$= -\frac{t^2}{2} + c$

$= -\frac{\left(\log\left(1+\frac{1}{x}\right)\right)^2}{2} + C$

8. $I = \int \frac{ax^3+bx}{x^4+c^2}dx$

$= \int\left[\frac{ax^3}{x^4+c^2} + \frac{bx}{x^4+c^2}\right]dx$

$= \frac{a}{4}\int \frac{4x^3}{x^4+c^2}dx + \frac{b}{2}\int \frac{2x}{x^4+c^2}dx$

$I_1 \quad + \quad I_2$

$= \frac{a}{4}\log\left(x^4+c^2\right) + \frac{b}{2}\int \frac{dt}{t^2+c^2}$ (In I_2, put $x^2 = t$)

$= \frac{a}{4}\log\left(x^4+c^2\right) + \frac{b}{2}\frac{1}{c}\tan^{-1}\frac{x^2}{c} + k$

9. $I = \int \frac{dx}{x^{2/3}(1+x^{2/3})}$

Let $t^3 = x$, then $dx = 3t^2\,dt$

$\therefore \quad I = \int \frac{3t^2dt}{t^2(1+t^2)}$

$= 3\int \frac{dt}{1+t^2} = 3\tan^{-1}(t) + C$

$= 3\tan^{-1}(x^{1/3}) + C$

10. $\int e^{3\log x}(x^4+1)^{-1}dx = \int e^{\log x^3}\frac{dx}{x^4+1} = \int \frac{x^3dx}{x^4+1}$

$= \frac{1}{4}\int \frac{4x^3}{x^4+1}dx$

$= \frac{1}{4}\log(x^4+1) + C$

11. $\int \frac{\sec x\,dx}{\sqrt{\cos 2x}} = \int \frac{\sec x}{\sqrt{\cos^2 x - \sin^2 x}}dx$

$= \int \frac{\sec^2 x\,dx}{\sqrt{1-\tan^2 x}}$

$= \sin^{-1}(\tan x) + C$

12. [Here, power of sin x is odd positive integer. Therefore, put $z = \cos x$.]

Let $z = \cos x$. Then $dz = -\sin x\,dx$. Now,

$\int \sin^3 x\cos^2 x\,dx = \int \sin^2 x\cos^2 x\sin x\,dx$

$= \int (1-\cos^2 x)\cos^2 x\sin x\,dx$

$= \int (1-z^2)z^2(-dz)$

$= -\int (z^2-z^4)dz$

$= -\left(\frac{z^3}{3} - \frac{z^5}{5}\right) + C = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$

13. $I = \int \frac{x\,dx}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}$

$= \int \frac{x\,dx}{\sqrt{(1+x^2)\left(1+\sqrt{1+x^2}\right)}}$

Let $1 + \sqrt{1+x^2} = t^2$

$\therefore \quad 2t\,dt = \frac{x}{\sqrt{1+x^2}}dx$

$\therefore \quad I = \int 2dt = t + c = 1 + \sqrt{1+x^2} + c$

14. $I = \int \frac{(x-x^3)^{1/3}}{x^4}dx = \int \frac{\left(\frac{1}{x^2}-1\right)^{1/3}}{x^3}dx$

Putting $\frac{1}{x^2} = t$, $\frac{1}{x^3}dx = -\frac{dt}{2}$, we get

$I = -\frac{1}{2}\int t^{1/3}dt = -\frac{3}{8}t^{4/3} + C = -\frac{3}{8}\left(\frac{1}{x^2}-1\right)^{4/3} + C$

15. $I = \int \frac{dx}{x^2(1+x^5)^{4/5}} = \int \frac{dx}{x^6\left(\frac{1}{x^5}+1\right)^{4/5}}$

Let $t = 1 + \frac{1}{x^5}$ or $dt = -\frac{5dx}{x^6}$

$\therefore \quad I = -\frac{1}{5}\int \frac{dt}{t^{4/5}} = -t^{1/5} + C = -\left(1+\frac{1}{x^5}\right)^{1/5} + C$

$= -\frac{(1+x^5)^{1/5}}{x} + C$

16. $\int \frac{1}{x^2(x^4+1)^{3/4}}dx = \int \frac{1}{x^5\left(1+\frac{1}{x^4}\right)^{3/4}}dx$

$= -\frac{1}{4}\int \frac{1}{t^{3/4}}dt$, where $t = 1+\frac{1}{x^4}$

$= -\frac{1}{4}\frac{t^{1/4}}{1/4} + C = -t^{1/4} + C$

$= -\left(1+\frac{1}{x^4}\right)^{1/4} + C$

17. $\int \frac{(x^4-x)^{1/4}}{x^5}dx = \int \frac{1}{x^4}\left(1-\frac{1}{x^3}\right)^{1/4}dx,$

Putting $1-\frac{1}{x^3} = t$, we get

$$I = \frac{1}{3}\int t^{1/4}dt = \frac{4}{15}t^{5/4} + C = \frac{4}{15}\left(1-\frac{1}{x^3}\right)^{5/4} + C$$

18. $I = \int x^x(\ln ex)\,dx = \int x^x(1+\ln x)\,dx$

Let $t = x^x = e^{x\ln x}$ then $\frac{dt}{dx} = x^x\left(x\times\frac{1}{x}+\ln x\right)$

or $dt = x^x(1+\ln x)\,dx$ or $I = \int dt = t + C = x^x + C$

19. $\sqrt{\frac{x-q}{x-p}} = t$

$\therefore \quad \frac{1}{2}\left(\frac{x-p}{x-q}\right)^{1/2}\frac{(x-p)1-(x-q)1}{(x-p)^2}dx = dt.$

or $\frac{1}{2}\frac{q-p}{\sqrt{x-q}\,(x-p)^{3/2}}dx = dt$

or $\frac{dx}{(x-p)^{3/2}\sqrt{(x-q)}} = \frac{2\,dt}{q-p}$

or $I = -\int\frac{2\,dt}{p-q}\quad t = -\frac{2}{p-q}\sqrt{\frac{x-q}{x-p}} + C$

20. Let $(\sqrt{1+x^2}+x)^n = z$

or $n(\sqrt{1+x^2}+x)^{n-1}\left(\frac{x}{\sqrt{1+x^2}}+1\right)dx = dz$

or $\frac{(\sqrt{1+x^2}+x)^n}{\sqrt{1+x^2}}dx = \frac{dz}{n}$

$\therefore$ Given integral $= \int\frac{dz}{n} = \frac{1}{n}z + C = \frac{1}{n}(\sqrt{1+x^2}+x)^n + C$

Exercise 7.5

1. $I = -\int\frac{dx}{x^2+x-1} = -\int\frac{dx}{\left(x+\frac{1}{2}\right)^2-\frac{5}{4}}$

$= -\frac{1}{2\cdot\frac{\sqrt{5}}{2}}\log_e\left|\frac{\left(x+\frac{1}{2}\right)-\frac{\sqrt{5}}{2}}{\left(x+\frac{1}{2}\right)+\frac{\sqrt{5}}{2}}\right| + C$

$= -\frac{1}{\sqrt{5}}\log_e\left|\frac{2x+1-\sqrt{5}}{2x+1+\sqrt{5}}\right| + C$

2. $\int\frac{x^2-\sqrt{3}x+1}{x^4-x^2+1}dx = \int\frac{x^2-\sqrt{3}x+1}{(x^2+1)^2-3x^2}dx$

$= \int\frac{x^2-\sqrt{3}x+1}{(x^2-\sqrt{3}x+1)(x^2+\sqrt{3}x+1)}dx$

$= \int\frac{1}{x^2+\sqrt{3}x+1}dx$

$= \int\frac{1}{\left(x+\frac{\sqrt{3}}{2}\right)^2+\frac{1}{4}}dx$

$= \frac{1}{1/2}\tan^{-1}\frac{x+\frac{\sqrt{3}}{2}}{\frac{1}{2}} + C$

$= 2\tan^{-1}(2x+\sqrt{3}) + C$

3. $I = \int\frac{x^2}{x^6-a^6}dx$

Let $x^3 = t$

$\therefore \quad 3x^2\,dx = dt$

$\therefore \quad I = \frac{1}{3}\int\frac{dt}{t^2-a^6} = \frac{1}{3}\frac{1}{2a^3}\log_e\left|\frac{t-a^3}{t+a^3}\right| + C$

$= \frac{1}{3}\frac{1}{2a^3}\log_e\left|\frac{x^3-a^3}{x^3+a^3}\right| + C$

4. $I = \int\frac{\sin x}{\cos 2x}dx = \int\frac{\sin x}{2\cos^2 x-1}dx$

Let $\cos x = t$

$\therefore \quad -\sin x\,dx = dt$

$\therefore \quad I = \int\frac{-dt}{2t^2-1} = -\frac{1}{2}\int\frac{dt}{t^2-\frac{1}{2}}$

$= -\frac{1}{2}\cdot\frac{1}{2\frac{1}{\sqrt{2}}}\log_e\left|\frac{t-\frac{1}{\sqrt{2}}}{t+\frac{1}{\sqrt{2}}}\right| + C$

$= -\frac{1}{2\sqrt{2}}\log_e\left|\frac{\sqrt{2}t-1}{\sqrt{2}t+1}\right| + C$

5. $I = \int\frac{e^x}{e^{2x}+6e^x+5}dx = \int\frac{e^x}{(e^x)^2+6e^x+5}dx$

Let $e^x = t$ or $e^x\,dx = dt$

$\therefore \quad I = \int\frac{dt}{t^2+6t+5}$

$= \int\frac{1}{(t+3)^2-2^2}dt$

$= \frac{1}{2\times 2}\log\left|\frac{t+3-2}{t+3+2}\right| + C = \frac{1}{4}\log\left|\frac{e^x+1}{e^x+5}\right| + C$

6. $\int \frac{x^3+x+1}{x^2-1}dx = \int\left(x+\frac{2x+1}{x^2-1}\right)dx$

$= \int x\,dx + \int \frac{2x}{x^2-1}dx + \int \frac{1}{x^2-1}dx$

$= \frac{x^2}{2} + \log\left|x^2-1\right| + \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C$

7. $I = \int \frac{1}{x^4+1}dx = \int \frac{\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$

$= \frac{1}{2}\int \frac{\frac{2}{x^2}}{x^2+\frac{1}{x^2}}dx$

$= \frac{1}{2}\int\left(\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}} - \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\right)dx$

$= \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$

$= \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}dx$

Putting $x-\frac{1}{x} = u$ in first integral and $x+\frac{1}{x} = v$ in second integral, we get

$I = \frac{1}{2}\int \frac{du}{u^2+\left(\sqrt{2}\right)^2} - \frac{1}{2}\int \frac{dv}{v^2-\left(\sqrt{2}\right)^2}$

$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) - \frac{1}{2}\times\frac{1}{2\sqrt{2}}\log\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right| + C$

$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{2}}\right) - \frac{1}{4\sqrt{2}}\log\left|\frac{x+1/x-\sqrt{2}}{x+1/x+\sqrt{2}}\right| + C$

$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}\,x}\right) - \frac{1}{4\sqrt{2}}\log\left|\frac{x^2-\sqrt{2}\,x+1}{x^2+x\sqrt{2}+1}\right| + C$

8. $I = \int \frac{1}{\sin^4 x+\cos^4 x}dx = \int \frac{1/\cos^4 x}{\frac{\sin^4 x+\cos^4 x}{\cos^4 x}}dx$

$= \int \frac{\sec^4 x}{\tan^4 x+1}dx = \int \frac{\sec^2 x\sec^2 x}{\tan^4 x+1}dx$

$= \int\left(\frac{1+\tan^2 x}{1+\tan^4 x}\right)\sec^2 x\,dx$

Putting $\tan x = t$ and $\sec^2 x\,dx = dt$, we get

$I = \int \frac{1+t^2}{1+t^4}dt = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}\,t}\right) + C$

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan^2 x-1}{\sqrt{2}\tan x}\right) + C$

9. $I = \int \frac{x^4+1}{x^6+1}dx = \int \frac{(x^2+1)^2-2x^2}{(x^2+1)(x^4-x^2+1)}dx$

$= \int \frac{x^2+1}{x^4-x^2+1}dx - 2\int \frac{x^2dx}{x^6+1}$

$= \int \frac{1+\frac{1}{x^2}}{x^2-1+\frac{1}{x^2}}dx - 2\int \frac{x^2dx}{\left(x^3\right)^2+1}$

$= \tan^{-1}\left(x-\frac{1}{x}\right) - \frac{2}{3}\tan^{-1}(x^3) + C$

Exercise 7.6

1. $I = \int \frac{x^2}{\sqrt{1-x^6}}dx = \int \frac{x^2}{\sqrt{1^2-\left(x^3\right)^2}}dx$

Let $x^3 = t$ then $3x^2\,dx = dt$

$\therefore \quad I = \frac{1}{3}\int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{3}\sin^{-1}(t) + C = \frac{1}{3}\sin^{-1}\left(x^3\right) + C$

2. $I = \int \sqrt{\frac{x}{a^3-x^3}}\,dx = \int \frac{\sqrt{x}}{\sqrt{\left(a^{3/2}\right)^2-\left(x^{3/2}\right)^2}}dx$

Let $x^{3/2} = t$ or $\frac{3}{2}x^{1/2}dx = dt$ or $dx = \frac{2}{3\sqrt{x}}dt$

$\therefore \quad I = \int \frac{2/3\,dt}{\sqrt{\left(a^{3/2}\right)^2-t^2}}dt = \frac{2}{3}\int \frac{dt}{\sqrt{\left(a^{3/2}\right)^2-t^2}}$

$= \frac{2}{3}\sin^{-1}\left(\frac{t}{a^{3/2}}\right) + C = \frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right) + C$

3. $I = \int \frac{1}{\sqrt{1-e^{2x}}}dx = \int \frac{1}{\sqrt{1-\frac{1}{e^{-2x}}}}dx = \int \frac{e^{-x}}{\sqrt{e^{-2x}-1}}dx$

$= \int \frac{e^{-x}}{\sqrt{\left(e^{-x}\right)^2-1^2}}dx$

Let $e^{-x} = t$ or $-e^{-x}\,dx = dt$

$\therefore \quad I = -\int \frac{dt}{\sqrt{t^2-1^2}} = -\log\left|t+\sqrt{t^2-1}\right| + C$

$= -\log\left|e^{-x}+\sqrt{e^{-2x}-1}\right| + C$

4. $I = \int \frac{2x+3}{\sqrt{x^2+4x+1}}dx = \int \frac{(2x+4)-1}{\sqrt{x^2+4x+1}}dx$

$= \int \frac{2x+4}{\sqrt{x^2+4x+1}}dx - \int \frac{1}{\sqrt{x^2+4x+1}}dx$

$= \int \frac{dt}{\sqrt{t}} - \int \frac{1}{\sqrt{(x+2)^2-\left(\sqrt{3}\right)^2}}dx$, where $t = x^2+4x+1$

$= 2\sqrt{t} - \log\left|(x+2)+\sqrt{x^2+4x+1}\right| + C$

$= 2\sqrt{x^2+4x+1} - \log\left|x+2+\sqrt{x^2+4x+1}\right| + C$

5. Put $x^{7/2} = t$

$\therefore \quad \frac{7}{2}x^{5/2}dx = dt$

$\therefore \quad I = \int \frac{2}{7}\frac{dt}{\sqrt{1+t^2}} = \frac{2}{7}\log\left(t+\sqrt{1+t^2}\right)+C$

$= \frac{2}{7}\log(x^{7/2}+\sqrt{1+x^7})+C$

6. $I = \int \sqrt{1+\text{cosec } x}\,dx$

$= \int \frac{\sqrt{1+\sin x}}{\sqrt{\sin x}}\,dx$

$= \int \frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\sqrt{2\sin\frac{x}{2}\cos\frac{x}{2}}}\,dx \qquad (\because 0 < x < \pi/2)$

$= \int \frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\sqrt{1-\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2}}\,dx$

Put $\sin\frac{x}{2}-\cos\frac{x}{2} = t$ or $\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)dx = 2dt$

$\therefore \quad I = \int \frac{2dt}{\sqrt{1-t^2}} = 2\sin^{-1}t + c$

$= 2\sin^{-1}\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)+c$

Exercise 7.7

1. $I = \int \frac{dx}{3+\sin 2x}$

$= \int \frac{dx}{3+2\sin x\cos x}$

$= \int \frac{\sec^2 x dx}{3\sec^2 x+2\tan x}$ (Dividing Nr. and Dr. by $\cos^2 x$)

$= \int \frac{\sec^2 x dx}{3\tan^2 x+2\tan x+3}$

$= \int \frac{dt}{3t^2+2t+3}$ (Putting $\tan x = t$)

$= \frac{1}{3}\int \frac{dt}{t^2+\frac{2}{3}t+1}$

$= \frac{1}{3}\int \frac{dt}{\left(t+\frac{1}{3}\right)^2+\frac{8}{9}}$

$= \frac{1}{3}\cdot\frac{1}{\sqrt{8}/3}\tan^{-1}\frac{\left(t+\frac{1}{3}\right)}{\frac{\sqrt{8}}{3}}+c$

$= \frac{1}{\sqrt{8}}\tan^{-1}\frac{3\tan x+1}{\sqrt{8}}+c$

2. $\int \frac{dx}{9+16\sin^2 x} = \int \frac{dx}{9\cos^2 x+25\sin^2 x} = \int \frac{\sec^2 x\,dx}{9+25\tan^2 x}$

$= \int \frac{dz}{9+25z^2} = \frac{1}{25}\int \frac{dz}{z^2+\left(\frac{3}{5}\right)^2} \qquad (z = \tan x)$

$= \frac{1}{25}\times\frac{1}{3/5}\tan^{-1}\frac{z}{3/5}+C$

$= \frac{1}{15}\tan^{-1}\frac{5z}{3}+C = \frac{1}{15}\tan^{-1}\frac{5\tan x}{3}+C$

3. $I = \int \frac{x+1}{(x-1)\sqrt{x+2}}\,dx$

Let $x+2 = t^2$

$\therefore \quad dx = 2t\,dt$

$\therefore \quad I = \int \frac{t^2-1}{(t^2-3)t}2t\,dt$

$= 2\int \frac{t^2-3+2}{(t^2-3)}dt$

$= 2\int\left(1+\frac{2}{(t^2-3)}\right)dt$

$= 2t+\frac{2}{\sqrt{3}}\log\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right|+C$

$= 2\sqrt{x+2}+\frac{2}{\sqrt{3}}\log\left|\frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}\right|+C$

4. $I = \int \frac{dx}{\sqrt{1+e^x+e^{2x}}} = \int \frac{e^x dx}{e^x\sqrt{1+e^x+e^{2x}}}$

$= \int \frac{dt}{t\sqrt{1+t+t^2}}$

Put $t = \frac{1}{y}$

$\therefore \quad I = \int \frac{-\frac{dy}{y^2}}{\frac{1}{y}\sqrt{1+\frac{1}{y}+\frac{1}{y^2}}}$

$= -\int \frac{dy}{\sqrt{y^2+y+1}}$

$= -\int \frac{dy}{\sqrt{\left(y+\frac{1}{2}\right)^2+\frac{3}{4}}}$

$= -\log_e\left|\left(y+\frac{1}{2}\right)+\sqrt{\left(y+\frac{1}{2}\right)^2+\frac{3}{4}}\right|+c$

$= -\log_e\left|\left(e^{-x}+\frac{1}{2}\right)+\sqrt{e^{-2x}+e^{-x}+1}\right|+c$

5. Let $I = \int \frac{1}{(x+1)\sqrt{x^2-1}}\,dx$

Putting $x+1 = \frac{1}{t}$ and $dx = -\frac{1}{t^2}dt$, we get

$$I=\int\frac{1}{\frac{1}{t}\sqrt{\left(\frac{1}{t}-1\right)^2-1}}\left(-\frac{1}{t^2}\right)dt$$

$$=-\int\frac{dt}{\sqrt{1-2t}}=-\int(1-2t)^{-1/2}\,dt$$

$$=-\frac{(1-2t)^{1/2}}{(-2)\left(\frac{1}{2}\right)}+C=\sqrt{1-2t}+C$$

$$=\sqrt{1-\frac{2}{x+1}}+C=\sqrt{\frac{x-1}{x+1}}+C$$

6. $I=\int\frac{x^2-1}{(x^2+1)\sqrt{x^4+1}}dx$

$$=\int\frac{x^2(1-1/x^2)}{x^2(x+1/x)\sqrt{x^2+1/x^2}}dx$$

$$=\int\frac{(1-1/x^2)dx}{(x+1/x)\sqrt{(x+1/x)^2-2}}$$

Putting $x+1/x=t$, we have

$$I=\int\frac{dt}{t\sqrt{t^2-2}}$$

Again putting $t^2-2=y^2$, $2t\,dt=2y\,dy$,

$$I=\int\frac{y\,dy}{(y^2+2)y}=\frac{1}{\sqrt{2}}\tan^{-1}\frac{y}{\sqrt{2}}+C$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\frac{\sqrt{x^2+1/x^2}}{\sqrt{2}}+C$$

7. $I=\int\frac{x}{(x^2+4)\sqrt{x^2+1}}dx$

Let $x^2+1=t^2$ or $x\,dx=t\,dt$

$$\therefore\quad I=\int\frac{t\,dt}{(t^2+3)t}=\frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}+C$$

$$=\frac{1}{\sqrt{3}}\tan^{-1}\sqrt{\frac{x^2+1}{3}}+C$$

8. $I=\int\frac{1}{(x+1)\sqrt{x^2+x+1}}dx$

Let $x+1=\frac{1}{t}$ or $dx=-\frac{1}{t^2}dt$

$$\therefore\quad I=\int\frac{1}{\frac{1}{t}\sqrt{\left(\frac{1}{t}-1\right)^2+\left(\frac{1}{t}-1\right)+1}}\left(-\frac{1}{t^2}\right)dt$$

$$=-\int\frac{dt}{\sqrt{(1-t)^2+(t-t^2)+t^2}}$$

$$=-\int\frac{dt}{\sqrt{t^2-t+1}}$$

$$=-\int\frac{dt}{\sqrt{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}}}$$

$$=-\log\left|\left(t-\frac{1}{2}\right)+\sqrt{t^2-t+1}\right|+C$$

$$=-\log\left|\frac{1}{x+1}-\frac{1}{2}+\frac{\sqrt{x^2+x+1}}{x+1}\right|+C$$

9. $I=\int\frac{(x-1)dx}{(x+1)\sqrt{x^3+x^2+x}}$

$$=\int\frac{(x^2-1)}{(x^2+2x+1)\sqrt{x^3+x^2+x}}$$

$$=\int\frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}+2\right)\sqrt{x+\frac{1}{x}+1}}$$

Putting $x+\frac{1}{x}+1=u^2$,

$$I=\int\frac{2u\,du}{(u^2+1).u}=2\tan^{-1}u+C$$

$$=2\tan^{-1}\sqrt{\left(x+\frac{1}{x}+1\right)}+C$$

10. $\int(\sqrt{\tan x}+\sqrt{\cot x})\,dx=\int\frac{\sin x+\cos x}{\sqrt{\sin x\cos x}}dx$

$$=\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{\sin 2x}}dx$$

$$=\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{1-(\sin x-\cos x)^2}}dx$$

$$=\sqrt{2}\int\frac{dt}{\sqrt{1-t^2}}$$

$$=\sqrt{2}\sin^{-1}t+C$$

$$=\sqrt{2}\sin^{-1}(\sin x-\cos x)+C$$

Exercise 7.8

1. Let $I=\int\frac{1}{(x^2-4)\sqrt{x+1}}dx$

Putting $x+1=t^2$ and $dx=2t\,dt$, we get

$$I=\int\frac{2t\,dt}{\left[(t^2-1)^2-4\right]\sqrt{t^2}}$$

$$=2\int\frac{dt}{(t^2-1-2)(t^2-1+2)}$$

$$=2\int\frac{dt}{(t^2-3)(t^2+1)}$$

$$=\frac{2}{4}\int\left(\frac{1}{t^2-3}-\frac{1}{t^2+1}\right)dt$$

$$=\frac{1}{4\sqrt{3}}\log\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right|-\frac{1}{2}\tan^{-1}t+C,\text{ where }t=\sqrt{x+1}$$

2. $\int \frac{x^2+1}{x(x^2-1)}dx = \int \frac{x^2+1}{x(x-1)(x+1)}dx$

$= \int\left(\frac{-1}{x} + \frac{1}{x-1} + \frac{1}{x+1}\right)dx$

$= \log|x-1| + \log|x+1| - \log|x| + C$

$= \log\left|\frac{x^2-1}{x}\right| + C$

3. $\int \frac{1}{x^4-1}dx = \int \frac{1}{(x^2+1)(x^2-1)}dx$

$= \frac{1}{2}\int\left(\frac{1}{x^2-1} - \frac{1}{x^2+1}\right)dx$

$= \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{-1}x + C$

4. Here, the degree of numerator is greater than that of denominator. So, we divide the numerator by denominator to obtain

$\int \frac{x^3}{(x-1)(x-2)}dx = \int\left(x+3+\frac{7x-6}{(x-1)(x-2)}\right)dx$

$= \frac{x^2}{3} + 3x + \int\left(\frac{-1}{(x-1)} + \frac{8}{(x-2)}\right)dx$

$= \frac{x^2}{3} + 3x - \log|x-1| + 8\log|x-2| + C$

5. $\int \frac{dx}{\sin x(3+\cos^2 x)} = \int \frac{\sin x\,dx}{\sin^2 x(3+\cos^2 x)}$

$= \int \frac{\sin x\,dx}{(1-\cos^2 x)(3+\cos^2 x)}$

$= \int \frac{dy}{(y^2-1)(y^2+3)}$ (Putting $\cos x = y$)

$= \frac{1}{4}\int\left[\frac{1}{y^2-1} - \frac{1}{y^2+3}\right]dy$

$= \frac{1}{4}\log\left|\frac{y-1}{y+1}\right| - \frac{1}{4\sqrt{3}}\tan^{-1}\frac{y}{\sqrt{3}} + C$

6. $I = \int \frac{\cos 2x \sin 4x\,dx}{\cos^4 x(1+\cos^2 2x)}$

$= \int \frac{2\cos^2 2x \sin 2x\,dx}{\left(\frac{1+\cos 2x}{2}\right)^2(1+\cos^2 2x)}$

Let $\cos 2x = t$. then $dt = -2\sin 2x\,dx$.

$\therefore \quad I = -\int \frac{t^2 dt}{\left(\frac{1+t}{2}\right)^2(1+t^2)} = -4\int \frac{t^2 dt}{(1+t)^2(1+t^2)}$

Now, $\frac{t^2}{(1+t)^2(1+t^2)} = \frac{A}{1+t} + \frac{B}{(1+t)^2} + \frac{Ct+D}{1+t^2}$

or $t^2 = A(1+t)(1+t^2) + B(1+t^2) + (Ct+D)(1+t)^2$

Put $t = -1$. Then $B = 1/2$

Put $t = 0$. Then $0 = A + 1/2 + D$ (1)

Put $t = 1$. Then $1 = 4A + 1 + 4C + 4D$ or $A + C + D = 0$ (2)

From equations (1) and (2), $C = 1/2$

Comparing coefficients of t^3, $A + C = 0$ (3)

or $A = -1/2$

From equations (2) and (3), $D = 0$

Hence, $I = \int\left(-\frac{1/2}{1+t} + \frac{1/2}{(1+t)^2} + \frac{(1/2)t}{1+t^2}\right)dt$

$= -\frac{1}{2}\log|t| - \frac{1}{2(1+t)} + \frac{1}{4}\log(1+t^2) + C$

$= -\frac{1}{2}\log|t| - \frac{1}{2(1+t)} + \frac{1}{4}\log(1+t^2) + C$, where $t = \cos 2x$

7. Let $I = \int \frac{1-\cos x}{\cos x(1+\cos x)}dx$.

Let $\cos x = y$. Then

$\frac{1-\cos x}{\cos x(1+\cos x)} = \frac{1-y}{y(1+y)}$

$= \frac{1}{y} - \frac{2}{1+y}$

$= \frac{1}{\cos x} - \frac{2}{1+\cos x}$

$\therefore \quad I = \int \frac{1-\cos x}{\cos x(1+\cos x)}dx$

$= \int \frac{1}{\cos x}dx - \int \frac{2}{1+\cos x}dx$

$= \int \sec x\,dx - \int \frac{2}{2\cos^2 x/2}dx$

$= \int \sec x - \int \sec^2 x/2\,dx$

$= \log|\sec x + \tan x| - 2\tan x/2 + C$

Exercise 7.9

1. $\int x\sin^2 x\,dx$

$= \int x\left\{\frac{1-\cos 2x}{2}\right\}dx = \frac{1}{2}\int x\,dx - \frac{1}{2}\int \underset{\text{I}}{x}\,\underset{\text{II}}{\cos 2x}\,dx$

$= \frac{1}{2}\left(\frac{x^2}{2}\right) - \frac{1}{2}\left[x\left\{\int \cos 2x\,dx\right\} - \int\left\{\frac{d}{dx}(x)\int \cos 2x\,dx\right\}dx\right]$

$= \frac{1}{4}x^2 - \frac{1}{2}\left\{\frac{x}{2}\sin 2x - \int 1 \times \frac{\sin 2x}{2}dx\right\}$

$= \frac{x^2}{4} - \frac{1}{2}\left\{\frac{x}{2}\sin 2x - \frac{1}{2}\int \sin 2x\,dx\right\}$

$= \frac{x^2}{4} - \frac{1}{2}\left\{\frac{x}{2}\sin 2x - \frac{1}{2}\left(-\frac{1}{2}\cos 2x\right)\right\} + C$

$= \frac{1}{4}x^2 - \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x + C$

2. Put $\sqrt{x} = t$ and $\frac{1}{2\sqrt{x}}dx = dt$ or $dx = 2t\,dt$

$\therefore \quad \int \cos\sqrt{x}\,dx = \int 2t\cos t\,dt$

$= 2\left[t\sin t - \int \sin t\,dt\right]$

$= 2t\sin t + 2\cos t + C$

$= 2[\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}] + C$

3. $\int \tan^{-1}\sqrt{x}\cdot 1\,dx = (\tan^{-1}\sqrt{x}) - \int \frac{1}{1+x}\frac{1}{2\sqrt{x}}x\,dx$

(Integrating by parts)

$= x\tan^{-1}\sqrt{x} - \frac{1}{2}\int \frac{\sqrt{x}\,dx}{1+x}$

$= x\tan^{-1}\sqrt{x} - \frac{1}{2}\left[2\left(\sqrt{x} - \tan^{-1}\sqrt{x}\right)\right] + C$

$= x\tan^{-1}\sqrt{x} - \sqrt{x} + \tan^{-1}\sqrt{x} + C$

$= (x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + C$

4. $\int f(x)dx = g(x)$

$I = \int f^{-1}(x)\cdot 1\,dx$

$= f^{-1}(x)\int dx - \int\left\{\frac{d}{dx}f^{-1}(x)\int dx\right\}dx$

$= xf^{-1}(x) - \int x\left(\frac{d}{dx}f^{-1}(x)\right)dx$

$= xf^{-1}(x) - \int x\,d\{f^{-1}(x)\}$

Let $f^{-1}(x) = t$, i.e., $x = f(t)$ and $d\{f^{-1}(x)\} = dt$

$\therefore \quad I = xf^{-1}(x) - \int f(t)dt = xf^{-1}(x) - g(t) + C$

$= xf^{-1}(x) - g\{f^{-1}(x)\} + C$

5. $\int [f(x)g''(x) - f''(x)g(x)]dx$

$= \int f(x)g''(x)dx - \int f''(x)g(x)\,dx$

$= \left(f(x)g'(x) - \int f'(x)g'(x)dx\right)$

$\qquad -\left(g(x)f'(x) - \int g'(x)f'(x)dx\right)$

$= f(x)\,g'(x) - f'(x)\,g(x) + C$

6. $\int g(x)\{f(x) + f'(x)\}dx$

$= \int g(x)f(x)dx + \int g(x)f'(x)dx$

$= f(x)\left(\int g(x)dx\right) - \int\left(f'(x)\int g(x)dx\right)dx + \int g(x)f'(x)\,dx$

$= f(x)\,g(x) - \int g(x)f'(x)dx + \int g(x)f'(x)dx + C$

$= f(x)\,g(x) + C \qquad [\because \int g(x)dx = g(x)]$

7. $\int 1\cdot(\log_e x)^2\,dx$

$= x\cdot(\log_e x)^2 - \int 2\frac{\log_e x}{x}\cdot x dx$

$= x\cdot(\log_e x)^2 - 2\int \log_e x dx$

$= x\cdot(\log_e x)^2 - 2(x\log_e x - x) + c$

8. Putting $\sin^{-1}x = t$ and $\frac{1}{\sqrt{1-x^2}}dx = dt$, we get

$\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx = \int t\sin t\,dt = -t\cos t + \sin t + C$

$= -\sin^{-1}x\cos(\sin^{-1}x) + \sin(\sin^{-1}x) + C$

$= x - (\sin^{-1}x)\sqrt{1-x^2} + C$

9. $I = \int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx$

Let $x = \cos 2\theta$ or $dx = -2\sin 2\theta\,d\theta$

$\therefore \quad I = \int \tan^{-1}\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}(-2\sin 2\theta)\,d\theta$

$= -2\int \tan^{-1}(\tan\theta)\sin 2\theta\,d\theta$

$= -2\int \theta\sin 2\theta\,d\theta$

$= -2\left[-\frac{\theta\cos 2\theta}{2} + \int \frac{\cos 2\theta}{2}d\theta\right]$

$= \theta\cos 2\theta - \frac{\sin 2\theta}{2} + C$

10. $\int \underset{II}{\cos x}\log\underset{I}{\left(\tan\frac{x}{2}\right)}dx$

$= \log\left(\tan\frac{x}{2}\right)\sin x - \int \frac{1}{\tan\frac{x}{2}}\sec^2\frac{x}{2}\times\frac{1}{2}\sin x\,dx + C$

(Integrating by parts)

$= \sin x\log\left(\tan\frac{x}{2}\right) - \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}\sin x\,dx + C$

$= \sin x\log\left(\tan\frac{x}{2}\right) - \int dx + C$

$= \sin x\log\left(\tan\frac{x}{2}\right) - x + C$

11. $I = \int \sin^2(\log x)dx$

Let $t = \log x$ or $dt = \frac{dx}{x}$

or $\quad dx = e^t\,dt$

$\therefore \quad I = \int e^t\sin^2 t\,dt = \frac{1}{2}\int e^t(1-\cos 2t)\,dt$

or $\quad 2I = e^t - \int e^t\cos 2t\,dt$

$= e^t - \frac{e^t}{5}(2\sin 2t + \cos 2t) + C$

or $\quad I = \frac{1}{10}x(5 - 2\sin(2\log x) - \cos(2\log x)) + C$

12. $\int e^x\left(1 + \tan x + \tan^2 x\right)dx = \int e^x\left(\tan x + \sec^2 x\right)dx$

$= e^x\tan x + C$

13. $\int \frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}}dx = \int \frac{e^x(1-x^2+1)}{(1-x)\sqrt{1-x^2}}dx$

$= \int e^x\left[\frac{(1-x^2)}{(1-x)\sqrt{1-x^2}} + \frac{1}{(1-x)\sqrt{1-x^2}}\right]dx$

$= \int e^x\left[\frac{1+x}{\sqrt{1-x^2}} + \frac{1}{(1-x)\sqrt{1-x^2}}\right]dx$

$= \int e^x\left[\sqrt{\frac{1+x}{1-x}} + \frac{d}{dx}\left(\sqrt{\frac{1+x}{1-x}}\right)\right]dx$

$= e^x\sqrt{\frac{1+x}{1-x}} + C$

14. Put $\log x = t$, i.e., $x = e^t$ so that $dx = e^t\,dt$

$$\therefore \int \left(\frac{(t-1)}{1+t^2}\right)^2 e^t\,dt = \int e^t\left[\frac{1}{1+t^2} - \frac{2t}{(1+t^2)^2}\right]dt$$

$$= e^t \frac{1}{1+t^2} + C$$

$$= \frac{x}{1+(\log x)^2} + C$$

15. $\int \sqrt{x^2+2x+5}\,dx$

$$= \int \sqrt{(x+1)^2+4}\,dx$$

$$= \frac{1}{2}(x+1)\sqrt{(x+1)^2+2^2} + \frac{1}{2}\cdot(2)^2 \log\left|(x+1)+\sqrt{(x+1)^2+2^2}\right| + C$$

$$= \frac{1}{2}(x+1)\sqrt{x^2+2x+5} + 2\log\left|(x+1)+\sqrt{x^2+2x+5}\right| + C$$

Exercises

Single Correct Answer Type

1. (2) $\int \frac{\sin^8 x - \cos^8 x}{1-2\sin^2 x\cos^2 x}\,dx = \int \frac{(\sin^2 x - \cos^2 x)(\sin^4 x + \cos^4 x)}{1-2\sin^2 x\cos^2 x}$

$$= \int -\cos 2x\,dx = -\frac{1}{2}\sin 2x + C$$

2. (2) $\int \frac{\cos 4x + 1}{\cot x - \tan x}\,dx = \int \frac{2\cos^2 2x}{\cos^2 x - \sin^2 x}\sin x\cos x\,dx$

$$= \int \cos 2x \sin 2x\,dx$$

$$= \frac{1}{4}\int \sin 4x\,dx = -\frac{1}{8}\cos 4x + C$$

3. (2) $\int \left(\sqrt{\frac{a+x}{a-x}} + \sqrt{\frac{a-x}{a+x}}\right)dx = \int \frac{2a}{\sqrt{a^2-x^2}}\,dx$

$$= 2a\sin^{-1}\left(\frac{x}{a}\right) + c$$

4. (1) $I = \int \frac{\sqrt{1+\sin x}\,\sqrt{1-\sin x}}{\sqrt{1-\sin x}}\,dx$

$$= \int \frac{\cos x}{\sqrt{1-\sin x}}\,dx = -2\sqrt{1-\sin x} + C$$

5. (2) $I = \int 4\sin x \cos\frac{x}{2}\cos\frac{3x}{2}\,dx$

$$= \int 2\sin x(\cos 2x + \cos x)\,dx$$

$$= \int (\sin 3x - \sin x + \sin 2x)\,dx$$

$$= \cos x - \frac{1}{3}\cos 3x - \frac{1}{2}\cos 2x + C$$

6. (1) Differentiating both sides, we get

$$\sqrt{1+\sin x}\,f(x) = \frac{2}{3}\frac{3}{2}(1+\sin x)^{1/2}\cos x$$

or $f(x) = \cos x$

7. (4) $I = \int \frac{\sqrt{x-1}}{x\sqrt{x+1}}\,dx = \int \frac{x-1}{x\sqrt{x^2-1}}\,dx$

$$= \int \frac{dx}{\sqrt{x^2-1}} - \int \frac{dx}{x\sqrt{x^2-1}}$$

$$= \ln\left|x+\sqrt{x^2+1}\right| - \sec^{-1}x + c$$

8. (4) $I = \int \frac{\sin x\cos x}{\sin x+\cos x}\,dx$

$$= \frac{1}{2}\int \frac{(\sin x+\cos x)^2 - 1}{\sin x+\cos x}\,dx$$

$$= \frac{1}{2}\int \left[\sin x + \cos x - \frac{1}{\sqrt{2}\sin(x+\pi/4)}\right]dx$$

$$= \frac{1}{2}[\sin x - \cos x] - \frac{1}{2\sqrt{2}}\log\left|\operatorname{cosec}(x+\pi/4) - \cot(x+\pi/4)\right| + C$$

9. (4) $\int \frac{\sin x}{\sin(x-\pi/4)}\,dx = \int \frac{\sin\left(\left(x-\frac{\pi}{4}\right)+\frac{\pi}{4}\right)}{\sin\left(x-\frac{\pi}{4}\right)}\,dx$

$$= \frac{1}{\sqrt{2}}\int \frac{\sin\left(x-\frac{\pi}{4}\right)+\cos\left(x-\frac{\pi}{4}\right)}{\sin\left(x-\frac{\pi}{4}\right)}\,dx$$

$$= \frac{1}{\sqrt{2}}\int \left(1+\cot\sin\left(x-\frac{\pi}{4}\right)\right)dx$$

$$= \frac{1}{\sqrt{2}}\left(x+\log_e\left|\sin\left(x-\frac{\pi}{4}\right)\right|\right) + c$$

10. (3) $I = \int \frac{\cos 5x + \cos 4x}{1-2\cos 3x}\,dx = \int \frac{2\cos\frac{9x}{2}\cos\frac{x}{2}}{1-2\left(2\cos^2\frac{3x}{2}-1\right)}\,dx$

$$= \int \frac{2\cos\frac{9x}{2}\cos\frac{x}{2}}{3-4\cos^2\frac{3x}{2}}\,dx$$

$$= \int \frac{2\cos\frac{9x}{2}\cos\frac{x}{2}\cos\frac{3x}{2}}{3\cos\frac{3x}{2}-4\cos^3\frac{3x}{2}}\,dx$$

$$= \int \frac{2\cos\frac{9x}{2}\cos\frac{3x}{2}\cos\frac{x}{2}}{-\cos\frac{9x}{2}}\,dx$$

$$= -\int 2\cos\frac{3x}{2}\cos\frac{x}{2}\,dx$$

$$= -\int (\cos 2x + \cos x)\,dx$$

$$= -\frac{\sin 2x}{2} - \sin x + c$$

11. (2) Put $2 + x = t^2$, so that $dx = 2t\,dt$. Then

$$I = \int \frac{\sqrt{7-t^2}}{t}(2t)\,dt = 2\int \sqrt{7-t^2}\;dt$$

$$= t\sqrt{7-t^2} + 7\sin^{-1}\left(\frac{t}{\sqrt{7}}\right) + C$$

$$= \sqrt{x+2}\,\sqrt{5-x} + 7\sin^{-1}\left(\frac{\sqrt{x+2}}{\sqrt{7}}\right) + C$$

12. (3) $\displaystyle\int \frac{\sin 2x}{\sin 5x \sin 3x}dx = \int \frac{\sin(5x-3x)}{\sin 5x \sin 3x}$

$$= \int \frac{\sin 5x\cos 3x - \cos 5x \sin 3x}{\sin 5x \sin 3x}\,dx$$

$$= \frac{1}{3}\log \sin 3x - \frac{1}{5}\log \sin 5x + C$$

13. (1) $\displaystyle I = \int \frac{dx}{x\left(x^n+1\right)} = \int \frac{x^{n-1}}{x^n\left(x^n+1\right)}\,dx$

Putting $x^n = t$ so that $n\,x^{n-1}\,dx = dt$, i.e.,

we get $x^{n-1}\,dx = \dfrac{1}{n}\,dt$

$$I = \int \frac{\frac{1}{n}dt}{t(t+1)} = \frac{1}{n}\int\left(\frac{1}{t} - \frac{1}{t+1}\right)dt$$

$$= \frac{1}{n}\left(\log t - \log(t+1)\right) + C$$

$$= \frac{1}{n}\log\left(\frac{x^n}{x^n+1}\right) + C$$

14. (3) $\sin^3 x \sin(x+\alpha) = \sin^3 x(\sin x\cos\alpha + \cos x \sin\alpha)$

$$= \sin^4 x(\cos\alpha + \cot x \sin\alpha)$$

$$I = \int \frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}\,dx$$

$$= \int \frac{1}{\sin^2 x\sqrt{\cos\alpha + \cot x \sin\alpha}}\,dx$$

$$= \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cos\alpha + \cot x \sin\alpha}}\,dx$$

Putting $\cos\alpha + \cot x \sin\alpha = t$ and $-\operatorname{cosec}^2 x \sin\alpha\,dx = dt$, we have

$$I = \int -\frac{1}{\sin\alpha\sqrt{t}}\,dt = -\frac{1}{\sin\alpha}\int t^{-1/2}\,dt$$

$$= \frac{1}{\sin\alpha}\left(\frac{t^{1/2}}{1/2}\right) + C$$

$$= -2\operatorname{cosec}\alpha\sqrt{t} + C$$

$$= -2\operatorname{cosec}\alpha(\cos\alpha + \cot x \sin\alpha)^{1/2} + C$$

15. (3) $\displaystyle\int \frac{px^{p+2q-1} - qx^{q-1}}{(x^{p+q}+1)^2}dx = \int \frac{px^{p-1} - qx^{-q-1}}{(x^p + x^{-q})^2}dx$

(Dividing N^r and D^r by x^{2q})

$$= \int \frac{dt}{t^2} = -\frac{1}{t} + C, \text{ where } t = x^p + x^{-q}$$

$$= -\frac{1}{x^p + x^{-q}} + C$$

$$= -\frac{x^q}{x^{p+q}+1} + C$$

16. (1) Let $x = \tan\theta$. Then $dx = \sec^2\theta\,d\theta$. Now,

$$y = \int \frac{dx}{(1+x^2)^{\frac{3}{2}}} = \int \frac{\sec^2\theta}{(1+\tan^2\theta)^{\frac{3}{2}}}d\theta$$

$$= \int \frac{\sec^2\theta}{(\sec^2\theta)^{\frac{3}{2}}}d\theta$$

$$= \int \frac{\sec^2\theta}{\sec^3\theta}d\theta = \int \frac{d\theta}{\sec\theta} = \int \cos\theta\,d\theta$$

Hence, $y = \sin\theta + c = \dfrac{x}{\sqrt{1+x^2}} + c$ (1)

$$\left[\because \tan\theta = x = \frac{x}{1},\ \sin\theta = \frac{x}{\sqrt{1^2+x^2}}\right]$$

Given when $x = 0$, $y = 0$ from equation (1),

$0 = 0 + c$ or $c = 0$

Thus, from equation (1), $y = \dfrac{x}{\sqrt{1+x^2}}$.

Hence, when $x = 1$, $y = \dfrac{1}{\sqrt{2}}$.

17. (3) Let $x = t^6$ or $dx = 6t^5\,dt$

$$\therefore \quad I = \int t^3(1+t^2)^4\,6t^5\,dt$$

$$= 6\int t^8(1 + 4t^2 + 6t^4 + 4t^6 + t^8)\,dt$$

$$= 6\int (t^8 + 4t^{10} + 6t^{12} + 4t^{14} + t^{16})\,dt$$

$$= 6\left\{\frac{t^9}{9} + \frac{4t^{11}}{11} + \frac{6t^{13}}{13} + \frac{4t^{15}}{15} + \frac{t^{17}}{17}\right\} + C$$

$$= 6\left\{\frac{1}{9}x^{3/2} + \frac{4}{11}x^{11/6} + \frac{6}{13}x^{13/6} + \frac{4}{15}x^{5/2} + \frac{1}{17}x^{17/6}\right\} + C$$

18. (3) $\displaystyle I = \int \frac{\ln(\tan x)}{\sin x\cos x}dx$

Let $t = \ln(\tan x)$

or $\dfrac{dt}{dx} = \dfrac{\sec^2 x}{\tan x}$

or $dt = \dfrac{dx}{\sin x\cos x}$

$$\therefore \quad I = \int t\,dt = \frac{1}{2}t^2 + C = \frac{1}{2}(\ln(\tan x))^2 + C$$

19. (2) $\displaystyle I = \int \frac{(x^{5m-1} + 2x^{4m-1})\,dx}{(1 + x^{-m} + x^{-2m})^3} = \int \frac{x^{-(m+1)} + 2\cdot x^{-(2m+1)}}{(1 + x^{-m} + x^{-2m})^3}dx$

Put $1 + x^{-m} + x^{-2m} = t$

$$\therefore \quad I = \frac{-1}{m}\int \frac{dt}{t^3} = \frac{1}{2mt^2} + c = \frac{x^{4m}}{2m(x^{2m} + x^m + 1)^2} + c$$

20. (1) Putting $I^{r+1}(x) = t$

and $\dfrac{1}{xI(x)I^2(x)\cdots I^r(x)}dx = dt$, we get

$$\int \frac{dx}{xI^2(x)I^3(x)\cdots I^r(x)} = \int 1\,dt = t + C = I^{r+1}(x) + C$$

21. (2) $I = \int \dfrac{\cos x - \sin x}{\sqrt{\cos x \sin x}}dx$

Put $\sin x + \cos x = t$, so that $2 \sin x \cos x = t^2 - 1$

$$\therefore \quad I = \sqrt{2}\int \frac{dt}{\sqrt{t^2-1}} = \sqrt{2}\log\left|t + \sqrt{t^2-1}\right| + C$$

$$= \sqrt{2}\log|\sin x + \cos x + \sqrt{\sin 2x}| + C$$

22. (2) $I = \int \dfrac{\sin 2x}{(3+4\cos x)^3}dx$

Put $3 + 4 \cos x = t$, so that $-4 \sin x\, dx = dt$. Then

$$I = \frac{-1}{8}\int \frac{(t-3)}{t^3}dt = \frac{1}{8}\left(\frac{1}{t} - \frac{3}{2}\frac{1}{t^2}\right) + C$$

$$= \frac{2t-3}{16t^2} = \frac{8\cos x + 3}{16(3+4\cos x)^2} + C$$

23. (3) $I = \int \dfrac{\ln\left(\dfrac{x-1}{x+1}\right)}{x^2-1}dx$

Let $t = \ln\left(\dfrac{x-1}{x+1}\right)$

or $\dfrac{dt}{dx} = \dfrac{x+1}{x-1}\left\{\dfrac{x+1-(x-1)}{(x+1)^2}\right\} = \dfrac{2}{(x^2-1)}$

or $\dfrac{dx}{x^2-1} = \dfrac{dt}{2}$

$$\therefore \quad I = \frac{1}{2}\int t\,dt = \frac{1}{4}t^2 + C = \frac{1}{4}\left(\ln\left(\frac{x-1}{x+1}\right)\right)^2 + C$$

24. (1) $I = \int \sqrt{e^x - 1}\,dx$

Let $e^x - 1 = t^2$ or $e^x\,dx = 2t\,dt$ or $dx = \dfrac{2t}{t^2+1}dt$

$$\therefore \quad I = \int t\,\frac{2t}{t^2+1}dt = \int \frac{2t^2}{t^2+1}dt$$

$$= \int \frac{2(t^2+1)-2}{t^2+1}dt = \int 2\,dt - \int \frac{2\,dt}{t^2+1}$$

$$= 2t - 2\tan^{-1}t + C$$

$$= 2\sqrt{e^x - 1} - 2\tan^{-1}\sqrt{e^x - 1} + C$$

25. (1) $\displaystyle\int \sqrt{\frac{x^2+1}{x^2(1-x^2)}}dx = \int \sqrt{\frac{1+x^2}{x^2(1-x^2)}} \times \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}}dx$

$$= \int \frac{1+x^2}{x\sqrt{1-x^4}}dx$$

$$= \int \frac{1}{x\sqrt{1-x^4}}dx + \int \frac{x}{\sqrt{1-x^4}}dx$$

$$= \frac{1}{4}\log_e\left|\frac{1-\sqrt{1-x^4}}{1+\sqrt{1-x^4}}\right| + \frac{1}{2}\sin^{-1}(x^2) + c$$

26. (4) $I = \displaystyle\int \frac{\sqrt{x^2+10x+24}}{x+5}dx = \int \frac{\sqrt{(x+5)^2-1}}{x+5}dx$

Put $x + 5 = \sec\theta \quad \Rightarrow dx = \sec\theta \tan\theta\, d\theta$

$$\therefore \quad I = \int \frac{\tan\theta}{\sec\theta}\cdot \sec\theta\tan\theta\,d\theta$$

$$= \int (\sec^2\theta - 1)\,d\theta$$

$$= \tan\theta - \theta + C$$

$$= \sqrt{x^2+10x+24} - \sec^{-1}(x+5) + c$$

27. (1) $I = \displaystyle\int \frac{x^x(1+\log x)}{x^x\sqrt{(x^x)^2-1}}dx$

Let $x^x = t$

$\therefore \quad x\log x = \log t$

$\Rightarrow \quad (\log x + 1)\,dx = \dfrac{1}{t}dt$

$\Rightarrow \quad x^x(1 + \log x)dx = dt$

$$\therefore \quad I = \int \frac{1}{t\sqrt{t^2-1}}dt$$

$$= \sec^{-1}(t) = \sec^{-1}(x^x) + C$$

28. (2) Here, $\int x^5(1+x^3)^{2/3}\,dx$

Let $1 + x^3 = t^2$ and $3x^2\,dx = 2t\,dt$

$$\therefore \quad \int x^5(1+x^3)^{2/3}\,dx = \int x^3(1+x^3)^{2/3}x^2\,dx$$

$$= \int (t^2-1)(t^2)^{2/3}x^2\,dx$$

$$= \frac{2}{3}\int (t^2-1)\,t^{7/3}\,dt$$

$$= \frac{2}{3}\int (t^{13/3} - t^{7/3})\,dt$$

$$= \frac{2}{3}\left\{\frac{3}{16}t^{16/3} - \frac{3}{10}t^{10/3}\right\} + C$$

$$= \frac{1}{8}(1+x^3)^{8/3} - \frac{1}{5}(1+x^3)^{5/3} + C$$

29. (2) $I = \displaystyle\int \frac{\sin 2x}{\sin^4 x + \cos^4 x}dx$

$$= \int \frac{2\sin x\cos x}{\sin^4 x + \cos^4 x}dx$$

$$= \int \frac{2\tan x\sec^2 x}{1+\tan^4 x}dx$$

Let $\tan^2 x = t \quad$ or $2\tan x\sec^2 x\,dx = dt$

$$\therefore \quad I = \int \frac{dt}{1+t^2} = \tan^{-1}t + C = \tan^{-1}(\tan^2 x) + C$$

30. (2) Let $I = \displaystyle\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}dx$

Putting $x + 1 = t^2$, $dx = 2t\,dt$, we get

$$I = 2\int \frac{t^2+1}{t^4+t^2+1}dt = 2\int \frac{1+(1/t)^2}{\left(t-\frac{1}{t}\right)^2+3}dt$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{3}}\right) + C$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3(x+1)}}\right) + C$$

31. (3) $I = \int \frac{\sec x\, dx}{\sqrt{2\sin(x+A)\cos x}}$

$$= \int \frac{\sec^2 x\, dx}{\sqrt{\frac{2\sin(x+A)}{\cos x}}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{\sec^2 x\, dx}{\sqrt{\tan x \cos A + \sin A}}$$

$$= \frac{\sec A}{\sqrt{2}} \int \frac{2p\,dp}{p} \qquad (\tan x \cos A + \sin A = p^2, \text{ then } \cos A \sec^2 x\, dx = 2p\,dp)$$

$$= \sqrt{2} \sec A \int dp$$

$$= \sqrt{2}\, \sec A \sqrt{\tan x \cos A + \sin A} + C$$

32. (4) $\int \frac{\cos 2x}{(e^{-x} + \cos x)\sqrt{1+\sin 2x}}\, dx$

$$= \int \frac{\cos^2 x - \sin^2 x}{(e^{-x} + \cos x)\sqrt{(\cos x + \sin x)^2}}\, dx$$

$$= \int \frac{\cos x - \sin x}{(e^{-x} + \cos x)}\, dx$$

$$= \int \frac{e^x(\cos x - \sin x)}{1 + e^x c \cos x}\, dx$$

$$= \int \frac{(1 + e^x \cos x)'}{1 + e^x c \cos x}\, dx$$

$$= \log_e |1 + e^x \cos x| + c$$

33. (3) $I = \int \frac{\cos 4x - 1}{\cot x - \tan x}\, dx = \int \frac{-2\sin^2 2x(\sin x \cos x)}{(\cos^2 x - \sin^2 x)}\, dx$

$$= -\int \frac{\sin^2 2x \sin 2x}{\cos 2x} x$$

$$= \int \frac{(\cos^2 2x - 1)\sin 2x}{\cos 2x}\, dx$$

Let $t = \cos 2x$ or $dt = -2 \sin 2x\, dx$

$$\therefore \quad I = \frac{1}{2}\int \frac{(1-t^2)}{t}\, dt = \frac{1}{2}\ln|t| - \frac{t^2}{4} + C$$

$$= \frac{1}{2}\ln|\cos 2x| - \frac{1}{4}\cos^2 2x + c$$

34. (2) We have $\int \frac{dx}{x^2(x^n+1)^{(n-1)/n}} = \int \frac{dx}{x^2 x^{n-1}\left(1+\frac{1}{x^n}\right)^{(n-1)/n}}$

$$= \int \frac{dx}{x^{n+1}(1+x^{-n})^{(n-1)/n}}$$

Put $1 + x^{-n} = t$

$$\therefore \quad -nx^{-n-1}\, dx = dt \text{ or } \frac{dx}{x^{n+1}} = -\frac{dt}{n}$$

$$\therefore \quad \int \frac{dx}{x^2(x^n+1)^{(n-1)/n}} = -\frac{1}{n}\int \frac{dt}{t^{(n-1)/n}}$$

$$= -\frac{1}{n}\int t^{\left(\frac{1}{n}-1\right)} dt$$

$$= -\frac{1}{n}\frac{t^{\frac{1}{n}-1+1}}{\frac{1}{n}-1+1} + C$$

$$= -t^{1/n} + C$$

$$= -(1+x^{-n})^{1/n} + C$$

35. (3) $\int \sqrt{\frac{\cos x - \cos^3 x}{1-\cos^3 x}}\, dx = \int \sqrt{\frac{\cos x}{1-\cos^3 x}} \sin x\, dx$

$$= \int \sqrt{\frac{t}{1-t^3}}\, dt$$

$$= -\int \frac{\sqrt{t}}{\sqrt{1-(t^{3/2})^2}}\, dt, \text{ where } t = \cos x$$

$$= -\frac{2}{3}\int \frac{\frac{3}{2}\sqrt{t}}{\sqrt{1-(t^{3/2})^2}}\, dt = \frac{2}{3}\cos^{-1}(t^{3/2}) + C$$

36. (2) $I = \int x\left(\frac{\ln a^{a^{x/2}}}{3a^{5x/2}b^{3x}} + \frac{\ln b^{b^x}}{2a^{2x}b^{4x}}\right) dx = \int \frac{\ln a^{2x}b^{3x}}{6a^{2x}b^{3x}}\, dx$

Let $a^{2x}\, b^{3x} = t$. Then $t \ln a^2 b^3\, dx = dt$.

Therefore, $I = \int \frac{1}{6\ln a^2 b^3}\frac{\ln t}{t^2} dt$

$$= \frac{1}{6\ln a^2 b^3}\left(\frac{-\ln t}{t} - \int \frac{-1}{t^2} dt\right)$$

$$= -\frac{1}{6\ln a^2 b^3}\left(\frac{\ln et}{t}\right) + k$$

$$= -\frac{1}{6\ln a^2 b^3}\left(\frac{\ln a^{2x}b^{3x}e}{a^{2x}b^{3x}}\right) + k$$

37. (1) Let $I = \int \frac{3+2\cos x}{(2+3\cos x)^2}\, dx$.

Multiplying N^r and D^r by cosec^2 x, we get

$$I = \int \frac{\left(3\,\text{cosec}^2 x + 2\cot x\, \text{cosec}\, x\right)}{(2\,\text{cosec}\, x + 3\cot x)^2}\, dx$$

$$= -\int \frac{-3\,\text{cosec}^2 x - 2\cot x\, \text{cosec}\, x}{(2\,\text{cosec}\, x + 3\cot x)^2}\, dx$$

$$= \frac{1}{2\,\text{cosec}\, x + 3\cot x} + C = \left(\frac{\sin x}{2 + 3\cos x}\right) + C$$

38. (3) $I = \int \frac{xf'(x) - 2f(x)}{x^2\sqrt{f(x)}}\, dx$

$$= \int \frac{\frac{x^2 f'(x) - 2x f(x)}{x^4}}{\sqrt{\frac{f(x)}{x^2}}}\, dx$$

Put $\frac{f(x)}{x^2} = t^2$

$\therefore \quad \frac{x^2 f'(x) - 2x f(x)}{x^4} dx = 2t\,dt$

$\therefore \quad I = \int \frac{2t\,dt}{t} = 2t + c = \frac{2\sqrt{f(x)}}{|x|} + c$

(1) Let $I = \int \frac{(1-\cos\theta)^{2/7}}{(1+\cos\theta)^{9/7}} d\theta$

$= \int \frac{(2\sin^2\theta/2)^{2/7}}{(2\cos^2\theta/2)^{9/7}} d\theta$

$= \frac{1}{2}\int \frac{(\sin\theta/2)^{4/7}}{(\cos\theta/2)^{18/7}} d\theta$

Put $\frac{\theta}{2} = t$ or $\frac{d\theta}{2} = dt$

$\therefore \quad I = \int \frac{(\sin t)^{4/7}}{(\cos t)^{18/7}} dt$ (Here, $m + n = -2$)

$= \int (\tan t)^{4/7} \sec^2 t\,dt$

Put $\tan t = u$ or $\sec^2 t\,dt = du$

$\therefore \quad I = \int u^{4/7} du = \frac{u^{11/7}}{11/7} + c = \frac{7}{11}(\tan t)^{11/7} + C$

$= \frac{7}{11}\left(\tan\frac{\theta}{2}\right)^{11/7} + C$

40. (4) $I = \int \frac{dx}{\sqrt{\sin^3 x \cos^5 x}}$

$= \int \frac{dx}{\sqrt{\frac{\sin^3 x}{\cos^3 x}\cos^8 x}}$

$= \int \frac{\sec^4 x}{\sqrt{\tan^3 x}} dx$

$= \int \frac{(1+\tan^2 x)\sec^2 x}{\sqrt{\tan^3 x}} dx$

Let $t = \tan x$ or $dt = \sec^2 x\,dx$

$\therefore \quad I = \int \frac{1+t^2}{t^{3/2}} dt$

$= \int (t^{-3/2} + t^{1/2})\,dt$

$= -2t^{-\frac{1}{2}} + \frac{2}{3}t^{3/2} + C$

$= -2\sqrt{\cot x} + \frac{2}{3}\sqrt{\tan^3 x} + C$

$\therefore \quad a = -2, b = \frac{2}{3}$

41. (2) $I = \int \frac{dx}{\cos^3 x \sqrt{\sin 2x}}$

$= \int \frac{dx}{\cos^3 x \sqrt{\frac{2\sin x\cos x}{\cos^2 x}\cos^2 x}}$

$= \int \frac{\sec^4 dx}{\sqrt{2\tan x}} = \frac{1}{\sqrt{2}}\int \frac{\sec^2 x(1+\tan^2 x)}{\sqrt{\tan x}} dx$

Let $t = \sqrt{\tan x}$

or $dt = \frac{\sec^2 x\,dx}{2\sqrt{\tan x}}$

$\therefore \quad I = \frac{2}{\sqrt{2}}\int (1+t^4) dt = \sqrt{2}\left(t + \frac{t^5}{5}\right) + C$

$= \frac{\sqrt{2}}{5} t(t^4 + 5) + C$

$= \frac{\sqrt{2}}{5}\sqrt{\tan x}(\tan^2 x + 5) + C$

or $a = \frac{\sqrt{2}}{5}, b = 5$

42. (3) $\int \frac{dx}{(x+2)(x^2+1)} = a\ln(1+x^2) + b\tan^{-1}x + \frac{1}{5}\ln|x+2| + C$

Differentiating both sides, we get

$\frac{1}{(x+2)(x^2+1)} = \frac{2ax}{(1+x^2)} + \frac{b}{(1+x^2)} + \frac{1}{5(x+2)}$

or $\frac{1}{(x+2)(x^2+1)} = \frac{(x+2)(5b+10ax)+1+x^2}{5(1+x^2)(x+2)}$

or $5 = (1+x^2) + 5(b+2ax)(x+2)$

Comparing the like powers of x on both sides, we get

$1 + 10a = 0,\ b + 4a = 0,\ 10b + 1 = 5$

or $a = -\frac{1}{10}, b = \frac{2}{5}$

43. (1) $\int \frac{3e^x - 5e^{-x}}{4e^x + 5e^{-x}} = ax + b\ln(4e^x + 5e^{-x}) + C$

Differentiating both sides, we get

$\frac{3e^x - 5e^{-x}}{4e^x + 5e^{-x}} = a + b\frac{(4e^x - 5e^{-x})}{4e^x + 5e^{-x}}$

or $3e^x - 5e^{-x} = a(4e^x + 5e^{-x}) + b(4e^x - 5e^{-x})$

Comparing the coefficients of like terms on both sides, we get

$3 = 4(a+b), -5 = 5a - 5b$ or $a = -\frac{1}{8}, b = \frac{7}{8}$

44. (1) Differentiating, we get

$\frac{f'(x)}{f(x)^2} = 2(b^2 - a^2)\sin x\cos x$

Integrating both sides w.r.t. x, we get

$-\frac{1}{f(x)} = -b^2\cos^2 x - a^2\sin^2 x$

or $f(x) = \frac{1}{a^2\sin^2 x + b^2\cos^2 x}$

45. (4) $I = \int \frac{x^9 dx}{(4x^2+1)^6} = \int \frac{dx}{x^3\left(4+\frac{1}{x^2}\right)^6}$

$= -\frac{1}{2}\int \frac{d\left(4+\frac{1}{x^2}\right)}{\left(4+\frac{1}{x^2}\right)^6} = -\frac{1}{2}\frac{\left(4+\frac{1}{x^2}\right)^{-5}}{-5} + C$

$= \frac{1}{10}\left(4+\frac{1}{x^2}\right)^{-5} + C$

46. (1) Putting $1-x^3=y^2$, $-3x^2dx=2y\,dy$, we get

$$\int\frac{1}{x\sqrt{1-x^3}}dx=-\frac{2}{3}\int\frac{1}{1-y^2}dy$$

$$=\frac{1}{3}\log\left|\frac{y-1}{y+1}\right|+C$$

$$=\frac{1}{3}\log\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right|+C \text{ or } a=\frac{1}{3}$$

47. (1) Given that

$$I=\int(x^2+x)(x^{-8}+2x^{-9})^{1/10}dx$$

$$=\int(x+1)(x^2+2x)^{1/10}dx$$

Now, put $x^2+2x=t$ and $(x+1)dx=\frac{dt}{2}$.

$$\therefore\quad I=\int t^{1/10}\frac{dt}{2}=\frac{1}{2}\times\frac{10}{11}t^{11/10}+C=\frac{5}{11}t^{11/10}+C$$

$$=\frac{5}{11}(x^2+2x)^{11/10}+C$$

48. (4) $I=\int\frac{x^3dx}{\sqrt{1+x^2}}=\int\frac{x\times x^2dx}{\sqrt{1+x^2}}$

Let $t=\sqrt{1+x^2}$

or $\quad\frac{dt}{dx}=\frac{x}{\sqrt{1+x^2}}$

$$\therefore\quad I=\int(t^2-1)dt$$

$$=\frac{t^3}{3}-t+C=\frac{t}{3}(t^2-3)+C$$

$$=\frac{1}{3}\sqrt{1+x^2}(x^2-2)+C$$

49. (2) Write $I=\int\frac{dx}{x^3\left(a^2/x^2-b^2\right)^{3/2}}$

and put $a^2/x^2=t+b^2$, so that $(-2a^2/x^3)\,dx=dt$

$$\therefore\quad I=\int\frac{\left(-1/2a^2\right)dt}{t^{3/2}}$$

$$=-\frac{1}{2a^2}\int t^{-3/2}\,dt=\frac{1}{a^2\sqrt{t}}+C$$

$$=\frac{1}{a^2\left(a^2/x^2-b^2\right)^{1/2}}+C$$

$$=\frac{x}{a^2\left(a^2-b^2x^2\right)^{1/2}}+C$$

50. (1) $I=\int\frac{x^4-1}{x^2\cdot x\sqrt{x^2+\frac{1}{x^2}+1}}dx=\int\frac{\left(x-\frac{1}{x^3}\right)dx}{\sqrt{x^2+\frac{1}{x^2}+1}}$

Put $x^2+\frac{1}{x^2}+1=t$ or $2\left(x-\frac{1}{x^3}\right)dx=dt$

$$\therefore\quad I=\frac{1}{2}\int\frac{dt}{\sqrt{t}}=\frac{1}{2}\cdot2\sqrt{t}+C=\sqrt{t}+C$$

$$=\sqrt{x^2+\frac{1}{x^2}+1}+C=\frac{\sqrt{x^4+x^2+1}}{x}+C$$

51. (1) Put $x=\tan\theta$ so that $\sqrt{x^2+1}=\sec\theta$, $dx=\sec^2\theta\,d\theta$

$$\therefore\quad I=\int\frac{\sec\theta\sec^2\theta}{\tan^4\theta}d\theta=\int\frac{\cos\theta}{\sin^4\theta}d\theta$$

$$=-\frac{1}{3}\frac{1}{\sin^3\theta}+C$$

$$=-\frac{1}{3}\frac{\sec^3\theta}{\tan^3\theta}+C$$

$$=-\frac{1}{3}\frac{(x^2+1)^{3/2}}{x^3}+C$$

52. (2) $I=\int\frac{1+2x^6}{(1-x^6)^{3/2}}dx=\int\frac{\frac{1}{x^3}+2x^3}{\left(\frac{1}{x^2}-x^4\right)^{\frac{3}{2}}}dx$

put $\quad\frac{1}{x^2}-x^4=t$

$$\therefore\quad\left(\frac{-2}{x^3}-4x^3\right)dx=dt$$

or $\quad\left(\frac{1}{x^3}+2x^3\right)dx=-\frac{dt}{2}$

$$\therefore\quad I=-\frac{1}{2}\int\frac{dt}{t^{\frac{3}{2}}}$$

$$=-\frac{1}{2}\frac{t^{-1/2}}{-1/2}+c$$

$$=\frac{1}{\sqrt{t}}+c$$

$$=\frac{1}{\sqrt{\frac{1}{x^2}-x^4}}+c$$

53. (3) $I=\int\frac{2x^{12}+5x^9}{(x^5+x^3+1)^3}dx$

$$=\int\frac{\left(\frac{2}{x^3}+\frac{5}{x^6}\right)dx}{\left(1+\frac{1}{x^2}+\frac{1}{x^5}\right)^3}$$

$$=\int\frac{-dt}{t^3}\text{, where t}=1+\frac{1}{x^2}+\frac{1}{x^5}$$

$$=\frac{1}{2t^2}+C$$

$$=\frac{1}{2\left(1+\frac{1}{x^2}+\frac{1}{x^5}\right)^2}+c$$

$$=\frac{x^{10}}{2\left(x^5+x^3+1\right)^2}+c$$

54. (3) $I_n=x(\ln x)^n-\int\frac{x(n)(\ln x)^{n-1}}{x}dx+C=x(\ln x)^n-nI_{(n-1)}$

or $\quad I_n+nI_{n-1}=x(\ln x)^n+C$

(3) Here, $\int e^x\{f(x)-f'(x)\}dx=\phi(x)$

and $\int e^x\{f(x)+f'(x)\}dx=e^x f(x)$

On adding, we get $2\int e^x f(x)dx=\phi(x)+e^x f(x)$

(4) $I=\int\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Let $x=\tan\theta$

or $dx=\sec^2\theta\,d\theta$

$\therefore\quad I=\int\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta d\theta$

$=2\int\theta\sec^2\theta\,d\,\theta$

$=2\,(\theta\,\tan\theta-\ln|\sec\theta|)+C$

$=2(x\tan^{-1}x-\ln|\sec(\tan^{-1}x)|)+C$

(1) $I=\int x\frac{\ln(x+\sqrt{x^2+1})}{\sqrt{x^2+1}}dx$

Let $t=\sqrt{x^2+1}$

or $\frac{dt}{dx}=\frac{x}{\sqrt{x^2+1}}$

$\therefore\quad I=\int\ln(t+\sqrt{t^2-1})dt$

$=\ln(t+\sqrt{t^2-1})\,t-\int\frac{1+\frac{t}{\sqrt{t^2-1}}}{t+\sqrt{t^2-1}}t\,dt$

$=t\ \ln(t+\sqrt{t^2-1})-\frac{1}{2}\int\frac{2t}{\sqrt{t^2-1}}dt$

$=t\ln(t+\sqrt{t^2-1})-\sqrt{t^2-1}+C$

$=\sqrt{1+x^2}\ \ln(x+\sqrt{1+x^2})-x+C$

or $a=1,\ b=-1$

(4) $\int x\log\left(1+\frac{1}{x}\right)dx$

$=\int x\log(x+1)dx-\int x\log x dx$

$=\frac{x^2}{2}\log(x+1)-\frac{1}{2}\int\frac{x^2}{x+1}dx-\frac{x^2}{2}\log x+\frac{1}{2}\int\frac{x^2}{x}dx+C$

$=\frac{x^2}{2}\log(x+1)-\frac{1}{2}\int\left(x-1+\frac{1}{x+1}\right)dx-\frac{x^2}{2}\log x+\frac{1}{4}x^2+C$

$=\frac{x^2}{2}\log(x+1)-\frac{x^2}{2}\log x-\frac{1}{2}\left(\frac{x^2}{2}-x\right)$

$-\frac{1}{2}\log(x+1)+\frac{1}{4}x^2+C$

$=\frac{x^2}{2}\log(x+1)-\frac{x^2}{2}\log x-\frac{1}{2}\log(x+1)+\frac{1}{2}x+C$

Hence, $f(x)=\frac{x^2}{2}-\frac{1}{2}$, $g(x)=-\frac{1}{2}\log x$, and $A=\frac{1}{2}$.

59. (3) $I=-e^{-x}\log\left(e^x+1\right)+\int\frac{e^{-x}e^x}{e^x+1}dx$

$=-e^{-x}\log(e^x+1)+\int\frac{e^{-x}}{e^{-x}+1}dx$

$=-e^{-x}\log(e^x+1)-\log(e^{-x}+1)+C$

$=-e^{-x}\log(e^x+1)-\log(1+e^x)+x+C$

$=-(e^{-x}+1)\log(e^x+1)+x+C$

60. (2) $I=\int xe^x\cos x dx$

$=xe^x\sin x-\int(xe^x+e^x)\sin x dx$

$=xe^x\sin x-xe^x(-\cos x)-\int(xe^x+e^x)\cos x\,dx-\int e^x\sin x dx$

$=xe^x\sin x+x\,e^x\cos x-\int xe^x\cos x dx-\int e^x(\cos x+\sin x)dx$

or $2I=xe^x(\sin x+\cos x)-e^x\sin x+d$

$=e^x((x-1)\sin x+x\cos x)+d$

or $I=\frac{1}{2}e^x((x-1)\sin x+x\cos x)+d$

or $a=\frac{1}{2},\ b=-1,\ c=1$

61. (2) $\int x\sin x\sec^3 x\,dx=\int x\sin x\frac{1}{\cos^3 x}dx$

$=\int x\tan x\sec^2 x\,dx$

$=x\int\sec x(\sec x\tan x)dx-\int\left[\int\sec x(\sec x\tan x)dx\right]dx+C$

$=x\frac{\sec^2 x}{2}-\int\frac{\sec^2 x}{2}dx+C$

$=x\frac{\sec^2 x}{2}-\frac{\tan x}{2}+C$

62. (3) $I=\int e^{\tan^{-1}x}(1+x+x^2)\left(-\left(\frac{1}{1+x^2}\right)dx\right)$

$=-\int e^{\tan^{-1}x}\left(1+\frac{x}{1+x^2}\right)dx$

$=-\int e^{\tan^{-1}x}dx-\int x\frac{e^{\tan^{-1}x}}{1+x^2}dx$

$=-\int e^{\tan^{-1}x}dx-x\,e^{\tan^{-1}x}+\int e^{\tan^{-1}x}dx+C$

$=-x\,e^{\tan^{-1}x}+C$

63. (2) $\int e^x\left(\frac{2\tan x}{1+\tan x}+\tan^2\left(x-\frac{\pi}{4}\right)\right)dx$

$=\int e^x\left(\tan\left(x-\frac{\pi}{4}\right)+\sec^2\left(x-\frac{\pi}{4}\right)\right)dx$

$=e^x\tan\left(x-\frac{\pi}{4}\right)+C$

64. (4) Putting $x^2=t$,

$I=\frac{1}{2}\int e^{t^2}(1+t+2t^2)e^t dt$

$=\frac{1}{2}\int e^t[te^{t^2}+(e^{t^2}+2t^2e^{t^2})]dt$

$=\frac{1}{2}\int e^t[f(t)+f'(t)]dt=\frac{1}{2}e^t(te^{t^2})+C$ where $t=x^2$

65. (1) $\int e^x\left(\frac{1}{\sqrt{1+x^2}}-\frac{x}{\sqrt{(1+x^2)^3}}+\frac{x}{\sqrt{(1+x^2)^3}}+\frac{1-2x^2}{\sqrt{(1+x^2)^5}}\right)$

$=e^x\frac{1}{\sqrt{1+x^2}}+e^x\frac{x}{\sqrt{(1+x^2)^3}}=e^x\left(\frac{1}{\sqrt{1+x^2}}+\frac{x}{\sqrt{(1+x^2)^3}}\right)+C$

Using $\int e^x(f(x)+f'(x))dx$, we get $e^x f(x)+c$

66. (1) $\int\frac{e^x(x^2+1)}{(x+1)^2}dx=\int\frac{e^x(x^2-1+2)}{(x+1)^2}dx$

$=\int e^x\left[\frac{x-1}{x+1}+\frac{2}{(x+1)^2}\right]dx$

$=\int e^x[f(x)+f'(x)]\,dx,$

$=e^x\left(\frac{x-1}{x+1}\right)+C$

67. (1) $I=\int\left(\frac{x+2}{x+4}\right)^2 e^x\,dx=\int e^x\left[\frac{x^2+4x+4}{(x+4)^2}\right]dx$

$=\int e^x\left[\frac{x(x+4)}{(x+4)^2}+\frac{4}{(x+4)^2}\right]dx$

$=\int e^x\left[\frac{x}{x+4}+\frac{4}{(x+4)^2}\right]dx$

$=e^x\left(\frac{x}{x+4}\right)+C$

68. (3) $I=\int e^{\tan x}(\sin x-\sec x)dx$

$=\int\sin x\,e^{\tan x}dx-\int\sec x\,e^{\tan x}dx$

$=-e^{\tan x}\cos x+\int\cos x\,e^{\tan x}\sec^2 x\,dx-\int\sec x\,e^{\tan x}dx$

$=-\cos x\;e^{\tan x}+C$

69. (4) $\int\frac{\operatorname{cosec}^2 x-2005}{\cos^{2005}x}dx$

$=\int(\cos x)^{-2005}\operatorname{cosec}^2 x\,dx-2005\int\frac{dx}{\cos^{2005}x}$

$=(\cos x)^{-2005}(-\cot x)$

$-\int(-2005)(\cos x)^{-2006}(-\sin x)(-\cot x)dx-2005\int\frac{dx}{\cos^{2005}x}$

$=-\frac{\cot x}{(\cos x)^{2005}}+C$

70. (2) $\int\left(1+2x^2+\frac{1}{x}\right)e^{x^2-\frac{1}{x}}dx$

$=\int\left[e^{x^2-\frac{1}{x}}+x\left(2x+\frac{1}{x^2}\right)e^{x^2-\frac{1}{x}}\right]dx$

$=xe^{x^2-\frac{1}{x}}-\int e^{x^2-\frac{1}{x}}\left(2x+\frac{1}{x^2}\right)xdx+\int x\left(2x+\frac{1}{x^2}\right)e^{x^2-\frac{1}{x}}dx$

$=xe^{x^2-\frac{1}{x}}+c$

71. (1) $\int e^{\sin^{-1}x}\left(\frac{\log_e x}{\sqrt{1-x^2}}+\frac{1}{x}\right)dx$

$=\int e^{\sin^{-1}x}\frac{\log_e x}{\sqrt{1-x^2}}dx+\int\frac{e^{\sin^{-1}x}}{x}dx$

$=\log_e x\int\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}dx-\int\left((\log_e x)'\int\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}dx\right)dx+\int\frac{e^{\sin^{-1}x}}{x}dx$

$=\log_e x\cdot e^{\sin^{-1}x}-\int\frac{1}{x}e^{\sin^{-1}x}dx+\int\frac{e^{\sin^{-1}x}}{x}dx$

$=\log_e x\cdot e^{\sin^{-1}x}+c$

72. (1) $f'(x)=\frac{f(x)}{6f(x)-x}$

Now, $I=\int\frac{2x(x-6f(x))+f(x)}{(6f(x)-x)(x^2-f(x))^2}dx$

$=-\int\frac{2x-f'(x)}{(x^2-f(x))^2}dx$

$=\frac{1}{x^2-f(x)}+C$

73. (3) Let $I=\int\frac{\left(ax^2-b\right)dx}{x\sqrt{c^2x^2-\left(ax^2+b\right)^2}}$

$=\int\frac{\left(a-\frac{b}{x^2}\right)dx}{\sqrt{c^2-\left(ax+\frac{b}{x}\right)^2}},$

Put $ax+\frac{b}{x}=t$

$\therefore\left(a-\frac{b}{x^2}\right)dx=dt$

$=\int\frac{dt}{\sqrt{c^2-t^2}}$

$=\sin^{-1}\left(\frac{t}{c}\right)+k$

$=\sin^{-1}\left(\frac{ax+\frac{b}{x}}{c}\right)+C$

74. (4) Let $I=\int\frac{dx}{\left(1+\sqrt{x}\right)\sqrt{\left(x-x^2\right)}}$

If $\sqrt{x}=\sin p$, then $\frac{1}{2\sqrt{x}}dx=\cos p\,dp$

$\therefore\quad I=\int\frac{2\sin p\cos p\,dp}{(1+\sin p)\sin p\cos p}$

$=2\int\frac{dp}{(1+\sin p)}$

$=2\int\frac{(1-\sin p)\,dp}{\cos^2 p}$

$=2\left\{\int\sec^2 p\,dp-\int(\tan p\sec p)\,dp\right\}$

$=2(\tan p-\sec p)+C$

$=2\left(\sqrt{\frac{x}{(1-x)}}-\frac{1}{\sqrt{(1-x)}}\right)+C=\frac{2\left(\sqrt{x}-1\right)}{\sqrt{(1-x)}}+C$

(3) $I=\int\frac{2\sin x}{(3+\sin 2x)}dx$

$=\int\frac{\sin x+\cos x+\sin x-\cos x}{(3+\sin 2x)}dx$

$=\int\frac{\sin x+\cos x}{3+\sin 2x}dx-\int\frac{-\sin x+\cos x}{(3+\sin 2x)}dx$

$\downarrow I_1$ $\downarrow I_2$

Putting $t_1=\sin x-\cos x$ in I_1 and $t_2=\sin x+\cos x$ in I_2, we get

$$I=\int\frac{dt_1}{[3+(1-t_1^2)]}-\int\frac{dt_2}{[3+(t_2^2-1)]}=\int\frac{dt_1}{4-t_1^2}-\int\frac{dt_2}{2+t_2^2}$$

$$=\frac{1}{4}\ln\left|\frac{2+t_1}{2-t_1}\right|-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t_2}{\sqrt{2}}\right)+C$$

$$=\frac{1}{4}\ln\left|\frac{2+\sin x-\cos x}{2-\sin x+\cos x}\right|-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)+C$$

(3) Putting $a^6+x^8=t^2$, we get

$$I=\int\frac{t^2}{t^2-a^6}dt=t+\frac{a^3}{2}\ln\left|\frac{t-a^3}{t+a^3}\right|+C$$

(3) $I_{4,3}=\int\cos^4 x\sin 3x dx$

Integrating by parts, we have

$$I_{4,3}=-\frac{\cos 3x\cos^4 x}{3}-\frac{4}{3}\int\cos^3 x\sin x\cos 3x dx$$

But $\sin x\cos 3x=-\sin 2x+\sin 3x\cos x$. So,

$$I_{4,3}=-\frac{\cos x\cos^4 x}{3}+\frac{4}{3}\int\cos^3 x\sin 2x dx$$

$$-\frac{4}{3}\int\cos^4 x\sin 3x dx+C$$

$$=-\frac{\cos 3x\cos^4 x}{3}+\frac{4}{3}I_{3,2}-\frac{4}{3}I_{4,3}+C$$

Therefore, $\frac{7}{3}I_{4,3}-\frac{4}{3}I_{3,2}=-\frac{\cos 3x\cos^3 x}{3}+C$

or $7I_{4,3}-4I_{3,2}=-\cos 3x\cos^4 x+C$

Multiple Correct Answers Type

1. **(1), (2), (4)**

$$I=\int\frac{dx}{\sqrt{2e^x-1}}$$

Let $2e^x-1=t^2$

$\therefore\quad 2e^x dx=2t dt$

$\therefore\quad I=\int\frac{t\,dt}{\frac{t^2+1}{2}\cdot t}$

$=2\tan^{-1}t+C$

$=2\tan^{-1}\sqrt{2e^x-1}+C$

$=-2\tan^{-1}\frac{1}{\sqrt{2e^x-1}}+C$

Also, $\tan^{-1}\sqrt{2e^x-1}=\sec^{-1}\sqrt{2e^x}$

2. **(2), (4)**

$$\int\sin x d(\sec x)=\int\sin x\frac{d(\sec x)}{dx}dx$$

$$=\int\sin x\sec x\tan x dx$$

$$=\int\tan^2 x dx$$

$$=\int(\sec^2 x-1)dx$$

$$=\tan x-x+C$$

$\therefore\quad f(x)=\tan x,\ g(x)=x$

3. **(1), (4)**

$$I=\int\frac{\sqrt{(1+\sin x)(1-\sin x)}}{\sqrt{\sin x(1-\sin x)}}dx$$

$$=\int\frac{\cos x}{\sqrt{\sin x(1-\sin x)}}dx$$

$$=\int\frac{\cos x}{\sqrt{\frac{1}{4}-\left(\frac{1}{2}-\sin x\right)^2}}dx$$

$$=\int\frac{-dt}{\sqrt{\left(\frac{1}{2}\right)^2-t^2}}\qquad\left(\text{Putting }\frac{1}{2}-\sin x=t\right)$$

$$=-\sin^{-1}\left(\frac{t}{1/2}\right)+C$$

$$=-\sin^{-1}(1-2\sin x)+C$$

$$=\cos^{-1}(1-2\sin x)+C-\frac{\pi}{2}$$

$$=\cos^{-1}(1-2\sin x)+C$$

$$=\cos^{-1}\left(1-2\left(\sqrt{\sin x}\right)^2\right)+C$$

$$=\cos^{-1}(1-2\sin^2 t)+C\qquad(\text{Putting }\sqrt{\sin x}=\sin t)$$

$$=\cos^{-1}(\cos 2t)+C$$

$$=2t+C\qquad\left[\because\sqrt{\sin x}>0\Rightarrow\sin t>0\Rightarrow t\in\left(0\,\frac{\pi}{2}\right)\right]$$

$$=2\sin^{-1}(\sqrt{\sin x})+C$$

4. **(1), (3)**

$$I=\int\sec^2 x\,\mathrm{cosec}^4 x dx$$

$$=\int\frac{(\sin^2 x+\cos^2 x)^2}{\cos^2 x\sin^4 x}dx$$

$$=\int\frac{\sin^4 x+\cos^4 x+2\sin^2 x\cos^2 x}{\cos^2 x\sin^4 x}dx$$

$$=\int\left(\sec^2 x+2\,\mathrm{cosec}^2 x+\frac{\cos^2 x}{\sin^4 x}\right)dx$$

$$=\tan x-2\cot x+\int\cot^2 x\,\mathrm{cosec}^2 x dx$$

$$=\tan x-2\cot x-\frac{\cot^3 x}{3}+D$$

5. (1), (3)

$$g(x) = \int x^{27}(1+x+x^2)^6(6x^2+5x+4)\,dx$$

$$= \int (x^4+x^5+x^6)^6(6x^5+5x^4+4x^3)\,dx$$

Let $x^6+x^5+x^4 = t$ or $(6x^5+5x^4+4x^3)dx = dt$

$$\therefore \quad g(x) = \int t^6 dt = \frac{t^7}{7} + C = \frac{1}{7}(x^4+x^5+x^6)^7 + C$$

$$g(0) = 0 \Rightarrow C = 0 \Rightarrow g(1) = \frac{3^7}{7} \text{ and } g(-1) = \frac{1}{7}$$

6. (2), (4)

$$I = \int \sqrt{\operatorname{cosec} x + 1}\,dx = \int \frac{\cot x}{\sqrt{\operatorname{cosec} x - 1}}\,dx$$

Put $\operatorname{cosec} x - 1 = t^2$ or $-\operatorname{cosec} x \cot x\,dx = 2t\,dt$

$$\therefore \quad I = -\int \frac{-\cot x \operatorname{cosec} x}{\operatorname{cosec} x\sqrt{\operatorname{cosec} x - 1}}\,dx$$

$$= -\int \frac{2dt}{1+t^2} = -2\tan^{-1} t + c$$

$$= -2\tan^{-1}\sqrt{\operatorname{cosec} x - 1} + C$$

$$= -2\left[\frac{\pi}{2} - \cot^{-1}\sqrt{\operatorname{cosec} x - 1}\right] + C$$

$$= 2\cot^{-1}\sqrt{\operatorname{cosec} x - 1} + C$$

$$= 2\cot^{-1}\frac{\cot x}{\sqrt{\operatorname{cosec} x + 1}} + C$$

7. (1), (3)

Let $\cos x = t$ so, $\cos 2x = 2t^2 - 1$

and $dt = -\sin x\,dx$. Thus,

$$I = \int \frac{t^2-2}{2t^2-1}dt$$

$$= \frac{1}{2}\int \frac{2t^2-4}{2t^2-1}dt$$

$$= \frac{1}{2}\int dt - \frac{3}{2}\int \frac{dt}{2t^2-1}$$

$$= \frac{1}{2}t - \frac{3}{2\sqrt{2}} \times \frac{1}{2}\log\left|\frac{\sqrt{2}\,t-1}{\sqrt{2}\,t+1}\right| + C$$

$$= \frac{1}{2}\cos x - \frac{3}{4\sqrt{2}}\log\left|\frac{\sqrt{2}\cos x - 1}{\sqrt{2}\cos x + 1}\right| + C$$

So, $P = 1/2, Q = -\frac{3}{4\sqrt{2}}, f(x) = \frac{\sqrt{2}\cos x - 1}{\sqrt{2}\cos x + 1}$

or $P = 1/2, Q = \frac{3}{4\sqrt{2}}, f(x) = \frac{\sqrt{2}\cos x + 1}{\sqrt{2}\cos x - 1}$

8. (1), (4)

$$\frac{2x}{(x-1)(x-4)} = \frac{C}{x-1} + \frac{D}{x-4}$$

$$2x = C(x-4) + D(x-1)$$

$\therefore \quad C = -2/3, D = 8/3$

$$\therefore \quad \int \frac{e^{x-1}}{(x-1)(x-4)}2x\,dx = \int e^{x-1}\left(\frac{-2/3}{x-1} + \frac{8/3}{x-4}\right)dx$$

$$= -\frac{2}{3}F(x-1) + \frac{8}{3}e^3 F(x-4) + C$$

$\therefore \quad A = -2/3, B = 8/3\,e^3$

9. (1), (3), (4)

$$\int x^2 e^{-2x}dx = e^{-2x}(ax^2+bx+c) + d$$

Differentiating both sides, we get

$$x^2 e^{-2x} = e^{-2x}(2ax+b) + (ax^2+bx+c)(-2e^{-2x})$$

$$= e^{-2x}(-2ax^2 + 2(a-b)x + b - 2c)$$

or $\quad a = \frac{-1}{2}, 2(a-b) = 0, b - 2c = 0$

or $\quad b = \frac{-1}{2}, c = \frac{-1}{4}$

10. (1), (3), (4)

Let $I = \int \frac{(x^4+1)}{(x^6+1)}dt = \int \frac{(x^2+1)^2 - 2x^2}{(x^2+1)(x^4-x^2+1)}dx$

$$= \int \frac{(x^2+1)dx}{(x^4-x^2+1)} - 2\int \frac{x^2dx}{(x^6+1)}$$

$$= \int \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x^2-1+\frac{1}{x^2}\right)} - 2\int \frac{x^2dx}{(x^3)^2+1}$$

In the first integral, put $x - \frac{1}{x} = t$, i.e., $\left(1+\frac{1}{x^2}\right)dx = dt$

and in the second integral put $x^3 = u$, i.e., $x^2dx = \frac{du}{3}$

Then $I = \int \frac{dt}{1+t^2} - \frac{2}{3}\int \frac{du}{1+u^2} = \tan^{-1} t - \frac{2}{3}\tan^{-1} u + C$

$$= \tan^{-1}\left(x - \frac{1}{x}\right) - \frac{2}{3}\tan^{-1}(x^3) + C$$

Here, $f(x) = x - \frac{1}{x}$ and $g(x) = x^3$

Draw the graphs of $f(x)$ and $g(x)$.

We find that $f(x)$ is many-one and $f(x) = g(x)$ has no real roots.

$$\int \frac{f(x)}{g(x)}dx = \int \frac{x - \frac{1}{x}}{x^3}dx = \int\left(\frac{1}{x^2} - \frac{1}{x^4}\right)dx = -\frac{1}{x} + \frac{3}{x^3} + C$$

11. (1), (2), (3)

$$I = \int \frac{x^2-x+1}{(x^2+1)^{3/2}}e^x dx$$

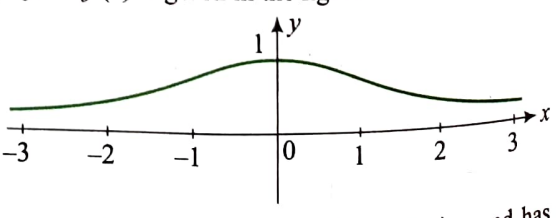
$$= \int e^x\left[\frac{x^2+1}{(x^2+1)^{3/2}} - \frac{x}{(x^2+1)^{3/2}}\right]dx$$

$$= \int e^x\left[\frac{1}{\sqrt{x^2+1}} + \left\{\frac{-x}{(x^2+1)^{3/2}}\right\}\right]dx$$

$$= \int e^x[f(x) + f'(x)]dx, \text{ where } f(x) = \frac{1}{\sqrt{x^2+1}}$$

$$= e^x f(x) + c = \frac{e^x}{\sqrt{x^2+1}} + c$$

The graph of $f(x)$ is given in the figure.



From the graph, $f(x)$ is even, bounded function and has t
range $(0, 1]$.

12. (1), (3)

$$\int \frac{\cos^2 2x \sin 2x\, dx}{\cos 2x} = \frac{1}{2}\int \sin 4x\, dx = -\frac{1}{8}\cos 4x + B$$

13. (1), (4)

$$\int \sin^{-1} x \cos^{-1} x\, dx = \int \left[\frac{\pi}{2}\sin^{-1} x - \left(\sin^{-1} x\right)^2\right] dx$$

$$= \frac{\pi}{2}\left(x\sin^{-1}x + \sqrt{1-x^2}\right) - \left(x\left(\sin^{-1}x\right)^2 + 2\sin^{-1}x\sqrt{1-x^2} - 2x\right) + C$$

(Integrating by parts)

$$= \sin^{-1} x\left[\frac{\pi}{2}x - x\sin^{-1}x - 2\sqrt{1-x^2}\right] + \frac{\pi}{2}\sqrt{1-x^2} + 2x + C$$

$\therefore \quad f^{-1}(x) = \sin^{-1} x, f(x) = \sin x$

14. (1), (2), (3), (4)

$$\int \frac{\left(x^8 + 4 + 4x^4\right) - 4x^4}{x^4 - 2x^2 + 2} dx$$

$$= \int \frac{\left(x^4+2\right)^2 - \left(2x^2\right)^2}{\left(x^4 - 2x^2 + 2\right)} dx$$

$$= \int \frac{\left(x^4 + 2 - 2x^2\right)\left(x^4 + 2 + 2x^2\right)}{\left(x^4 - 2x^2 + 2\right)} dx$$

$$= \frac{x^5}{5} + \frac{2x^3}{3} + 2x + C$$

15. (1), (2), (4)

$$\int \frac{dx}{x^2 + ax + 1} = \int \frac{dx}{\left(x + \frac{a}{2}\right)^2 + \left(1 - \frac{a^2}{4}\right)}$$

16. (1), (4)

$$I = \int \frac{1 - x^7}{x(1 + x^7)} dx = a\log_e |x| + b\log_e |1 + x^7| + c$$

Differentiating both sides w.r.t. x, we get

$$\frac{1 - x^7}{x(1 + x^7)} = \frac{a}{x} + b \cdot \frac{7x^6}{1 + x^7}$$

$\Rightarrow \quad 1 - x^7 = a(1 + x^7) + 7bx^7$

$\Rightarrow \quad a = 1, a + 7b = -1$

$\Rightarrow \quad b = -2/7$

17. (3), (4)

Differentiating both sides w.r.t. x, we get

$$\frac{3\sin x + 2\cos x}{3\cos x + 2\sin x} = a + \frac{b(2\cos x - 3\sin x)}{(2\sin x + 3\cos x)}$$

$$= \frac{\sin x \,.\, (2a - 3b) + \cos x \,.\, (3a + 2b)}{(3\cos x + 2\sin x)}$$

Comparing like terms on both sides, we get

$3 = 2a - 3b, 2 = 3a + 2b$

$\Rightarrow \quad a = \frac{12}{13}, b = -\frac{15}{39}$

Linked Comprehensive Type

For Problems 1–3

1. (4), 2. (2), 3. (1)

From the given data, we can conclude that $\frac{dy}{dx} = 0$, at $x = 1, 2, 3$.

Hence, $f'(x) = a(x-1)(x-2)(x-3), a > 0$

$\Rightarrow \quad f(x) = \int a(x^3 - 6x^2 + 11x - 6)dx$

$$= a\left(\frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 6x\right) + C$$

Also, $f(0) = 1 \Rightarrow c = 1$

$$\therefore \quad f(x) = a\left(\frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 6x\right) + 1 \qquad (1)$$

So, graph is symmetrical about line $x = 2$ and range is $[f(1), \infty)$ or $[f(3), \infty)$.

1. (4)

$f(1) = -8$

$$\Rightarrow \quad a\left(-\frac{9}{4}\right) + 1 = -8$$

$\Rightarrow \quad a = 4$

$\therefore \quad f(2) = -7$

2. (2)

For $f(1) = -8, a = 4$.

Clearly, $x = 1$ and 3 are points of minima and $x = 2$ is point of maxima.

Also, $f(3) = -8$

So, range is $[f(1), \infty)$ or $[-8, \infty)$.

3. (1)

For $f(x) = 0$, we have $\frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 6x = -\frac{1}{a}$

For $x = 1, \; -\frac{9}{4} = -\frac{1}{a}$ or $a = \frac{4}{9}$

For $x = 2, \; -2 = -\frac{1}{a}$ or $a = \frac{1}{2}$

So, $f(x) = 0$ has four roots if $a \in \left[\frac{4}{9}, \frac{1}{2}\right]$.

For Problems 4–6

4. (1), 5. (2), 6. (3)

$$A = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$$

$$\Rightarrow \quad A^2 = \begin{bmatrix} 2x^2 & 2x^2 \\ 2x^2 & 2x^2 \end{bmatrix}, A^3 = \begin{bmatrix} 2^2x^3 & 2^2x^3 \\ 2^2x^3 & 2^2x^3 \end{bmatrix} \text{ and so on}$$

Then $e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots +$

$$= \begin{bmatrix} 1 + x + \frac{2x^2}{2!} + \frac{2^2x^3}{3!} + \cdots & x + \frac{2x^2}{2!} + \frac{2^2x^3}{3!} + \cdots \\ x + \frac{2x^2}{2!} + \frac{2^2x^3}{3!} + \cdots & 1 + x + \frac{2x^2}{2!} + \frac{2^2x^3}{3!} + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}\left(1+2x+\frac{2^2x^2}{2!}+\frac{2^3x^3}{3!}+\cdots\right)+\frac{1}{2} & \frac{1}{2}\left(1+2x+\frac{2^2x^2}{2!}+\cdots\right)-\frac{1}{2} \\ \frac{1}{2}\left(1+2x+\frac{2^2x^2}{2!}+\frac{2^3x^3}{3!}+\cdots\right)-\frac{1}{2} & \frac{1}{2}\left(1+2x+\frac{2^2x^2}{2!}+\cdots\right)+\frac{1}{2} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e^{2x}+1 & e^{2x}-1 \\ e^{2x}-1 & e^{2x}+1 \end{bmatrix}$$

$\therefore \quad f(x) = e^{2x} + 1$ and $g(x) = e^{2x} - 1$

4. (1) $\int \frac{e^{2x}-1}{e^{2x}+1}dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx = \log|e^x - e^{-x}| + C$

5. (2) $\int (g(x)+1)\sin x dx = \int e^{2x}\sin x dx = \frac{e^{2x}}{5}(2\sin x - \cos x)$

6. (3) $\int \frac{e^{2x}+1}{\sqrt{e^{2x}-1}}dx = \int \frac{e^{2x}}{\sqrt{e^{2x}-1}}dx + \int \frac{1}{\sqrt{e^{2x}-1}}dx$

$$= \int \frac{e^{2x}}{\sqrt{e^{2x}-1}}dx + \int \frac{e^x}{e^x\sqrt{e^{2x}-1}}dx$$

$$= \frac{1}{2\sqrt{e^{2x}-1}} + \sec^{-1}(e^x) + C$$

For Problems 7–9

7. (4) Here, $a = 1 > 0$. Therefore, we make the substitution $\sqrt{x^2+2x+2} = t - x$. Squaring both sides, we get

$$2x + 2tx = t^2 - 2 \text{ or } x = \frac{t^2-2}{2(1+t)} \text{ or } dx = \frac{t^2+2t+2}{2(1+t)^2}dt$$

$$1 + \sqrt{x^2+2x+2} = 1 + t - \frac{t^2-2}{2(1+t)} = \frac{t^2+4t+4}{2(1+t)}.$$

Substituting into the integral, we get

$$I = \int \frac{2(1+t)(t^2+2t+2)}{(t^2+4t+4)\,2(1+t)^2}dt = \int \frac{(t^2+2t+2)\,dt}{(1+t)(t+2)^2}$$

Now, let us expand the obtained proper rational fraction into partial fractions:

$$\frac{t^2+2t+2}{(t+1)(t+2)^2} = \frac{A}{t+1} + \frac{B}{t+2} + \frac{D}{(t+2)^2}.$$

8. (2) $I = \int \frac{dx}{x + \sqrt{x^2 - x + 1}}$

Since here $c = 1 > 0$, we can apply the second Euler substitution:

$$\sqrt{x^2 - x + 1} = tx - 1$$

or $\quad (2t-1)x = (t^2-1)x^2; x = \frac{2t-1}{t^2-1}$

Substituting into I, we get an integral of a rational fraction:

$$\int \frac{dx}{x + \sqrt{x^2 - x + 1}} = \int \frac{-2t^2 + 2t - 2}{t(t-1)(t+1)^2}dt$$

Now, $\frac{-2t^2+2t-2}{t(t-1)(t+1)} = \frac{A}{t} + \frac{B}{t-1} + \frac{D}{(t+1)^2} + \frac{E}{t+1}$

9. (1) In this case, $a < 0$ and $c < 0$. Therefore, neither the first nor the second Euler substitution is applicable. But the quadratic $7x - 10 - x^2$ has real roots $\alpha = 2$, $\beta = 5$. Therefore, we use the third Euler substitution:

$$\sqrt{7x - 10 - x^2} = \sqrt{(x-2)(5-x)} = (x-2)t$$

or $\quad 5 - x = (x-2)t^2$

or $\quad x = \frac{5+2t^2}{t^2+1}$

For Problems 10 and 11

10. (2) 11. (1)

$$f(x) = \int \frac{x^2dx}{(1+x^2)(1+\sqrt{1+x^2})}$$

let $\quad x = \tan\theta$

$\Rightarrow \quad dx = \sec^2\theta\, d\theta = (1+x^2).\, d\theta$

$$\Rightarrow \quad f(x) = \int \frac{x^2dx}{(1+x^2)(1+\sqrt{1+x^2})} = \int \frac{\tan^2\theta\sec^2\theta d\theta}{\sec^2\theta(1+\sec\theta)}$$

$$= \int \frac{\tan^2\theta\, d\theta}{1+\sec\theta}$$

$$= \int \frac{\sin^2\theta\, d\theta}{\cos\theta(1+\cos\theta)}$$

$$= \int \frac{1-\cos^2\theta\, d\theta}{\cos\theta(1+\cos\theta)}$$

$$= \int \frac{(1-\cos\theta)\,d\theta}{\cos\theta}$$

$$= \int \sec\theta d\theta - \int d\theta$$

$$= \log(x + \sqrt{1+x^2}) - \tan^{-1}x + c$$

Given $f(0) = 0$

$\Rightarrow \quad 0 = \log 1 - 0 + c$

$\Rightarrow \quad c = 0$

$\Rightarrow \quad f(1) = \log(1 + \sqrt{1+1^2}) - \tan^{-1}(1)$

$$= \log(1+\sqrt{2}) - \frac{\pi}{4}.$$

$$f'(x) = \frac{\sqrt{1+x^2}-1}{(1+x^2)} > 0 \text{ for all real } x.$$

So, $f(x)$ is an increasing function.

For Problems 12 and 13

12. (3), 13.(1)

$$\int \frac{x(x-1)}{(x^2+1)(x+1)\sqrt{x^3+x^2+x}}dx$$

$$= \int \frac{x(x-1)}{(x^2+1)(x+1)x\sqrt{x+1+\frac{1}{x}}}dx$$

$$= \int \frac{x^2-1}{(x^2+1)(x+1)^2\sqrt{x+1+\frac{1}{x}}}dx$$

$$= \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)\left(x+2+\frac{1}{x}\right)\sqrt{x+1+\frac{1}{x}}}dx$$

Put $x+\frac{1}{x}+1=t^2$

$\therefore \quad I=\int\frac{2tdt}{(t^2-1)(t^2+1)t}$

$=2\int\frac{dt}{(t^2-1)(t^2+1)}$

$=\int\left(\frac{1}{t^2-1}-\frac{1}{t^2+1}\right)dt$

$=\frac{1}{2}\log\left|\frac{t-1}{t+1}\right|-\tan^{-1}t+C$

$=\frac{1}{2}\log_e\left|\frac{\sqrt{x+\frac{1}{x}+1}-1}{\sqrt{x+\frac{1}{x}+1}+1}\right|-\tan^{-1}\sqrt{x+\frac{1}{x}+1}+C$

For Problems 14 and 15

14. (2), 15. (2)

$f(x)f''(x)-f(x)f'(x)=(f'(x))^2$

$\therefore \quad \frac{f''(x)-f'(x)}{f'(x)}=\frac{f'(x)}{f(x)}$

$\Rightarrow \quad \frac{f''(x)}{f'(x)}-1=\frac{f'(x)}{f(x)}$

On integrating both sides w.r.t. x, we get

$\log_e(f'(x))-x=\log_e f(x)+c$

$f(0)=f'(0)=1$

$\therefore \quad 0-0=0+c$

$\therefore \quad c=0$

$\therefore \quad \log_e(f'(x))-x=\log_e f(x)$

$\Rightarrow \quad \log_e(f'(x))-\log_e f(x)=x$

$\Rightarrow \quad \frac{f'(x)}{f(x)}=e^x$

On integrating both sides w.r.t. x, we get

$\therefore \quad \log_e f(x)=e^x+c$

Putting $x=0$, we get $c=-1$

$\therefore \quad \log_e f(x)=e^x-1$

$\therefore \quad f(x)=e^{e^x-1}$

For Problems 16 and 17

16. (2), 17. (4)

We have, $6\int f(x)g(x)dx=x^6+3x^4+3x^2+c$

Differentiating both sides w.r.t. x, we get

$6f(x)\,g(x)=6x^5+12x^3+6x$

$\therefore \quad f(x)g(x)=x^5+2x^3+x \qquad \ldots(1)$

$2\int\frac{g(x)dx}{f(x)}=x^2+c$

Differentiating both sides w.r.t. x, we get

$\frac{g(x)}{f(x)}=x \qquad \ldots(2)$

Multiplying (1) and (2), we get

$(g(x))^2=x^6+2x^4+x^2=(x^3+x)^2$

$\therefore \quad g(x)=x^3+x$

$\therefore \quad f(x)=x^2+1 \quad (\text{as } f(x)>0).$

Now $\int(g(x)-f(x))dx=\int(x^3+x-x^2-1)dx$

$=\frac{x^4}{4}+\frac{x^2}{2}-\frac{x^3}{3}-x+c$

$\lim_{x\to0}\frac{\log(f(x))}{g(x)}=\lim_{x\to0}\frac{\log(1+x^2)}{x+x^3}$

$=\lim_{x\to0}\frac{\log(1+x^2)}{x^2}\cdot\frac{x^2}{x+x^3}$

$=\lim_{x\to0}\frac{x}{1+x^2}=0$

Matrix Match Type

1. a → p, q; b → r, s; c → p.; d → p, q.

a. Let $I=\int\frac{2^x}{\sqrt{1-4^x}}dx=\frac{1}{\log 2}\int\frac{1}{\sqrt{1-t^2}}dt$

Putting $2^x=t$, $2^x\log 2\,dx=dt$, we get

$I=\frac{1}{\log 2}\sin^{-1}\left(\frac{t}{1}\right)+C=\frac{1}{\log 2}\sin^{-1}(2^x)+C$

$\therefore \quad k=\frac{1}{\log 2}$

b. $\int\frac{dx}{(\sqrt{x})^2+(\sqrt{x})^7}=\int\frac{dx}{(\sqrt{x})^7\left(1+\frac{1}{(\sqrt{x})^5}\right)}$

Put $\frac{1}{(\sqrt{x})^5}=y, \quad \frac{dy}{dx}=-\frac{5}{2(\sqrt{x})^7}$

$\therefore \quad I=\int\frac{-2dy}{5(1+y)}=-\frac{2}{5}\ln|1+y|+C=\frac{2}{5}\ln\left(\frac{1}{1+\frac{1}{(\sqrt{x})^5}}\right)$

$\therefore \quad a=\frac{2}{5}, k=\frac{5}{2}$

c. Add and subtract $2x^2$ in the numerator. Then $k=1$ and $m=1$.

d. $I=\int\frac{dx}{5+4\cos x}$

$=\int\frac{dx}{5\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\right)+4\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$

$=\int\frac{dx}{9\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}=\int\frac{\sec^2\frac{x}{2}}{9+\tan^2\frac{x}{2}}dx$

Let $t=\tan\frac{x}{2}$ or $2\,dt=\sec^2\frac{x}{2}dx$

$\therefore \quad I=\int\frac{2dt}{9+t^2}=\frac{2}{3}\tan^{-1}\left(\frac{t}{3}\right)+C$

$=\frac{2}{3}\tan^{-1}\left(\frac{\tan\left(\frac{x}{2}\right)}{3}\right)+C$

$\therefore \quad k=\frac{2}{3}, m=\frac{1}{3}$

2. $\mathbf{a \to r;\ b \to s;\ c \to q;\ d \to p.}$

a. $\int \frac{e^{2x}-1}{e^{2x}+1}dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$

$= \int \frac{(e^x + e^{-x})'}{e^x + e^{-x}}dx$

$= \log(e^x + e^{-x})$

$= \log(e^{2x} + 1) - x + C$

b. $I = \int \frac{1}{(e^x + e^{-x})^2}dx = \int \frac{e^{2x}}{(e^{2x}+1)^2}dx$

Put $e^{2x} + 1 = t$ or $2e^{2x}\,dx = dt$

$\therefore \quad I = \frac{1}{2}\int \frac{1}{t^2}dt = -\frac{1}{2}\frac{1}{t} + C = -\frac{1}{2(e^{2x}+1)} + C$

c. $I = \int \frac{e^{-x}}{1+e^x}dx = \int \frac{e^{-x}e^{-x}}{e^{-x}+1}dx$

Put $e^{-x} + 1 = t$ or $-e^{-x}\,dx = dt$

$\therefore \quad I = -\int \frac{(t-1)}{t}dt = \int\left(\frac{1}{t} - 1\right)dt$

$= \log t - t + C$

$= \log(e^{-x} + 1) - (e^{-x} + 1) + C$

$= \log(e^x + 1) - x - e^{-x} - 1 + C$

$= \log(e^x + 1) - x - e^{-x} + C$

d. $I = \int \frac{1}{\sqrt{1-e^{2x}}}dx = \int \frac{e^{-x}}{\sqrt{e^{-2x}-1}}dx$

Put $e^{-x} = t$ or $-e^{-x}\,dx = dt$

$\therefore \quad I = -\int \frac{1}{\sqrt{t^2-1}}dt$

$= -\log\left[t + \sqrt{t^2-1}\right] + C$

$= -\log\left[e^{-x} + \sqrt{e^{-2x}-1}\right] + C$

$= -\log\left[\frac{1}{e^x} + \frac{\sqrt{1-e^{2x}}}{e^x}\right] + C$

$= -\log\left[1 + \sqrt{1-e^{2x}}\right] + \log e^x + C$

$= x - \log\left[1 + \sqrt{1-e^{2x}}\right] + C$

3. $\mathbf{a \to p, q, r;\ b \to p, q, r;\ c \to p, q, r, s;\ d \to p, q, r, s.}$

a. $\int \frac{x^2 - x + 1}{x^3 - 4x^2 + 4x}dx = \int\left[\frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}\right]dx$

b. $\int \frac{x^2 - 1}{x(x-2)^3}dx = \int\left[\frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}\right]dx$

c. $\int \frac{x^3+1}{x(x-2)^2}dx = \int\left[\left(\frac{x^3+1}{x(x-2)^2} - 1\right) + 1\right]dx$

$= \int\left[\left(\frac{x^3 + 1 - x(x-2)^2}{x(x-2)^2}\right) + 1\right]dx$

$= \int\left[\left(\frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}\right) + 1\right]dx$

d. $\int \frac{x^5+1}{x(x-2)^3}dx = \int\left[x + k + \frac{g(x)}{x(x-2)^3}\right]dx,$

where k is constant $\neq 0$ and $g(x)$ is a polynomial of degree less than 4.

4 (4) $\int \frac{x^3 + 3x^2 + 2x + 1}{\sqrt{x^2 + x + 1}}dx$

$= (ax^2 + bx + c)\sqrt{x^2 + x + 1} + \lambda \int \frac{dx}{\sqrt{x^2 + x + 1}}$

Differentiating both side, we get

$\frac{x^3 + 3x^2 + 2x + 1}{\sqrt{x^2 + x + 1}} = \frac{(ax^2 + bx + c)(2x+1)}{2\sqrt{x^2 + x + 1}}$

$+ (2ax + b)\sqrt{x^2 + x + 1} + \lambda \frac{1}{\sqrt{x^2 + x + 1}}$

$= \frac{6ax^3 + (5a + 4b)x^2 + (4a + 3b + 2c)x + (c + 2b + 2\lambda)}{2\sqrt{x^2 + x + 1}}$

Now, compare coefficients on both sides and solve equations.

Numerical Value Type

1. (–1) $f(x) = \int x^{\sin x}(1 + x\cos x \cdot \ln x + \sin x)dx$

If $F(x) = x^{\sin x} = e^{\sin x \ln x}$, then

$f(x) = \int (F(x) + xF'(x)) = xF(x) + C$

$= x \cdot x^{\sin x} + C$

or $\quad f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\cdot\frac{\pi}{2} + C$ or $C = 0$

$\therefore \quad f(x) = x(x)^{\sin x};\ f(\pi) = \pi(\pi)^0 = \pi$

2. (0.5) $g(x) = \int \frac{\cos x(\cos x + 2) + \sin^2 x}{(\cos x + 2)^2}dx$

$= \int \underbrace{\cos x}_{II} \cdot \underbrace{\frac{1}{(\cos x + 2)}}_{I}dx + \int \frac{\sin^2 x}{(\cos x + 2)^2}dx$

$= \frac{1}{\cos x + 2}\cdot \sin x - \int \frac{\sin^2 x}{(\cos x + 2)^2}dx + \int \frac{\sin^2 x}{(\cos x + 2)^2}dx$

$\therefore \quad g(x) = \frac{\sin x}{\cos x + 2} + C$

$g(0) = 0 \quad$ or $\quad C = 0$

$\therefore \quad g(x) = \frac{\sin x}{\cos x + 2}$ or $g\left(\frac{\pi}{2}\right) = \frac{1}{2}$

3. (2) $k(x) = \int \frac{(x^2+1)dx}{(x^3 + 3x + 6)^{1/3}}$

Put $\quad x^3 + 3x + 6 = t^3$

or $\quad 3(x^2 + 1)\,dx = 3t^2 dt$

$k(x) = \int \frac{t^2\,dt}{t} = \frac{t^2}{2} + C = \frac{1}{2}(x^3 + 3x + 6)^{2/3} + C$

$k(-1) = \frac{1}{2}(2)^{2/3} + C \quad$ or $\quad C = 0$

$\therefore \quad k(x) = \frac{1}{2}(x^3 + 3x + 6)^{2/3};\ f(-2) = \frac{1}{2}(-8)^{2/3}$

$= \frac{1}{2}[(-2)^3]^{2/3} = 2$

4. (2010) $I = \int \frac{1-\cot^{2008}x}{\tan x+\cot^{2009}x}\,dx$

$= \int \frac{\sin^{2008}x-\cos^{2008}x}{\sin^{2008}x\left[\frac{\sin x}{\cos x}+\frac{\cos^{2009}x}{\sin^{2009}x}\right]}\,dx$

$= \int \frac{\sin x\cos x[\sin^{2008}x-\cos^{2008}x]}{\sin^{2010}x+\cos^{2010}x}\,dx$

$= \int \frac{\sin^{2009}x\cos x-\cos^{2009}x\sin x}{\sin^{2010}x+\cos^{2010}x}\,dx$

Put $\sin^{2010}x+\cos^{2010}x = t$.

$\therefore \quad I = \frac{1}{2010}\int \frac{dt}{t}$

$= \frac{1}{2010}\log|\sin^{2010}x+\cos^{2010}x|+C$

$\therefore \quad k = 2010$

5. (9) $f(x) = \int \frac{3x^2+1}{(x^2-1)^3}\,dx$

$= \int \frac{-(x^2-1)}{(x^2-1)^3}\,dx + \int \frac{4x^2}{(x^2-1)^3}\,dx$

$= \int\left[\frac{-1}{(x^2-1)^2}+x\cdot\frac{4x}{(x^2-1)^3}\right]dx$

$= -\int \frac{dx}{(x^2-1)^2} + x\int \frac{4x\,dx}{(x^2-1)^3} - \int\left((x)'\int \frac{4x}{(x^2-1)^3}\,dx\right)dx$

$= x\left(\frac{-1}{(x^2-1)^2}\right)+C$

$= -\frac{x}{(x^2-1)^2}+C$

$f(0) = 0 \quad \Rightarrow \quad C = 0$

$\therefore \quad f(x) = -\frac{x}{(x^2-1)^2}$

Now, $f(2) = -\frac{2}{9}$

6. (0) $fog(x) = \sqrt{e^x-1}$

$\therefore \quad I = \int \sqrt{e^x-1}\,dx$

$= \int \frac{2t^2}{t^2+1}\,dt$, where $\sqrt{e^x-1} = t$

$= 2t - 2\tan^{-1}t + C$

$= 2\sqrt{e^x-1} - 2\tan^{-1}\left(\sqrt{e^x-1}\right)+C$

$= 2\,fog(x) - 2\tan^{-1}(fog(x)) + C$

$\therefore \quad A+B = 2+(-2) = 0$

7. (3) $\frac{d}{dx}(A\ln|\cos x+\sin x-2|+Bx+C)$

$= A\frac{\cos x-\sin x}{\cos x+\sin x-2}+B$

$= \frac{A\cos x - A\sin x + B\cos x + B\sin x - 2B}{\cos x+\sin x-2}$

$\therefore \quad 2 = A+B,\ -1 = -A+B,\ \lambda = -2B$

$\therefore \quad A = 3/2,\ B = 1/2,\ \lambda = -1$

$\therefore \quad A+B+|\lambda| = 3$

8. (1) $\int e^{x^3+x^2-1}(3x^4+2x^3+2x)dx$

$= \int \underbrace{x^2}_{I}\underbrace{e^{x^3+x^2-1}(3x^2+2x)}_{II}dx + \int e^{x^3+x^2-1}2x\,dx$

$= x^2e^{x^3+x^2-1} - \int 2xe^{x^3+x^2-1}dx + \int e^{x^3+x^2-1}2x\,dx$

$= x^2e^{x^3+x^2-1}+C$

$\Rightarrow \quad f(1) = e$ and $f(-1) = \frac{1}{e}$

9. (7.5) Let $x = t^3$

$\therefore \quad dx = 3t^2dt$

$\therefore \quad f(x) = \int \frac{dx}{x^{1/3}+2}$

$= 3\int \frac{t^2dt}{t+2}$

$= 3\int \frac{(t^2-4+4)\,dt}{t+2}$

$= 3\int\left(t-2+\frac{4}{t+2}\right)dt$

$= 3\left(\frac{t^2}{2}-2t+4\log_e|t+2|\right)+c$

$= 3\left(\frac{x^{2/3}}{2}-2x^{1/3}+4\log_e|x^{1/3}+2|\right)+c$

$f(0) = 12\log_e 2$

$\therefore \quad c = 0$

$\therefore \quad f(-1) = \frac{3}{2}+6 = \frac{15}{2}$

10. (–3) $I = \int \frac{dx}{\sin^{11/3}x\cos^{1/3}x}$

$= \int \frac{dx}{\tan^{11/3}x\cos^4x}$

$= \int \frac{(1+\tan^2x)\sec^2x\,dx}{\tan^{11/3}x}$

$= \int \frac{(1+t^2)dt}{t^{11/3}}$ (where $\tan x = t$)

$= -\frac{3}{8}t^{-\frac{8}{3}} - \frac{3}{2}t^{-\frac{2}{3}} + c$, where $t = \tan x$

$\therefore \quad 4A+B = -3$

Archives

JEE Main

Single Correct Answer Type

1. (4) $\int \frac{5\tan x}{\tan x - 2}dx = \int \frac{5\sin x}{\sin x - 2\cos x}dx$

$= \int \frac{(\sin x - 2\cos x) + 2(\cos x + 2\sin x)}{(\sin x - 2\cos x)}dx$

$= \int dx + 2\int \frac{\cos x + 2\sin x}{\sin x - 2\cos x}dx$

$= x + 2\ln|(\sin x - 2\cos x)| + k$

2. (3) Given that $\int f(x)\,dx = \Psi(x)$. Let $I = \int x^5 f(x^3)\,dx$

or $I = \int tf(t)\frac{dt}{3}$ (Put $x^3 = t$)

$= \frac{1}{3}\left[t\,\Psi(t) - \int \Psi(t)dt\right] + c$

$\therefore \quad I = \frac{x^3}{3}\Psi(x^3) - \frac{1}{3}\int 3x^2\,\Psi(x^3)\,dx + c$

$= \frac{x^3}{3}\Psi(x^3) - \int x^2\Psi(x^3)\,dx + c$

3. (2) $\int\left(1 + x - \frac{1}{x}\right)e^{\left(x+\frac{1}{x}\right)}dx$

$= \int e^{\left(x+\frac{1}{x}\right)}dx + \int x\left(1 - \frac{1}{x^2}\right)e^{\left(x+\frac{1}{x}\right)}dx + c$

$= \int e^{\left(x+\frac{1}{x}\right)}dx + xe^{\left(x+\frac{1}{x}\right)} - \int e^{\left(x+\frac{1}{x}\right)}dx + c$

(integrating second integral using by parts)

$= xe^{\left(x+\frac{1}{x}\right)} + c$

4. (4) $I = \int \frac{dx}{x^2(x^4+1)^{3/4}}$

$\int \frac{dx}{x^5\left(1 + \frac{1}{x^4}\right)^{3/4}}$

Let $1 + \frac{1}{x^4} = t^4$

$\Rightarrow \quad \frac{-4}{x^5}dx = 4t^3dt$ or $\frac{dx}{x^5} = -t^3dt$

$\therefore \quad I = \int \frac{-t^3dt}{t^3} = -t + c = -\left(1 + \frac{1}{x^4}\right)^{1/4} + c$

5. (1) $I = \int \frac{2x^{12} + 5x^9}{(x^5 + x^3 + 1)^3}dx$

$= \int \frac{\left(\frac{2}{x^3} + \frac{5}{x^6}\right)}{\left(1 + \frac{1}{x^2} + \frac{1}{x^5}\right)^3}dx$

Let $1 + \frac{1}{x^2} + \frac{1}{x^5} = t$

$\therefore \quad \frac{dt}{dx} = \frac{-2}{x^3} - \frac{5}{x^6}$

$\therefore \quad \int \frac{-dt}{t^3} = \frac{1}{2t^2} + C$

$= \frac{1}{2\left(1 + \frac{1}{x^2} + \frac{1}{x^5}\right)^2} + C$

$= \frac{x^{10}}{2(x^5 + x^3 + 1)^2} + C$

6. (3) $I_n = \int \tan^n x\,dx$

$I_4 + I_6 = \int (\tan^4 x + \tan^6 x)\,dx$

$= \int \tan^4 x\sec^2 x\,dx$

$= \frac{1}{5}\tan^5 x + C$

$\therefore \quad a\tan^5 x + bx^5 + C = \frac{1}{5}\tan^5 x + C$ (given)

$a = \frac{1}{5}, b = 0$

7. (3) $I = \int \frac{\sin^2 x\cos^2 x}{(\sin^5 x + \cos^3 x\sin^2 x + \sin^3 x\cos^2 x + \cos^5 x)^2}dx$

Dividing numerator and denominator by $\cos^{10} x$, we get

$I = \int \frac{\tan^2 x\sec^6 x\,dx}{(\tan^5 x + \tan^2 x + \tan^3 x + 1)^2}$

$= \int \frac{\tan^2 x\sec^6 x}{(1+\tan^2 x)^2(1+\tan^3 x)^2}dx$

$= \int \frac{\tan^2 x\sec^6 x}{(\sec^2 x)^2(1+\tan^3 x)^2}dx$

$= \int \frac{\tan^2 x\sec^2 x}{(1+\tan^3 x)^2}dx$

$= \frac{1}{3}\int \frac{3\tan^2 x\sec^2 x}{(1+\tan^3 x)^2}dx$

$= \frac{1}{3}\int \frac{(1+\tan^3 x)'}{(1+\tan^3 x)^2}dx$

$= \frac{-1}{3(1+\tan^3 x)} + C$

JEE Advanced

Single Correct Answer Type

1. (3) $I = \int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}}dx$

Let $\sec x + \tan x = t$

or $\sec x - \tan x = 1/t$

Now, $(\sec x\tan x + \sec^2 x)\,dx = dt$

or $\sec x(\sec x + \tan x)\,dx = dt$

or $\sec x\,dx = \frac{dt}{t}, \frac{1}{2}\left(t + \frac{1}{t}\right) = \sec x$

$$\therefore \quad I = \frac{1}{2}\int \frac{\left(t+\frac{1}{t}\right)}{t^{9/2}}\frac{dt}{t}$$

$$= \frac{1}{2}\int (t^{-9/2} + t^{-13/2})\,dt$$

$$= \frac{1}{2}\left[\frac{t^{-9/2+1}}{-\frac{9}{2}+1} + \frac{t^{-13/2+1}}{-\frac{13}{2}+1}\right] + K$$

$$= \frac{1}{2}\left[\frac{t^{-7/2}}{-\frac{7}{2}} + \frac{t^{-11/2}}{-\frac{11}{2}}\right] + K$$

$$= -\frac{1}{7}t^{-7/2} - \frac{1}{11}t^{-11/2} + K$$

$$= -\frac{1}{7}\frac{1}{t^{7/2}} - \frac{1}{11}\frac{1}{t^{11/2}} + K$$

$$= -\frac{1}{t^{11/2}}\left(\frac{1}{11} + \frac{t^2}{7}\right) + K$$

$$= -\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$$

Concept Application Exercises

Exercise 8.1

1. (a) Here, $f(x) = \cos x$

$$\int_a^b \cos x\,dx = \lim_{n\to\infty} h\,[\cos a + \cos(a+h) + \cdots + \cos\{a+(n-1)h\}], \text{ where } nh = b-a$$

$$= \lim_{n\to\infty} h \frac{\cos\left\{a+\frac{1}{2}(n-1)h\right\}\sin\left(\frac{1}{2}nh\right)}{\sin\left(\frac{1}{2}h\right)}$$

$$= \lim_{n\to\infty}\left[2\cos\left\{a+\left(\frac{1}{2}nh-\frac{1}{2}h\right)\right\}\sin\left(\frac{1}{2}nh\right)\left(\frac{\frac{1}{2}h}{\sin\left(\frac{1}{2}h\right)}\right)\right]$$

$$= 2\cos\left\{a+\frac{1}{2}(b-a)-0\right\}.\sin\left(\frac{1}{2}(b-a)\right)\times 1$$

$$= 2\cos\frac{1}{2}(a+b)\sin\frac{1}{2}(b-a)$$

$$= \sin b - \sin a$$

(b) Here, $f(x) = x^3$

$$\therefore \quad \int_a^b x^3\,dx = \lim_{n\to\infty} h\sum_{r=0}^{n-1}(a+rh)^3, \qquad \text{where } nh = b-a$$

$$= \lim_{n\to\infty}\left[nha^3 + 3a^2h^2\sum_{r=1}^{n-1}r + 3ah^3\sum_{r=1}^{n-1}r^2 + h^4\sum_{r=1}^{n-1}r^3\right]$$

$$= \lim_{n\to\infty}\left[nha^3 + 3a^2h^2\left\{\frac{(n-1)n}{2}\right\} + 3ah^3\left\{\frac{n(n-1)(2n-1)}{6}\right\} + h^4\left\{\frac{(n-1)^2n^2}{4}\right\}\right]$$

$$= \lim_{h\to 0}\left[(nh)a^3 + 3a^2\left\{\frac{(nh-h)(nh)}{2}\right\} + 3a\left\{\frac{(nh)(nh-h)(2nh-h)}{6}\right\} + \left\{\frac{(nh-h)^2(nh)^2}{4}\right\}\right]$$

$$= \left[(b-a)a^3 + 3a^2\left\{\frac{(b-a-0)(b-a)}{2}\right\} + 3a\left\{\frac{(b-a)(b-a-0)(2(b-a)-0)}{6}\right\} + \left\{\frac{(b-a-0)^2(b-a)^2}{4}\right\}\right]$$

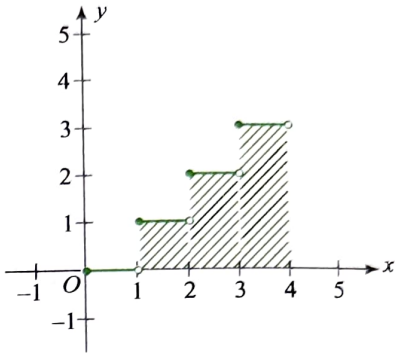
$$[\because \text{ as } n\to\infty,\ h\to 0,\ nh\to b-a]$$

$$= \frac{1}{4}(b-a)\left[4a^3 + 6a^2(b-a) + 4a(b-a)^2 + (b-a)^3\right]$$

$$= \frac{1}{4}(b-a)\left(a^3 + a^2b + ab^2 + b^3\right)$$

$$= \frac{1}{4}(b-a)(b+a)\left(b^2+a^2\right) = \frac{1}{4}\left(b^4 - a^4\right)$$

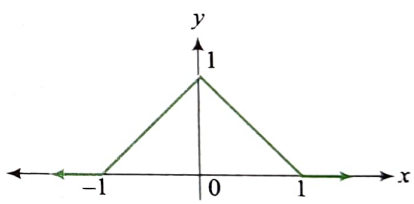
2. Graph of $y = [x]$ for $0 < x < 4$ is as shown in the following figure.



From the graph $\int_0^4 [x]dx$ = 0 (for $0 < x < 1$)
+ (1 × 1) (for $1 < x < 2$)
+ (1 × 2) (for $2 < x < 3$)
+ (1 × 3) (for $3 < x < 4$)
= 0 + 1 + 2 + 3
= 6

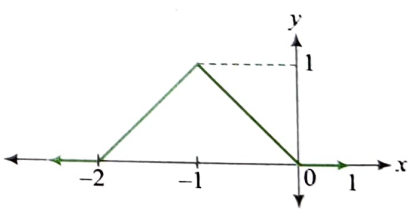
3.

(a)



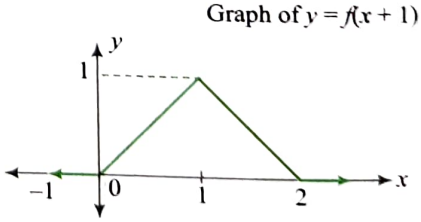
Graph of $y = f(x)$

(b)



Graph of $y = f(x+1)$

(c)



Graph of $y = f(x-1)$

$$\int_{-3}^{5} g(x)dx = \int_{-3}^{5} f(x-1)dx + \int_{-3}^{5} f(x-1)dx$$

= Area of triangle in the graph $y = f(x-1)$
+ Area of triangle in the graph $y = f(x+1)$

$$= 2\frac{1}{2}(2)(1) = 2$$

Exercise 8.2

1. Here, the mistake lies in the substitution $\tan\frac{1}{2}x = t$, because $\tan\frac{1}{2}x$ is discontinuous at $x = \pi$ which is a point in the interval $[0, 2\pi]$.

2. (a) $\int_0^{\pi} \frac{dx}{1+\sin x} = \int_0^{\pi} \frac{1-\sin x}{1-\sin^2 x}\,dx = \int_0^{\pi} \frac{1-\sin x}{\cos^2 x}\,dx$

$= \int_0^{\pi} (\sec^2 x - \sec x \tan x)\,dx$

$= |\tan x - \sec x|_0^{\pi}$

$= (\tan \pi - \sec \pi) - (\tan 0 - \sec 0)$

$= 0 - (-1) - (0 - 1) = 1 + 1 = 2$

(b) $I = \int_1^{\infty} \frac{dx}{(ee^x + e^3 e^{-x})}$

$= \int_1^{\infty} \frac{e^x\,dx}{e(e^{2x} + e^2)}$ (multiply N^r and D^r by e^x)

Put $e^x = t$ or $e^x\,dx = dt$

$\therefore \quad I = \frac{1}{e}\int_e^{\infty} \frac{dt}{t^2 + e^2}$

$= \frac{1}{e^2}\tan^{-1}\frac{t}{e}\Big|_e^{\infty}$

$= \frac{1}{e^2}\left[\frac{\pi}{2} - \frac{\pi}{4}\right] = \frac{\pi}{4e^2}$

(c) Put $x = \sin\theta$. So, $dx = \cos\theta\,d\theta$

When $x = 0$, $\theta = 0$, when $x = \frac{1}{\sqrt{2}}$, $\theta = \frac{\pi}{4}$.

$\therefore$ Given integral

$= \int_0^{\pi/4} \frac{\sin^{-1}(\sin\theta)\cos\theta\,d\theta}{(1-\sin^2\theta)^{3/2}}$

$= \int_0^{\pi/4} \frac{\theta\cos\theta}{\cos^3\theta}\,d\theta = \int_0^{\pi/4} \underset{\text{I}}{\theta}\cdot\underset{\text{II}}{\sec^2\theta}\,d\theta$

$= |\theta\tan\theta|_0^{\pi/4} - \int_0^{\pi/4} 1\cdot\tan\theta\,d\theta$

$= \frac{\pi}{4}\tan\frac{\pi}{4} + \log\cos\theta|_0^{\pi/4}$

$= \frac{\pi}{4} + \log\cos\frac{\pi}{4} - \log\cos 0 = \frac{\pi}{4} + \log\frac{1}{\sqrt{2}}$

$= \frac{\pi}{4} + \log 1 - \log(2)^{1/2} = \frac{\pi}{4} - \frac{1}{2}\log 2$

(d) Put $x = \sin\theta$. So, $dx = \cos\theta\,d\theta$

$\therefore$ Given integral

$= \int_0^{\pi/2} \frac{(2-\sin^2\theta)\cos\theta\,d\theta}{(1+\sin\theta)\cos\theta}$

$= \int_0^{\pi/2} \left(1 - \sin\theta + \frac{1}{1+\sin\theta}\right)d\theta$

$= |\theta + \cos\theta|_0^{\pi/2} + \int_0^{\pi/2} \frac{d\theta}{1+\sin\theta}$

$= \frac{\pi}{2} - 1 + \int_0^{\pi/2} \frac{1-\sin\theta}{\cos^2\theta}\,d\theta$

$= \frac{\pi}{2} - 1 + \int_0^{\pi/2} (\sec^2\theta - \sec\theta\tan\theta)\,d\theta$

$= \frac{\pi}{2} - 1 + |\tan\theta - \sec\theta|_0^{\pi/2}$

$= \frac{\pi}{2} - 1 + \lim_{\theta\to\pi/2} \frac{\sin\theta - 1}{\cos\theta} - \frac{\sin 0 - 1}{\cos 0}$

$= \frac{\pi}{2} - 1 + \lim_{\theta\to\pi/2} \frac{\cos\theta}{-\sin\theta} + 1$

$= \frac{\pi}{2}$

(e) $I = \int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

$= \int_0^{\pi/2} \frac{\sec^2 x\,dx}{a^2 + b^2\tan^2 x} = \frac{1}{b^2}\int_0^{\pi/2} \frac{\sec^2 x\,dx}{\left(\frac{a}{b}\right)^2 + \tan^2 x}$

Put $\tan x = z$ or $\sec^2 x\,dx = dz$

When $x = 0$, $z = 0$, $x \to \frac{\pi}{2}$, $z \to \infty$

$\therefore \quad I = \frac{1}{b^2}\int_0^{\infty} \frac{dz}{\left(\frac{a}{b}\right)^2 + z^2}$

$= \frac{1}{b^2}\frac{1}{\frac{a}{b}}\left|\tan^{-1}\frac{z}{a/b}\right|_0^{\infty}$

$= \frac{1}{ab}[\tan^{-1}\infty - \tan^{-1} 0]$

$= \frac{1}{ab}\frac{\pi}{2}$

$= \frac{\pi}{2ab}$

3. $I = \int_{\pi/6}^{\pi/4} e^{-x}(\operatorname{cosec} x + \cot x \operatorname{cosec} x)\,dx$

Put $-x = t \Rightarrow dx = -dt$

$\therefore \quad I = -\int_{-\pi/6}^{-\pi/4} e^t(-\operatorname{cosec} t + \cot t\cdot\operatorname{cosec} t)\,dt$

$= \int_{-\pi/6}^{-\pi/4} e^t(\operatorname{cosec} t - \cot t\cdot\operatorname{cosec} t)\,dt$

$= e^t\operatorname{cosec} t\Big|_{-\pi/6}^{-\pi/4}$

$= -\sqrt{2}e^{-\pi/4} + 2e^{-\pi/6}$

$= 2e^{-\pi/6} - \sqrt{2}e^{-\pi/4}$

4. $I = \int_0^1 \frac{e^{-x}\,dx}{1+e^x} = \int_0^1 \frac{dx}{e^x(1+e^x)}$

Put $e^x = z$.

$\therefore \quad e^x\,dx = dz$

$\Rightarrow \quad dx = \frac{dz}{e^x} = \frac{dz}{z}$

$\Rightarrow \quad I = \int_1^e \frac{dz}{z^2(1+z)}$

$= \int_1^e \left(\frac{1}{1+z} - \frac{z-1}{z^2}\right)dz$

$$= \left|\log_e(1+z) - \log_e z - \frac{1}{z}\right|_1^e$$

$$= \left(\log_e(1+e) - \log_e e - \frac{1}{e}\right) - \left(\log_e 2 - \log_e 1 - 1\right)$$

$$= \log(1+e) - \frac{1}{e} - \log 2$$

5. $I = \int_0^{102} (x-1)(x-2)\cdots(x-100) \times \left(\frac{1}{(x-1)} + \frac{1}{(x-2)} + \cdots + \frac{1}{(x-100)}\right) dx$

$$= \int_0^{102} \frac{d}{dx}((x-1)(x-2)\cdots(x-100))\,dx$$

$$= [(x-1)(x-2)\ldots(x-100)]_0^{102}$$

$$= 101! - 100!$$

6. Let $I = \int_0^1 \frac{\log x}{(1+x)} dx$

$$= [\log x \log(1+x)]_0^1 - \int_0^1 \frac{\log(1+x)}{x} dx$$

$$= 0 - \int_0^1 \frac{\log(1+x)}{x} dx$$

7. $a = \int_0^1 \frac{e^t}{1+t} dt = \left(\frac{1}{(1+t)} e^t\right)_0^1 + \int_0^1 \frac{e^t}{(1+t)^2} dt$

(Integrating by parts)

$$= \frac{e}{2} - 1 + \int_0^1 \frac{e^t}{(1+t)^2} dt$$

or $$\int_0^1 \frac{e^t}{(1+t)^2} dt = a + 1 - \frac{e}{2}$$

8. Let $y = f(x)$

$\therefore$ $x = f^{-1}(y)$ and $dy = f'(x)\,dx$

$$I = \int_3^7 f^{-1}(x)\,dx = \int_3^7 f^{-1}(y)\,dy$$

Given $f^{-1}(3) = 2$ and $f^{-1}(7) = 5$

$\therefore$ $I = \int_2^5 xf'(x)\,dx = [xf(x)]_2^5 - \int_2^5 f(x)\,dx$

$$= (5)(7) - (2)(3) - 17 = 12$$

9. (a) Given limit

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{n}{\sqrt{4n^2-1}} + \frac{n}{\sqrt{4n^2-2^2}} + \cdots + \frac{n}{\sqrt{4n^2-n^2}}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{\sqrt{4-\left(\frac{1}{n}\right)^2}} + \frac{1}{\sqrt{4-\left(\frac{2}{n}\right)^2}} + \cdots \frac{1}{\sqrt{4-\left(\frac{n}{n}\right)^2}}\right]$$

$$= \int_0^1 \frac{dx}{\sqrt{4-x^2}} = \left|\sin^{-1}\frac{x}{2}\right|_0^1 = \sin^{-1}\frac{1}{2} - 0 = \frac{\pi}{6}$$

(b) $\lim_{n\to\infty}\left[\frac{1}{n^2}\sec^2\frac{1}{n^2} + \frac{2}{n^2}\sec^2\frac{4}{n^2} + \cdots + \frac{1}{n}\sec^2 1\right]$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \frac{r}{n}\sec^2\left(\frac{r}{n}\right)^2$$

$$= \int_0^1 x\sec^2 x^2\,dx$$

Put $x^2 = t$ so that $2x\,dx = dt$

When $x = 0$, $t = 0$. When $x = 1$, $t = 1$

$\therefore$ Required limit $= \frac{1}{2}\int_0^1 \sec^2 t\,dt$

$$= \frac{1}{2}[\tan t]_0^1 = \frac{1}{2}[\tan 1 - 0]$$

$$= \frac{1}{2}\tan 1$$

(c) $\lim_{n\to\infty} \sum_{K=1}^{n} \frac{K}{n^2 + K^2}$

$$= \lim_{n\to\infty} \sum_{K=1}^{n} \frac{1}{n^2} \times \frac{K}{1+\left(\frac{K}{n}\right)^2}$$

$$= \lim_{n\to\infty} \sum_{K=1}^{n} \frac{1}{n} \times \frac{K/n}{1+\left(\frac{K}{n}\right)^2}$$

$$= \int_0^1 \frac{x}{1+x^2} dx$$

$$= \frac{1}{2}\log(1+x^2)\Big|_0^1 = \frac{1}{2}\log 2$$

(d) $\lim_{n\to\infty} \frac{\sum_{r=1}^{n}\sqrt{r}\sum_{r=1}^{n}\frac{1}{\sqrt{r}}}{\sum_{r=1}^{n} r}$

$\therefore$ Limit $= \lim_{n\to\infty} \frac{\left(\frac{1}{n}\sum_{r=1}^{n}\sqrt{\frac{r}{n}}\right)\left(\frac{1}{n}\sum_{r=1}^{n}\sqrt{\frac{n}{r}}\right)}{\frac{1}{n}\sum_{r=1}^{n}\frac{r}{n}}$

$$= \frac{\int_0^1 \sqrt{x}\,dx \int_0^1 \frac{dx}{\sqrt{x}}}{\int_0^1 x\,dx} = \frac{8}{3}$$

(e) Let $A = \lim_{n\to\infty}\left[\frac{n!}{n^n}\right]^{1/n}$

$$= \lim_{n\to\infty}\left(\frac{1\times 2\times 3\cdots n}{n\times n\times n\cdots n}\right)^{1/n}$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}\times\frac{2}{n}\times\frac{3}{n}\cdots\frac{n}{n}\right)^{1/n}$$

$\therefore$ $\log A = \lim_{n\to\infty}\frac{1}{n}\left[\log\frac{1}{n} + \log\frac{2}{n} + \cdots + \log\frac{n}{n}\right]$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\log\frac{r}{n}$$

$$= \int_0^1 \log x\,dx = [x \log x]_0^1 - \int_0^1 x \times 1/x\,dx$$

$$= 0 - \int_0^1 dx = 0 - [x]_0^1 = -1$$

$$\therefore \quad A = e^{-1} = 1/e$$

Exercise 8.3

1. Since $1 \le x \le 3$

or $1 \le x^2 \le 9$

or $4 \le x^2 + 3 \le 12$

or $2 \le \sqrt{3 + x^2} \le 2\sqrt{3}$

or $2(3-1) \le \int_1^3 \sqrt{3 + x^2}\,dx \le 2\sqrt{3}(3-1)$

or $4 \le \int_1^3 \sqrt{3 + x^2} \le 4\sqrt{3}$

2. For $0 < x < 1$, $x^2 > x^3$

or $2^{x^2} > 2^{x^3}$

or $\int_0^1 2^{x^2}\,dx > \int_0^1 2^{x^3}\,dx$

Hence, $I_1 > I_2$

Also, for $1 < x < 2$, $x^2 < x^3$

or $2^{x^2} < 2^{x^3}$

or $\int_1^2 2^{x^2}\,dx < \int_1^2 2^{x^3}\,dx$

or $I_3 < I_4$

3. We know that cos x is decreasing function in $(0, \pi/2)$.

Also, $x >$ sin x for $x \in (0, \pi/2)$.

Thus, cos $x <$ cos (sin x)

Further, $x >$ sin x and cos $x \in (0, 1)$ for $x \in (0, \pi/2)$

$\therefore$ cos $x >$ sin (cos x)

Thus, sin (cos x) < cos $x <$ cos (sin x)

Hence, $\int_0^{\pi/2} \sin(\cos x)dx < \int_0^{\pi/2} \cos x\,dx < \int_0^{\pi/2} \cos(\sin x)dx$

$\Rightarrow \quad I_2 < I_3 < I_1$

4. Since $0 < x^3 < x^2$, we have (for $0 < x < 1$)

$x^2 < x^2 + x^3 < 2x^2$

or $-2x^2 < -x^2 - x^3 < -x^2$

or $4 - 2x^2 < 4 - x^2 - x^3 < 4 - x^2$

or $\sqrt{4-2x^2} < \sqrt{4-x^2-x^3} < \sqrt{4-x^2}$

or $\frac{1}{\sqrt{4-x^2}} < \frac{1}{\sqrt{4-x^2-x^3}} < \frac{1}{\sqrt{4-2x^2}}$

or $\int_0^1 \frac{1}{\sqrt{4-x^2}}dx < \int_0^1 \frac{1}{\sqrt{4-x^2-x^3}}dx < \int_0^1 \frac{1}{\sqrt{4-2x^2}}dx$

or $\sin^{-1}\left(\frac{x}{2}\right)\Big]_0^1 < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{1}{\sqrt{2}}\sin^{-1}\frac{x}{\sqrt{2}}\Big]_0^1$

or $\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{\pi}{4\sqrt{2}}$

Exercise 8.4

1. $\int_0^{\pi/2} |\sin x - \cos x|\,dx$

$$= \int_0^{\pi/4} -(\sin x - \cos x)dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)dx$$

$$= |\cos x + \sin x|_0^{\pi/4} + |-\cos x - \sin x|_{\pi/4}^{\pi/2}$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 - 0\right) + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$= \frac{4}{\sqrt{2}} - 2 = 2\sqrt{2} - 2$$

$$= 2(\sqrt{2} - 1)$$

2. We have $\int_2^4 (3 - f(x))\,dx = 7$

$$\therefore \quad 6 - \int_2^4 f(x)\,dx = 7 \text{ or } \int_2^4 f(x)\,dx = -1$$

Now, $\int_2^{-1} f(x)\,dx = -\int_{-1}^2 f(x)\,dx = -\left[\int_{-1}^4 f(x)\,dx + \int_4^2 f(x)\,dx\right]$

$$= -\left[\int_{-1}^4 f(x)\,dx - \int_2^4 f(x)\,dx\right] = -[4 + 1] = -5$$

3. $\int_1^5 \sqrt{x - 2\sqrt{x-1}}\,dx = \int_1^5 \sqrt{(\sqrt{x-1} - 1)^2}\,dx$

$$= \int_1^5 \left|\sqrt{x-1} - 1\right| dx$$

$$= \int_1^2 \left(1 - \sqrt{x-1}\right)dx + \int_2^5 \left(\sqrt{x-1} - 1\right)dx$$

$$= \left|x - \frac{2(x-1)^{3/2}}{3}\right|_1^2 + \left|\frac{2(x-1)^{3/2}}{3} - x\right|_2^5$$

$$= \left(2 - \frac{2}{3} - 1 + 0\right) + \left(\frac{16}{3} - 5 - \frac{2}{3} + 2\right) = 2$$

4. $\int_{-1}^3 \left(\tan^{-1}\frac{x}{x^2+1} + \tan^{-1}\frac{x^2+1}{x}\right)dx$

$$= \int_{-1}^0 \left(\tan^{-1}\frac{x}{x^2+1} + \tan^{-1}\frac{x^2+1}{x}\right)dx$$

$$+ \int_0^3 \left(\tan^{-1}\frac{x}{x^2+1} + \tan^{-1}\frac{x^2+1}{x}\right)dx$$

$$= \int_{-1}^0 -\frac{\pi}{2}dx + \int_0^3 \frac{\pi}{2}dx$$

$$= \left[-\frac{\pi}{2}x\right]_{-1}^0 + \left[\frac{\pi}{2}x\right]_0^3$$

$$= 2\pi$$

5. $1 < x < a$

or $0 < \log_a x < 1$

or $[\log_a x] = 0$

$\therefore \quad I = \int_1^a x\,dx = \frac{1}{2}(a^2 - 1)$

6. When $1 < x < e^3$, $\left[\frac{\log x}{3}\right] = 0$

and when $e^3 < x < e^6$, $\left[\frac{\log x}{3}\right] = 1$.

$$\therefore \quad \int_1^{e^6}\left[\frac{\log x}{3}\right]dx = \int_1^{e^3}\left[\frac{\log x}{3}\right]dx + \int_{e^3}^{e^6}\left[\frac{\log x}{3}\right]dx$$

$$= \int_1^{e^3} 0\,dx + \int_{e^3}^{e^6} 1\,dx = (e^6 - e^3).$$

7. $\int_{-1}^{1}[x^2 + \{x\}]dx = \int_{-1}^{0}[x^2 + x + 1]dx + \int_0^1[x^2 + x]dx$

$$= 0 + \int_0^{\frac{\sqrt{5}-1}{2}} 0\,dx + \int_{\frac{\sqrt{5}-1}{2}}^{1} 1\,dx$$

$$= \frac{3-\sqrt{5}}{2}$$

8. We have $\int_{-\pi/2}^{2\pi}[\cot^{-1}x]dx$.

We know that $\cot^{-1}x \in [0, \pi]$.

So, $[\cot^{-1}x] = 0$ for $\cot^{-1}x \in (0, 1)$ or $x \in (\cot 1, \infty)$

$[\cot^{-1}x] = 1$ for $\cot^{-1}x \in [1, 2)$ or $x \in (\cot 2, \cot 1]$

$[\cot^{-1}x] = 2$ for $\cot^{-1}x \in [2, 3)$ or $x \in (\cot 3, \cot 2]$

$[\cot^{-1}x] = 3$ for $\cot^{-1}x \in [3, \pi)$ or $x \in (-\infty, \cot 3]$

$$\therefore \quad \int_{\cot 1}^{2\pi} 0dx + \int_{\cot 2}^{\cot 1} 1dx + \int_{-\pi/2}^{\cot 2} 2dx$$

$$= \cot 1 - \cot 2 + 2\cot 2 + \pi$$

$$= \cot 1 + \cot 2 + \pi$$

9. Let $x = n + f \;\forall\; n \in I$ and $0 \le f < 1$

$$\therefore \quad [x] = n \qquad (1)$$

$$\int_0^x [t]\,dt = \int_0^1 [t]\,dt + \int_1^2 [t]\,dt + \int_2^3 [t]\,dt + \cdots + \int_n^{n+f} [t]\,dt$$

$$= 0 + 1\int_1^2 dt + 2\int_2^3 dt + \cdots + n\int_n^{n+f} dt$$

$$= (2-1) + 2(3-2) + \cdots + n(n+f-n)$$

$$= 1 + 2 + 3 + \cdots + (n-1) + nf$$

$$= \frac{(n-1)n}{2} + nf$$

$$= \frac{[x]([x]-1)}{2} + [x](x - [x]) \quad \text{[from equation (1)]}$$

10. $f(x) = 2e^{-x}$ is decreasing for $x \in [0, \infty)$.

Also, when $x = 0$, $2e^{-x} = 2$,

and when $x \to \infty$, $2e^{-x} \to 0$.

Thus, $[2e^{-x}]$ is discontinuous when $2e^{-x} = 1$ or $x = \log 2$.

Also, for $x > \ln 2$, $[2e^{-x}] = 0$

and for $0 < x < \log 2$, $[2e^{-x}] = 1$

$$\therefore \quad \int_0^{\infty}[2e^{-x}]dx = \int_0^{\ln 2}[2e^{-x}]dx + \int_{\ln 2}^{\infty}[2e^{-x}]dx$$

$$= \int_0^{\ln 2} 1dx + \int_{\ln 2}^{0} 0dx = (x)_0^{\ln 2} = \ln 2$$

Exercise 8.5

1. $I = \int_a^b xf(x)dx = \int_a^b x\,f(a+b-x)\,dx$

$(\because$ given $f(x) = f(a+b-x))$

$$= \int_a^b (a+b-x)\,f((a+b)-(a+b-x))\,dx$$

$$= \int_a^b (a+b-x)\,f(x)\,dx = (a+b)\int_a^b f(x)\,dx - I$$

or $\quad 2I = (a+b)\int_a^b f(x)\,dx$ or $I = \frac{a+b}{2}\int_a^b f(x)\,dx$

2. $I = \int_3^6 \frac{\sqrt{x}}{\sqrt{9-x}+\sqrt{x}}dx$ (1)

$$\therefore \quad I = \int_3^6 \frac{\sqrt{9-x}}{\sqrt{x}+\sqrt{9-x}}\,dx \qquad (2)$$

Adding (1) and (2),

$$2I = \int_3^6 1\cdot dx = [x]_3^6 = 6 - 3 = 3$$

Hence, $I = \frac{3}{2}$.

3. Using $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$,

$I = -I$ or $I = 0$

4. Let $I = \int_0^{\pi} xf(\sin x)dx$ (1)

$$\therefore \quad I = \int_0^{\pi}(\pi - x)f\{\sin(\pi - x)\}dx$$

or $\quad I = \int_0^{\pi}(\pi - x)f(\sin x)dx$ (2)

Thus, adding (1) and (2), we get

$$2I = \pi\int_0^{\pi} f(\sin x)dx$$

or $\quad I = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx.$

5. $I = \int_0^1 (1-x)\,x^n\,dx$ (replacing x by $1-x$)

$$= \int_0^1 (x^n - x^{n+1})\,dx$$

$$= \left(\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right)_0^1$$

$$= \frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{(n+1)(n+2)}$$

6. $f(x)f(a-x) = 1$ or $f(a-x) = \frac{1}{f(x)}$

Now, $I = \int_0^a \frac{dx}{1+f(x)} = \int_0^a \frac{dx}{1+f(a-x)}$

$$= \int_0^a \frac{dx}{1+\frac{1}{f(x)}}$$

$$= \int_0^a \frac{f(x)dx}{1+f(x)}$$

$$\therefore \quad 2I = \int_0^a \frac{1+f(x)}{1+f(x)}dx = a \text{ or } I = a/2$$

7. $\int_0^a f(x)g(x)dx$

$= \int_0^a f(a-x)\, g(a-x)dx$

$= \int_0^a f(x)\cdot\{2-g(x)\}dx$

$= 2\int_0^a f(x)\,dx - \int_0^a f(x)g(x)dx$

or $2\int_0^a f(x)f(x)\,dx = 2\int_0^a f(x)\,dx$

or $\int_0^a f(x)\,g(x)\,dx = \int_0^a f(x)\,dx$

8. $I = \int_0^{\pi/2} \sin 2x \log\tan x\,dx$

$= \int_0^{\pi/2} \sin 2\left(\frac{\pi}{2}-x\right)\log\tan\left(\frac{\pi}{2}-x\right)dx$

$= -\int_0^{\pi/2} \sin 2x \log\tan x\,dx = -I$

or $2I = 0$

or $I = 0$

9. Let $I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}\,dx,\ a>0$ (1)

$\therefore \quad I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^{-x}}\,dx$ (Replacing x by $-x$)

or $I = \int_{-\pi}^{\pi} \frac{a^x\cos^2 x}{1+a^x}\,dx$ (2)

Adding (1) and (2), we get

$2I = 4\int_0^{\pi/2}\cos^2 x\,dx$

$= 2\int_0^{\pi/2}(1+\cos 2x)\,dx$

$= 2\left[x+\frac{\sin 2x}{2}\right]_0^{\pi/2}$

$= \pi$

or $I = \frac{\pi}{2}$

10. Let $I = \int_0^{\pi} \frac{x\sin x\,dx}{1+\cos^2 x}$ (1)

or $I = \int_0^{\pi} \frac{(\pi-x)\sin x\,dx}{1+\cos^2 x}$ (2)

Adding (1) and (2), we get

$2I = \pi\int_0^{\pi} \frac{\sin x\,dx}{1+\cos^2 x}$

or $I = -\frac{\pi}{2}\int_1^{-1}\frac{dt}{1+t^2} = -\frac{\pi}{2}\left[\tan^{-1}t\right]_1^{-1}$

[Putting $\cos x = t$, $-\sin x\,dx = dt$]

$= -\frac{1}{2}\pi\left[\tan^{-1}(-1)-\tan^{-1}1\right] = \pi^2/4$

11. Let $I = \int_0^{\pi} \frac{x\,dx}{1+\cos\alpha\sin x}$ (1)

$= \int_0^{\pi} \frac{(\pi-x)\,dx}{1+\cos\alpha\,(\sin(\pi-x))}$

[using $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$]

$\therefore \quad I = \int_0^{\pi} \frac{(\pi-x)\,dx}{1+\cos\alpha\sin x}$ (2)

Adding (1) and (2) , we get

$2I = \int_0^{\pi} \frac{x+\pi-x}{1+\cos\alpha\sin x}dx$

$= \int_0^{\pi} \frac{\pi}{1+\cos\alpha\sin x}\,dx$

$\therefore \quad I = \frac{\pi}{2}\int_0^{\pi}\frac{1}{1+\cos\alpha\sin x}\,dx$

$= \frac{\pi}{2}\int_0^{\pi}\frac{1}{1+\cos\alpha\times\frac{2\tan x/2}{1+\tan^2 x/2}}dx$

$= \frac{\pi}{2}\int_0^{\pi}\frac{\sec^2 x/2}{1+\tan^2 x/2+2\cos\alpha\tan x/2}dx$

Put $\tan x/2 = t$ or $\frac{1}{2}\sec^2\frac{x}{2}dx = dt$

Also, when $x\to 0$, $t\to 0$

and when $x\to\pi$, $t\to\infty$

$\therefore \quad I = \frac{\pi}{2}\int_0^{\infty}\frac{2dt}{t^2+(2\cos\alpha)t+1}$

$= \pi\int_0^{\infty}\frac{dt}{(t+\cos\alpha)^2+1-\cos^2\alpha}$

$= \pi\int_0^{\infty}\frac{dt}{(t+\cos\alpha)^2+\sin^2\alpha}$

$= \pi\cdot\frac{1}{\sin\alpha}\left[\tan^{-1}\left(\frac{t+\cos\alpha}{\sin\alpha}\right)\right]_0^{\infty}$

$= \frac{\pi}{\sin\alpha}\left[\tan^{-1}\infty-\tan^{-1}(\cot\alpha)\right]$

$= \frac{\pi}{\sin\alpha}\left[\frac{\pi}{2}-\tan^{-1}(\cot\alpha)\right]$

$= \frac{\pi}{\sin\alpha}\left[\cot^{-1}(\cot\alpha)\right]$

$= \frac{\pi\alpha}{\sin\alpha}$

Exercise 8.6

1. $I = \int_0^{2\pi}\frac{1}{1+\tan^4 x}dx = 2\int_0^{\pi}\frac{dx}{1+\tan^4 x}$

$= 4\int_0^{\pi/2}\frac{dx}{1+\tan^4 x}$ ...(1)

$= 4\int_0^{\pi/2}\frac{dx}{1+\cot^4 x}$ ...(2)

Adding (1) and (2), we get

$$2I = 4\int_0^{\pi/2} dx = 4 \cdot \frac{\pi}{2}$$

$$\Rightarrow \quad I = \pi$$

2. $I = \int_0^{2\pi} \sin^{100} x \cos^{99} x dx$

Here, $f(x) = \sin^{100} x \cos^{99} x$ for which $f(2\pi - x) = f(x)$

$$\therefore \quad I = 2\int_0^{\pi} \sin^{100} x \cos^{99} x dx$$

$$= 2\int_0^{\pi} \sin^{100}(\pi - x)\cos^{99}(\pi - x)dx$$

$$= -2\int_0^{\pi} \sin^{100} x \cos^{99} x dx$$

$$= -I$$

or $\quad 2I = 0$ or $I = 0$

3. We have $U_n = \int_0^1 x^n .(2-x)^n \, dx$

Put $x = 2t$.

$$\therefore \quad dx = 2dt$$

$$\therefore \quad U_n = 2\int_0^{1/2} 2^n .t^n 2^n (1-t)^n dt$$

$$\therefore \quad U_n = 2^{2n+1}\int_0^{1/2} x^n (1-x)^n dx \qquad ...(1)$$

Now $V_n = \int_0^1 x^n (1-x)^n \, dx$

$$= 2\int_0^{1/2} x^n (1-x)^n \, dx \qquad ...(2)$$

From (1) and (2)

$$U_n = 2^{2n}. V_n$$

4. $I = \int_0^{\pi} \log(1+\cos x)dx = \int_0^{\pi} \log\left(2\cos^2\frac{x}{2}\right)dx$

$$= \int_0^{\pi}\left(\log 2 + 2\log\cos\frac{x}{2}\right)dx = \pi\log 2 + 2\int_0^{\pi}\log\cos\frac{x}{2}dx$$

$= \pi \log 2 + 2 \times 2 \int_0^{\pi/2} \log\cos t \, dt$, where $t = \frac{x}{2}$ and $dx = 2\,dt$

$$= \pi\log 2 + 4 \times \left(-\frac{\pi}{2}\log 2\right)$$

$$= -\pi \log 2$$

5. $\int_0^1 \{(\sin^{-1} x)/x\}dx$

$$= \left[\left(\sin^{-1} x\right)(\log x)\right]_0^1 - \int_0^1 \frac{1}{\sqrt{1-x^2}}\log x \, dx$$

$$= 0 - \lim_{x\to 0}\frac{\sin^{-1} x}{x}(x\log x) - \int_0^{\pi/2}\frac{1}{\sqrt{1-\sin^2\theta}}(\log\sin\theta)\cos\theta \, d\theta$$

$$= -\lim_{x\to 0} x\log x - \int_0^{\pi/2}\log\sin\theta d\theta$$

$$= \frac{\pi}{2}\log 2$$

6. $I = \int_{-\infty}^{0} \frac{te^t}{\sqrt{1-e^{2t}}}dt$

Put $e^t = x$

$$\therefore \quad I = \int_0^1 \frac{\log_e x}{\sqrt{1-x^2}}dx$$

Put $x = \sin\theta$

$$\therefore \quad I = \int_0^{\pi/2} \log_e \sin\theta \, d\theta = \frac{\pi}{2}\log_e\frac{1}{2}$$

7. $I_1 = \int_0^{\pi} (\pi - x) f(\sin^3 x + \cos^2 x) \, dx$

$$\therefore \quad 2I = \pi \int_0^{\pi} f(\sin^3 x + \cos^2 x) \, dx$$

$$= 2\pi \int_0^{\pi/2} f(\sin^3 x + \cos^2 x) \, dx$$

or $\quad I_1 = \pi \int_0^{\pi/2} f(\sin^3 x + \cos^2 x) \, dx = \pi I_2$

Exercise 8.7

1. Let $I = \int_{-\pi/2}^{\pi/2} \sin^2 x\cos^2 x(\sin x + \cos x)dx$

$$= \int_{-\pi/2}^{\pi/2} \sin^3 x\cos^2 x \, dx + \int_{-\pi/2}^{\pi/2} \sin^2 x\cos^3 x \, dx \qquad (1)$$

Since $\sin^3 x \cos^2 x$ is an odd function and $\sin^2 x \cos^3 x$ is an even function. Therefore, $\int_{-\pi/2}^{\pi/2} \sin^3 x\cos^2 x \, dx = 0$

and $\quad \int_{-\pi/2}^{\pi/2} \sin^2 x\cos^3 x \, dx = 2\int_0^{\pi/2} \sin^2 x\cos^3 x \, dx.$

$$\therefore \quad I = 2\int_0^{\pi/2} \sin^2 x\cos^3 x \, dx$$

$$= 2\int_0^1 t^2(1-t^2)dt$$

$$= \int_0^1 (t^2 - t^4)dt = 2\left[\frac{1}{3} - \frac{1}{5}\right] = \frac{4}{15}$$

2. $I = \int_{-1}^{1} \frac{x^3 + |x| + 1}{x^2 + 2|x| + 1}dx$

$$= \int_{-1}^{1} \frac{x^3}{x^2 + 2|x| + 1}dx + \int_{-1}^{1} \frac{|x| + 1}{x^2 + 2|x| + 1}dx$$

$$= 0 + 2\int_0^1 \frac{(|x| + 1)}{(|x| + 1)^2}dx = 2\int_0^1 \frac{dx}{1+x}$$

$$= 2\ln(1 + x)|_0^1 = 2\ln 2$$

3. Value = 0 since $(1 - x^2) \sin x \cos^2 x$ is an odd function of x.

4. $\int_{-1}^{1} \frac{\sin x - x^2}{3 - |x|}dx$

$$= \int_{-1}^{1} \frac{\sin x}{3 - |x|}dx - \int_{-1}^{1} \frac{x^2}{3 - |x|}$$

$$= 0 - 2\int_0^1 \frac{x^2}{3 - |x|}dx \left[\because \frac{\sin x}{3 - |x|} \text{ is odd and } \frac{x^2}{3 - |x|} \text{ is even}\right]$$

$$= -2\int_0^1 \frac{x^2\,dx}{3-|x|} = 2\int_0^1 \frac{x^2}{x-3}\,dx$$

$$= 2\int_0^1 \left(x+3+\frac{9}{x-3}\right)dx$$

$$= 2\left[\frac{x^2}{2}+3x+9\log|x-3|\right]_0^1$$

$$= \left[7+18\log\frac{2}{3}\right]$$

5. Since $\cos x \log\dfrac{1-x}{1+x}$ is an odd function of x,

$$\int_{-1/2}^{1/2} \cos x \log\frac{1-x}{1+x}\,dx = 0.$$

6. Let $I = \int_{-3\pi/2}^{-\pi/2}[(x+\pi)^3 + \cos^2(x+3\pi)]\,dx$

Put $x+\pi = t$, so that $dx = dt$

When $x = -\dfrac{3\pi}{2}$, $t = -\dfrac{\pi}{2}$

When $x = -\dfrac{\pi}{2}$, $t = \dfrac{\pi}{2}$

$$\therefore \quad I = \int_{-\pi/2}^{\pi/2}\left[t^3+\cos^2(t+2\pi)\right]dt$$

$$= \int_{-\pi/2}^{\pi/2}\left[t^3+\cos^2 t\right]dt$$

$$= 0 + 2\int_0^{\pi/2}\cos^2 t\,dt = \int_0^{\pi/2}(1+\cos 2t)\,dt$$

$$= \frac{\pi}{2}+0 = \frac{\pi}{2}$$

Exercise 8.8

1. $x-[x] = \{x\}$ has period 1.

Thus, $\int_0^{100}(x-[x])\,dx = 100\int_0^1 \{x\}\,dx$

$$= 100\int_0^1 x\,dx = \frac{100}{2}[x^2]_0^1 = 50$$

2. We have $f(x) = \sqrt{(1-\cos 2x)} = \sqrt{2\sin^2 x} = \sqrt{2}\,|\sin x|$

Now, $f(x+\pi) = \sqrt{2}\,|\sin(x+\pi)| = \sqrt{2}|\sin x| = f(x)$

i.e., $f(x)$ is periodic function with period π.

$$\int_0^{100\pi}\sqrt{(1-\cos 2x)}\,dx = \int_0^{100\pi}\sqrt{2}|\sin x|\,dx$$

$$= 100\sqrt{2}\int_0^{\pi}|\sin x|\,dx$$

$$= 100\sqrt{2}\int_0^{\pi}\sin x\,dx$$

$$= 100\sqrt{2}\,[-\cos x]_0^{\pi} = 200\sqrt{2}$$

3. Since $\cos^2 x$ is a periodic function with period π, so is $f(\cos^2 x)$.

Hence, $\int_0^{n\pi} f(\cos^2 x)dx = n\int_0^{\pi} f(\cos^2 x)dx$ or $k = n$.

4. Let $I = \int_0^{n\pi+t}(|\cos x|+|\sin x|)dx$

$$= \int_0^{n\pi}(|\cos x|+|\sin x|)dx + \int_{n\pi}^{n\pi+t}(|\cos x|+|\sin x|)dx$$

$$= 2n\int_0^{\pi/2}(|\cos x|+|\sin x|)dx + \int_0^{t}(|\cos x|+|\sin x|)dx$$

$$= 2n\int_0^{\pi/2}(\cos x+\sin x)dx + \int_0^{t}(\cos x+\sin x)dx$$

$$= 4n + \sin t - \cos t + 1$$

5. $\int_0^{10} e^{2x-[2x]}\,d(x-[x])$

$$= \int_0^{10} e^{\{2x\}}\,dx$$

$$= 20\int_0^{1/2} e^{\{2x\}}\,dx \qquad (\{2x\} \text{ has period } 1/2)$$

$$= 20\int_0^{1/2} e^{2x}\,dx, \qquad [\text{for } x \in (0, 1/2),\ \{2x\} = 2x]$$

$$= 10\left(e^{2x}\right)_0^{1/2}$$

$$= 10(e-1)$$

6. $f(x+a)+f(x) = 0$

or $\quad f(x+2a)+f(x+a) = 0$

or $\quad f(x) = f(x+2a)$

Thus, $f(x)$ is periodic with period $2a$.

Since $\int_b^{c+b} f(x)dx$ is independent of b, then c must be $k(2a)$ where $k \in N$.

Hence, least positive value of c is $2a$.

7. Let $I = \int_0^{n\pi+v}|\sin x|\,dx$

$$= \int_0^{v}|\sin x|\,dx + \int_v^{n\pi+v}|\sin x|\,dx$$

$$= \int_0^{v}\sin x\,dx + n\int_0^{\pi}|\sin x|\,dx \qquad [\because |\sin x| \text{ has period } \pi]$$

$$= (-\cos x)_0^{v} + n(-\cos x)_0^{\pi}$$

$$= 2n + 1 - \cos v = \text{R.H.S.}$$

Exercise 8.9

1. We have $\int_{\pi/3}^{x}\sqrt{(3-\sin^2 t)}\,dt + \int_0^{y}\cos t\,dt = 0.$

Differentiating w.r.t. x,

$$\frac{d}{dx}\int_{\pi/3}^{x}\sqrt{(3-\sin^2 t)}\,dt + \frac{d}{dx}\int_0^{y}\cos t\,dt = 0$$

or $\quad \sqrt{(3-\sin^2 x)} + \cos y\,\dfrac{dy}{dx} = 0$

or $\quad \dfrac{dy}{dx} = -\dfrac{\sqrt{(3-\sin^2 x)}}{\cos y}$

2. $g(x) = \int_2^{x}\dfrac{t\,dt}{1+t^4}$ or $g'(x) = \dfrac{x}{1+x^4}$ or $g'(2) = \dfrac{2}{17}$

Now, $f(x) = e^{g(x)}$ or $f'(x) = e^{g(x)}\,g'(x)$ or $f'(2) = e^{g(2)}g'(2)$

$\therefore \quad f'(2) = e^0 \times \dfrac{2}{17} = \dfrac{2}{17}$ as $g(2) = 0$

3. $L = \lim_{x\to 4}\int_4^{x}\dfrac{(4t-f(t))}{(x-4)}dt = \lim_{x\to 4}\dfrac{\int_4^{x}(4t-f(t))dt}{x-4}$

(0/0 form, using L' Hopital's rule)

$$= \lim_{x\to 4}\frac{4x-f(x)}{1}$$

$$= 16 - f(4)$$

4. Given limit is of the form $\frac{0}{0}$.

Then by L'Hopital's rule,

Given limit $\lim_{x\to 0}\frac{\int_0^x \cos t^2\, dt}{x} = \lim_{x\to 0}\frac{\cos x^2}{1} = 1.$

5. $f(x) = \int_0^x t(t-1)(t-2)\,dt$

Let $f'(x) = x(x-1)(x-2) = 0$

$\therefore$ $x = 0, 1,$ or 2

At $x = 0$ and 2, $f'(x)$ changes sign from –ve to +ve.

Hence, $x = 0$ and 2 are points of minima.

6. $y|_{x=1} = 0, \frac{dy}{dx} = \frac{1}{\sqrt{1+x^6}}3x^2 - \frac{1}{\sqrt{1+x^4}}2x$

or $\left.\frac{dy}{dx}\right|_{x=1} = \frac{1}{\sqrt{2}}$

Thus, required equation is $y\sqrt{2} = x - 1$.

7. $f(x) = \sin x \int_{\pi^2/16}^{x^2} \frac{\sin\sqrt{\theta}}{1+\cos^2\sqrt{\theta}}d\theta$

$\therefore$ $f'(x) = \sin x\left[\frac{\sin x}{1+\cos^2 x}2x - 0\right] + \left(\int_{\pi^2/16}^{x^2}\frac{\sin\sqrt{\theta}}{1+\cos^2\sqrt{\theta}}d\theta\right)\cos x$

Therefore, $f'\left(\frac{\pi}{2}\right) = \pi$

8. $f(x) = \int_0^x \sin(t^2 - t + x)\,dt$

$= \cos x \int_0^x \sin(t^2-t)\,dt + \sin x \int_0^x \cos(t^2-t)\,dt.$

or $f'(x) = -\sin x \int_0^x \sin(t^2-t)\,dt + \cos x \sin(x^2-x)$

$+ \cos x \int_0^x \cos(t^2-t)\,dt + \sin x \cos(x^2-x)$

$= -\sin x \int_0^x \sin(t^2-t)\,dt + \cos x \int_0^x \cos(t^2-t)\,dt + \sin x^2$

or $f''(x) = -\sin x \sin(x^2-x) - \cos x \int_0^x \sin(t^2-t)\,dt - \sin x$

$\int_0^x \cos(t^2-t)\,dt + \cos x \cos(x^2-x) + 2x\sin x^2$

$= \cos x^2 - f(x) + 2x\sin x^2$

or $f''(x) + f(x) = \cos x^2 + 2x\sin x^2$

9. We have $f(x) = \int_0^x e^{x^2-(x-t)^2}\cos(x-t)\,dt$

$\therefore$ $f(x) = \int_0^x e^{x^2-(x-(x-t))^2}\cos(x-(x-t))\,dt$

$\Rightarrow$ $f(x) = \int_0^x e^{x^2-t^2}\cos t\,dt$

$\Rightarrow$ $f(x) = e^{x^2}\int_0^x e^{-t^2}\cos t\,dt$ .....(1)

Differentiating w.r.t. x, we get

$f'(x) = 2xe^{x^2}\int_0^x e^{-t^2}\cos t\,dt + \cos x$

Again differentiating w.r.t. x, we get

$f''(x) = 2(e^{x^2} + 2x^2e^{x^2})\int_0^x e^{-t^2}\cos t\,dt + 2x\cos x - \sin x$

$= 2(1+2x^2)f(x) + 2x\cos x - \sin x$

$\therefore$ $f''(0) = 2f(0) + 0 - 0 = 0$ ($\because f(0) = 0$ from (1))

Exercise 8.10

1. In $\int_{b-1}^{b}\frac{e^{-t}dt}{t-b-1}$, put $t - b - 1 = -y - 1$. So, $dt = -dy$.

$\therefore$ $\int_{b-1}^{b}\frac{e^{-t}dt}{t-b-1} = \int_1^0 \frac{e^{y-b}}{-y-1}(-dy)$

$= -e^{-b}\int_0^1 \frac{e^y}{y+1}dy = -ae^{-b}$

2. Given $f(x) = \int_1^x \frac{\log t}{1+t+t^2}dt$

or $f\left(\frac{1}{x}\right) = \int_1^{1/x}\frac{\log t}{1+t+t^2}\,dt$

Let $y = \frac{1}{t}$ or $dy = -\frac{dt}{t^2}$

$\therefore$ $f\left(\frac{1}{x}\right) = \int_1^x \frac{\log\frac{1}{y}}{1+\frac{1}{y}+\frac{1}{y^2}}\left(-\frac{1}{y^2}dy\right)$

$= \int_1^x \frac{\log y}{1+y+y^2}dy = f(x)$

3. $f(x) = \int_1^x \frac{\tan^{-1}(t)}{t}dt$

$\therefore$ $f\left(\frac{1}{x}\right) = \int_1^{1/x}\frac{\tan^{-1}(t)}{t}dt$

Put $t = 1/u$

$\therefore$ $dt = -\frac{du}{u^2}$

$\therefore$ $f(1/x) = \int_1^x \frac{\tan^{-1}\left(\frac{1}{u}\right)}{\frac{1}{u}}\left(-\frac{1}{u^2}\right)du$

$= -\int_1^x \frac{\tan^{-1}\left(\frac{1}{u}\right)}{u}du$

$= -\int_1^x \frac{\cot^{-1}(u)}{u}du$

$= -\int_1^x \frac{\cot^{-1}(t)}{t}dt$

Now, $f(x)-f(1/x)=\int_1^x \frac{\tan^{-1}t+\cot^{-1}t}{t}dt$

$=\int_1^x \frac{\pi}{2}\times\frac{1}{t}dt$

$=\frac{\pi}{2}\log(x)$

$\therefore \quad f(e^2)-f(1/e^2)=\frac{\pi}{2}\log_e e^2=\pi$

4. $I=\int_{\sqrt{2}-1}^{\sqrt{2}+1}\frac{(x^2-1)}{(x^2+1)^2}dx=\int_{1/a}^{a}\frac{(x^2-1)}{(x^2+1)^2}dx$, where $a=\sqrt{2}+1$

Put $x=\frac{1}{t}$. So, $dx=-\frac{1}{t^2}dt$

$\therefore \quad I=\int_a^{1/a}\frac{\frac{1}{t^2}-1}{\left(\frac{1}{t^2}+1\right)^2}\cdot\left(-\frac{1}{t^2}\right)dt$

$=-\int_a^{1/a}\frac{(1-t^2)t^4}{t^4(1+t^2)^2}dt$

$=-\int_a^{1/a}\frac{(1-t^2)}{(1+t^2)^2}dt$

$=\int_a^{1/a}\frac{t^2-1}{(t^2+1)^2}dt$

$=-\int_{1/a}^{a}\frac{t^2-1}{(t^2+1)^2}dt=-I$

or $\quad 2I=0$

or $\quad I=0$

5. Put $x+1=t$ in first integral. Then

$\int_1^e \frac{e^{\frac{t^2-2}{2}}}{t}dt+\int_1^e x\log x e^{\frac{x^2-2}{2}}dx$

$=\int_1^e \frac{e^{\frac{x^2-2}{2}}}{x}dt+\int_1^e x\log x\, e^{\frac{x^2-2}{2}}dx$

$=\left[e^{\frac{x^2-2}{2}}\log x\right]_1^e-\int_1^e xe^{\frac{x^2-2}{2}}\log x\,dx+\int_1^e x\log x\, e^{\frac{x^2-2}{2}}dx$

$=e^{\frac{e^2-2}{2}}$

6. Let $I=\int_{\frac{1}{2}}^{2}e^{\left|x-\frac{1}{x}\right|}dx$ ... (1)

Put $\quad x=\frac{1}{t}$;

$\therefore \quad I=-\int_2^{\frac{1}{2}}e^{\left|t-\frac{1}{t}\right|}\frac{dt}{t^2}=\int_{\frac{1}{2}}^{2}e^{\left|x-\frac{1}{x}\right|}\frac{dx}{x^2}$ ... (2)

Adding (1) and (2), we get

$2I=\int_{\frac{1}{2}}^{2}e^{\left|x-\frac{1}{x}\right|}\cdot\left(1+\frac{1}{x^2}\right)dx$

$=\int_{\frac{1}{2}}^{1}e^{-\left(x-\frac{1}{x}\right)}\cdot\left(1+\frac{1}{x^2}\right)dx+\int_1^2 e^{\left(x-\frac{1}{x}\right)}\cdot\left(1+\frac{1}{x^2}\right)dx$

$=\left[-e^{-\left(x-\frac{1}{x}\right)}\right]_{1/2}^{1}+\left[e^{\left(x-\frac{1}{x}\right)}\right]_1^2$

$=-1+e^{3/2}+e^{3/2}-1$

$=2(e\sqrt{e}-1)$

$\Rightarrow \quad I=e\sqrt{e}-1$

7. $I_2=\int_0^{\pi/4}\frac{e^{\tan^2\theta}\cdot\tan\theta}{(2-\tan^2\theta)}\sec^2\theta\,d\theta$

Put $\tan^2\theta=t$

$\therefore \quad 2\tan\theta\sec^2\theta\,d\theta=dt$

$\therefore \quad I_2=\frac{1}{2}\int_0^1\frac{e^t dt}{(2-t)}$

$=\frac{1}{2}\int_0^1\frac{e^{1-t}dt}{1+t}$

$=\frac{e}{2}\int_0^1\frac{dt}{e^t(t+1)}=\frac{e}{2}.I_1$

$\therefore \quad \frac{I_1}{I_2}=\frac{2}{e}$

Exercise 8.11

1. $I_k=\int_1^e(\ln x)^k dx=|x(\ln x)^k|_1^e-k\int_1^e(\ln x)^{k-1}dx$

or $\quad I_k=e-kI_{k-1}$

or $\quad I_4=e-4I_3$

$=e-4(e-3(e-3(e-2I_1)))$

$=9e-24$ $\quad (\because I_1=1)$

2. $I_m=\int_1^e(\log x)^m\,dx$

$=(x(\log x)^m)_1^e-\int_1^e x\frac{m(\log x)^{m-1}}{x}dx$ (Integrating by parts)

$=e-m\int_1^e(\log x)^{m-1}dx=e-mI_{m-1}$ (1)

Replacing m by $m-1$, we get

$I_{m-1}=e-(m-1)I_{m-2}$ (2)

From (1) and (2), we have $I_m=e-m[e-(m-1)I_{m-2}]$

or $\quad I_m-m(m-1)I_{m-2}=e(1-m)$

or $\quad \frac{I_m}{1-m}+mI_{m-2}=e$

3. $I_n = \int_0^{\pi} x^n \sin x dx$

$= \left[-x^n \cos x\right]_0^{\pi} + n\int_0^{\pi} x^{n-1} \cos x dx$

$= \pi^n + n[x^{n-1} \sin x]_0^{\pi} - n(n-1)\int_0^{\pi} x^{n-2} \sin x dx$

$\Rightarrow \quad I_n = \pi^n + n \cdot 0 - n(n-1) I_{n-2}$

Put $n = 5$

$I_5 = \pi^5 - 20I_3$

$I_5 + 20I_3 = \pi^5$

4. $L(m, n) = \int_0^1 t^m (1+t)^n dx$

$= \left[\frac{t^{m+1}}{m+1}(1+t)^n\right]_0^1 - \int_0^1 n(1+t)^{n-1} \frac{t^{m+1}}{m+1} dx$

$= \left[\frac{t^{m+1}}{m+1}(1+t)^n\right]_0^1 - \frac{n}{m+1}\int_0^1 t^{m+1}(1+t)^{n-1} dx$

$= \frac{2^n}{m+1} - \frac{n}{m+1} L(m+1, n-1)$

5. $I_n = \int_0^1 x^n (\tan^{-1} x) dx = \int_0^1 x^{n-1} (x \tan^{-1} x) dx$

$= \left[x^{n-1}\left(\frac{x^2}{2}\tan^{-1} x - \frac{x}{2} + \frac{\tan^{-1} x}{2}\right)\right]_0^1$

$-(n-1)\int_0^1 x^{n-2}\left(\frac{x^2}{2}\tan^{-1} x - \frac{x}{2} + \frac{\tan^{-1} x}{2}\right) dx$

$= \frac{\pi}{4} - \frac{1}{2} - \frac{(n-1)}{2} I_n + \frac{(n-1)}{2}\int_0^1 x^{n-1} dx - \frac{1}{2}(n-1) I_{n-2}$

or $\quad \frac{(n+1)}{2} I_n = \frac{\pi}{4} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2n} - \frac{1}{2}(n-1) I_{n-2}$

or $\quad (n+1) I_n + (n-1) I_{n-2} = -\frac{1}{n} + \frac{\pi}{2}$

6. $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^{m-1} x \left(\sin x \cos^n x\right) dx$

$= \left[-\frac{\sin^{m-1} x \cos^{n+1} x}{n+1}\right]_0^{\frac{\pi}{2}}$

$+ \int_0^{\frac{\pi}{2}} \frac{\cos^{n+1} x}{n+1} (m-1) \sin^{m-2} x \cos x \, dx$

$= \left(\frac{m-1}{n+1}\right)\int_0^{\frac{\pi}{2}} \sin^{m-2} x \cos^n x \cos^2 x \, dx$

$= \left(\frac{m-1}{n+1}\right)\int_0^{\frac{\pi}{2}} \left(\sin^{m-2} x \cos^n x - \sin^m x \cos^n x\right) dx$

$= \left(\frac{m-1}{n+1}\right) I_{m-2,n} - \left(\frac{m-1}{n+1}\right) I_{m,n}$

or $\quad \left(1 + \frac{m-1}{n+1}\right) I_{m,n} = \left(\frac{m-1}{n+1}\right) I_{m-2,n}$

or $\quad I_{m,n} = \left(\frac{m-1}{m+n}\right) I_{m-2,n}$

$= \left(\frac{m-1}{m+n}\right)\left(\frac{m-3}{m+n-2}\right)\left(\frac{m-5}{m+n-4}\right) \ldots I_{0,n} \text{ or } I_{1,n}$

According as m is even or odd

$I_{0,n} = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$ and $I_{1,n} = \int_0^{\frac{\pi}{2}} \sin x \cos^n x \, dx = \frac{1}{n+1}$

$\therefore \quad I_{m,n} = \begin{cases} \frac{(m-1)(m-3)(m-5)\ldots(n-1)(n-3)(n-5)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots 2} \cdot \frac{\pi}{2}, \\ \qquad \text{when both } m, n \text{ are even} \\ \frac{(m-1)(m-3)(m-5)\ldots(n-1)(n-3)(n-5)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots}, \\ \qquad \text{otherwise} \end{cases}$

Exercises

Single Correct Answer Type

1. (4) $f(x) = \lim_{n\to\infty} \frac{1}{n}\left(\left(x + \frac{1}{n}\right)^2 + \left(x + \frac{2}{n}\right)^2 + \ldots + \left(x + \frac{n-1}{n}\right)^2\right)$

$= \int_0^1 (x+y)^2 \, dy$

$= x^2 + x + \frac{1}{3}$

$\therefore \quad f(x) = \left(x + \frac{1}{2}\right)^2 + \frac{1}{3} - \frac{1}{4}$

$\therefore \quad f(x)|_{\min} = \frac{1}{12}$

2. (2) $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{1}{1+\sqrt{n}} + \frac{1}{2+\sqrt{2n}} + \cdots + \frac{1}{n+\sqrt{n^2}}\right]$

$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{\frac{1}{n}+\frac{1}{\sqrt{n}}} + \frac{1}{\frac{2}{n}+\sqrt{\frac{2}{n}}} + \cdots + \frac{1}{\frac{n}{n}+\sqrt{\frac{n}{n}}}\right]$

$= \int_0^1 \frac{dx}{\sqrt{x}\left(\sqrt{x}+1\right)}$

Put $\sqrt{x} = z$ or $\frac{1}{2\sqrt{x}} dx = dz$

$\therefore \quad \lim_{n\to\infty} S_n = \int_0^1 \frac{2\,dz}{z+1} = 2\,|\log(z+1)|_0^1$

$= 2(\log 2 - \log 1)$

$= 2 \log 2 = \log 4$

3. (3) $\lim_{n\to\infty}\sum_{r=1}^{4n}\frac{\sqrt{n}}{\sqrt{r}\left(3\sqrt{r}+4\sqrt{n}\right)^2}$

$$T_r=\frac{1}{\sqrt{\frac{r}{n}}n\left(3\sqrt{\frac{r}{n}}+4\right)^2}$$

$$\therefore\quad S=\lim_{n\to\infty}\frac{1}{n}\sum_{1}^{4n}\frac{1}{\left(3\sqrt{\frac{r}{n}}+4\right)^2\sqrt{\frac{r}{n}}}=\int_0^4\frac{dx}{\sqrt{x}(3\sqrt{x}+4)^2}$$

Put $3\sqrt{x}+4=t$

or $\frac{3}{2}\frac{1}{\sqrt{x}}dx=dt$

$$\therefore\quad S=\frac{2}{3}\int_4^{10}\frac{dt}{t^2}=\frac{2}{3}\left[-\frac{1}{t}\right]_{10}^{4}=\frac{1}{10}$$

4. (1) $\lim_{n\to\infty}\frac{\left(\frac{1}{n}\sum_{r=1}^{n}\frac{r^2}{n^2}\right)\left(\frac{1}{n}\sum_{r=1}^{n}\frac{r^3}{n^3}\right)\left(\frac{1}{n}\sum_{r=1}^{n}\frac{r^4}{n^4}\right)}{\left(\frac{1}{n}\sum_{r=1}^{n}\frac{r^5}{n^5}\right)^2}$

$$=\frac{\int_0^1 x^2\,dx\cdot\int_0^1 x^3\,dx\cdot\int_0^1 x^4\,dx}{\left(\int_0^1 x^5\,dx\right)^2}$$

$$=\frac{\frac{1}{3}\times\frac{1}{4}\times\frac{1}{5}}{\frac{1}{6}\times\frac{1}{6}}=\frac{36}{60}=\frac{3}{5}$$

5. (3) Let $A=\lim_{n\to\infty}\left[\tan\frac{\pi}{2n}\tan\frac{2\pi}{2n}\cdots\tan\frac{n\pi}{2n}\right]^{1/n}$

$$\therefore\quad \log A=\lim_{n\to\infty}\frac{1}{n}\left[\log\tan\frac{\pi}{2n}+\log\tan\frac{2\pi}{2n}+\cdots+\log\tan\frac{n\pi}{2n}\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\log\tan\frac{\pi r}{2n}$$

$$=\int_0^1\log\tan\left(\frac{\pi}{2}x\right)dx$$

$$=\frac{2}{\pi}\int_0^{\pi/2}\log\tan y\,dy \qquad (1)$$

$$\left[\text{Putting }\frac{1}{2}\pi x=y\text{ and }dx=(2/\pi)\,dy\right]$$

Now, let $I=\int_0^{\pi/2}\log\tan y\,dy$

$$\therefore\quad I=\int_0^{\pi/2}\log\tan\left(\frac{1}{2}\pi-y\right)dy$$

$$=\int_0^{\pi/2}\log\cot y\,dy$$

$$=-\int_0^{\pi/2}\log\tan y\,dy=-I$$

or $I+I=0$ or $2I=0$ or $I=0$

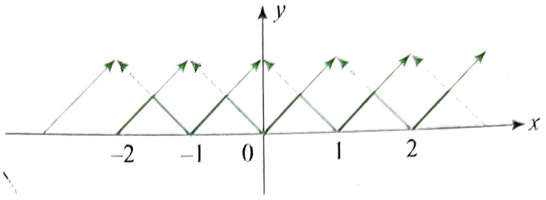
Thus, from equation (1), $\log A=0$ or $A=e^0=1$.

6. (1) $f(2-\alpha)=f(2+\alpha)$

Thus, function is symmetric about the line $x=2$.

$$\int_{2-a}^{2+a}f(x)dx=2\int_{2}^{2+a}f(x)dx$$

7. (1)



The graph with solid line is the graph of $f(x)=\{x\}$ and the graph with dotted lines is the graph of $f(x)=\{-x\}$. Now, the graph of $\min(\{x\},\{-x\})$ is the graph with dark solid lines.

$\int_{-100}^{100}f(x)dx=$ Area of 200 triangles shown as solid dark lines in the diagram

$$=200\,\frac{1}{2}(1)\left(\frac{1}{2}\right)=50$$

8. (4) $I_1=\int_{a+c}^{b+c}f(x)dx$. Putting $x=t+c$ and $dx=dt$, we get

$$I_1=\int_a^b f(t+c)\,dt=\int_a^b f(x+c)\,dx$$

Now, $I_2=\int_{ac}^{bc}f(x)dx$

Putting $x=tc$ and $dx=c\,dt$, we get

$$I_2=c\int_a^b f(ct)\,dt=c\int_a^b f(cx)\,dx$$

$$f(x)=\frac{1}{2}(f(x)+f(-x)+f(x)-f(-x))$$

$$\therefore\quad I_3=\int_{-a}^{a}f(x)dx$$

$$=\frac{1}{2}\int_{-a}^{a}(f(x)+f(-x)+f(x)-f(-x))dx$$

$$=\frac{1}{2}\int_{-a}^{a}(f(x)+f(-x))dx+\frac{1}{2}\int_{-a}^{a}(f(x)-f(-x))dx$$

$$=\frac{1}{2}\int_{-a}^{a}(f(x)+f(-x))dx$$

as $f(x)+f(-x)$ is even and $f(x)-f(-x)$ is odd.

9. (3) $\int_{-1}^{1/2}\frac{e^x\left(2-x^2\right)dx}{(1-x)\sqrt{1-x^2}}=\int_{-1}^{1/2}\frac{e^x\left(1-x^2+1\right)}{(1-x)\sqrt{1-x^2}}$

$$=\int_{-1}^{1/2}e^x\left[\sqrt{\frac{1+x}{1-x}}+\frac{1}{(1-x)\sqrt{1-x^2}}\right]dx$$

$$=e^x\sqrt{\frac{1+x}{1-x}}\Bigg|_{-1}^{1/2}=\sqrt{3e}$$

10. (3) $\frac{\pi}{6}=\int_{\log 2}^{x}\frac{dx}{\sqrt{e^x-1}}=\int_{\log 2}^{x}\frac{e^{x/2}dx}{e^{x/2}\sqrt{(e^{x/2})^2-1}}$

$$=2\left[\sec^{-1}e^{x/2}\right]_{\log 2}^{x}$$

$$=2\left[\sec^{-1}e^{x/2}-\sec^{-1}\sqrt{2}\right]$$

or $\frac{\pi}{12}+\frac{\pi}{4}=\sec^{-1}e^{x/2}$

or $\frac{\pi}{3}=\sec^{-1}e^{x/2}$

or $e^{x/2}=2$

or $x/2=\log 2$

or $x=\log 4$

11. (4) $I=\int_{5/2}^{5}\frac{\sqrt{(25-x^2)^3}}{x^4}dx$

Let $x=5\sin\theta$

$\therefore \quad dx=5\cos\theta\,d\theta$

$\therefore \quad I=\int_{\pi/6}^{\pi/2}\frac{\sqrt{(25-25\sin^2\theta)^3}}{5^4\sin^4\theta}\cdot 5\cos\theta\,d\theta$

$=\int_{\pi/6}^{\pi/2}\frac{5^3\cos^3\theta\cdot 5\cos\theta}{5^4\sin^4\theta}d\theta$

$=\int_{\pi/6}^{\pi/2}\cot^2\theta(\operatorname{cosec}^2\theta-1)\,d\theta$

$=\int_{\pi/6}^{\pi/2}\cot^2\theta\operatorname{cosec}^2\theta\,d\theta-\int_{\pi/6}^{\pi/2}\cot^2\theta\,d\theta$

$=\int_{\pi/6}^{\pi/2}\cot^2\theta\operatorname{cosec}^2\theta\,d\theta-\int_{\pi/6}^{\pi/2}(\operatorname{cosec}^2\theta-1)\,d\theta$

$=\left[-\frac{\cot^3\theta}{3}+\cot\theta+\theta\right]_{\pi/6}^{\pi/2}$

$=-0+0+\frac{\pi}{2}-\left(-\frac{3\sqrt{3}}{3}+\sqrt{3}+\frac{\pi}{6}\right)=\frac{\pi}{3}$

12. (3) As $f(x)$ satisfies the conditions of Rolle's theorem in [1, 2], $f(x)$ is continuous in the interval and $f(1)=f(2)$.

Therefore, $\int_1^2 f'(x)dx=[f(x)]_1^2=f(2)-f(1)=0$

13. (2) Putting $e^x-1=t^2$ in the given integral, we have

$$\int_0^{\log 5}\frac{e^x\sqrt{e^x-1}}{e^x+3}dx=2\int_0^2\frac{t^2}{t^2+4}dt=2\left(\int_0^2 1dt-4\int_0^2\frac{dt}{t^2+4}\right)$$

$$=2\left[\left(t-2\tan^{-1}\left(\frac{t}{2}\right)\right)_0^2\right]$$

$$=2[(2-2\times\pi/4)]=4-\pi$$

14. (3) Given integral

$$=\int_0^1\frac{dx}{(x+\cos\alpha)^2+(1-\cos^2\alpha)}$$

$$=\int_0^1\frac{dx}{(x+\cos\alpha)^2+\sin^2\alpha}$$

$$=\frac{1}{\sin\alpha}\left|\tan^{-1}\frac{x+\cos\alpha}{\sin\alpha}\right|_0^1$$

$$=\frac{1}{\sin\alpha}\left[\tan^{-1}\frac{1+\cos\alpha}{\sin\alpha}-\tan^{-1}\frac{\cos\alpha}{\sin\alpha}\right]$$

$$=\frac{1}{\sin\alpha}\left[\tan^{-1}\cot\frac{\alpha}{2}-\tan^{-1}(\cot\alpha)\right]$$

$$=\frac{1}{\sin\alpha}\left[\tan^{-1}\tan\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)-\tan^{-1}\tan\left(\frac{\pi}{2}-\alpha\right)\right]$$

$$=\frac{1}{\sin\alpha}\left[\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)-\left(\frac{\pi}{2}-\alpha\right)\right]$$

$$=\frac{\alpha}{2\sin\alpha}$$

15. (1) Putting $x=\tan\theta$, we get

$$\int_0^{\pi/2}\frac{dx}{[x+\sqrt{x^2+1}]^3}=\int_0^{\pi/2}\frac{\sec^2\theta d\theta}{(\tan\theta+\sec\theta)^3}$$

$$=\int_0^{\pi/2}\frac{\cos\theta}{(1+\sin\theta)^3}d\theta$$

$$=\left[-\frac{1}{2(1+\sin\theta)^2}\right]_0^{\pi/2}=-\frac{1}{8}+\frac{1}{2}=\frac{3}{8}$$

16. (1) We have $f(y)=e^y$, $g(y)=y$, $y>0$

$$F(t)=\int_0^t f(t-y)g(y)dy$$

$$=\int_0^t e^{t-y}y\,dy$$

$$=e^t\int_0^t e^{-y}y\,dy$$

$$=e^t\left(\left[-ye^{-y}\right]_0^t+\int_0^t e^{-y}\,dy\right)$$

$$=e^t\left(-te^{-t}-\left[e^{-y}\right]_0^t\right)$$

$$=e^t\left(-te^{-t}-e^{-t}+1\right)$$

$$=e^t-(1+t)$$

17. (3) The polynomial function is differentiable everywhere. Therefore, the points of extremum can only be the roots of the derivative. Further, the derivative of a polynomial is a polynomial. The polynomial of the least degree with roots $x=1$ and $x=3$ has the form $a(x-1)(x-3)$.

Hence, $P'(x)=a(x-1)(x-3)$.

Since at $x=1$, we must have $P(1)=6$.

$$P(x)=\int_1^x P'(x)dx+6$$

$$=a\int_1^x(x^2-4x+3)dx+6$$

$$=a\left(\frac{x^3}{3}-2x^2+3x-\frac{4}{3}\right)+6$$

Also, $P(3)=2$. So, $a=3$. Hence, $P(x)=x^3-6x^2+9x+2$.

Thus, $\int_0^1 P(x)dx=\frac{1}{4}-2+\frac{9}{2}+2=\frac{19}{4}$

18. (4) Since $a^2I_1-2aI_2+I_3=0$,

$$\int_0^1(a-x)^2f(x)dx=0$$

Hence, there is no such positive function $f(x)$.

19. (1) Let $I = \int_1^3 \frac{\sin 2x}{x}\,dx$

Put $2x = t$ and $dx = \frac{dt}{2}$

$\therefore \quad I = \frac{2}{2}\int_2^6 \frac{\sin t}{t}\,dt = \int_2^6 \frac{\sin t}{t}\,dt$

But given $\int \frac{\sin x}{x}\,dx = F(x)$

or $\int_2^6 \frac{\sin t}{t}\,dt = F(6) - F(2)$

20. (1) For $x \in \left(-\frac{\pi}{3}, 0\right)$, $2\cos x - 1 > 0$

$\therefore \quad I = \int_{-\pi/3}^{0} \frac{\pi}{2}\,dx = \frac{\pi^2}{6}$

21. (1) $I = \int_0^{\pi/4} \frac{\sin x + \cos x}{25 - 16(\sin x - \cos x)^2}\,dx$

Let $\sin x - \cos x = t$

$\therefore \quad I = \int_{-1}^{0} \frac{dt}{25 - 16t^2}$

$= \frac{1}{16}\int_{-1}^{0} \frac{dt}{\left(\frac{5}{4}\right)^2 - t^2}$

$= \frac{1}{16}\cdot\frac{1}{2.\frac{5}{4}} \log\left|\frac{\frac{5}{4}+t}{\frac{5}{4}-t}\right|_{-1}^{0}$

$= \frac{1}{40}\left[\log 1 - \log\frac{1}{9}\right]$

$= \frac{\log 9}{40}$

$= \frac{1}{20}\log 3$

22. (4) $I = \int_{-1}^{1} \frac{e^{-1/x}}{x^2(1+e^{-2/x})}\,dx = 2\int_0^1 \frac{e^{-1/x}}{x^2(1+e^{-2/x})}\,dx$

Put $e^{-\frac{1}{x}} = t$

$\Rightarrow \quad \frac{1}{x^2}e^{-\frac{1}{x}}\,dx = dt$

$\therefore \quad I = 2\int_0^{1/e} \frac{dt}{1+t^2}$

$= 2\left[\tan^{-1} t\right]_0^{1/e}$

$= 2\tan^{-1}\frac{1}{e}$

$= 2\left[\frac{\pi}{2} - \tan^{-1} e\right]$

$= \pi - 2\tan^{-1} e$

23. (2) $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$

$\therefore \quad \int_0^\infty \frac{\sin^3 x}{x}\,dx = \frac{3}{4}\int_0^\infty \frac{\sin x}{x}\,dx - \frac{1}{4}\int_0^\infty \frac{\sin 3x}{x}\,dx$

$= \frac{3}{4}\int_0^\infty \frac{\sin x}{x}\,dx - \frac{1}{4}\int_0^\infty \frac{\sin u}{u}\,du \qquad (u = 3x)$

$= \frac{3}{4}\frac{\pi}{2} - \frac{1}{4}\frac{\pi}{2} = \frac{\pi}{4}$

24. (4) We have

$f(x) = \int_{-1}^{1} \frac{\sin x\,dt}{\sin^2 x + (t - \cos x)^2}$

$= \frac{\sin x}{\sin x}\tan^{-1}\left(\frac{t-\cos x}{\sin x}\right)\Big|_{-1}^{1}$

$= \tan^{-1}\left(\frac{1-\cos x}{\sin x}\right) - \tan^{-1}\left(\frac{-1-\cos x}{\sin x}\right)$

$= \tan^{-1}(\tan x/2) + \tan^{-1}(\cot x/2)$

Now, we know that $\tan^{-1} x + \tan^{-1}\frac{1}{x} = \begin{cases} \frac{\pi}{2}, & x > 0 \\ -\frac{\pi}{2}, & x < 0 \end{cases}$

or $\tan^{-1}\left(\tan\frac{x}{2}\right) + \tan^{-1}\left(\frac{1}{\tan\frac{x}{2}}\right) = \begin{cases} \frac{\pi}{2}, & \tan\frac{x}{2} > 0 \\ -\frac{\pi}{2}, & \tan\frac{x}{2} < 0 \end{cases}$

Hence, range of $f(x)$ is $\left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$.

25. (3) Let $g(x) = \int_0^{x^3} f(t)dt$

Now, $\int_0^8 f(t)dt = g(2) = \frac{g(2)-g(1)}{2-1} + \frac{g(1)-g(0)}{1-0}$

$= g'(\alpha) + g'(\beta)$

$= 3[\alpha^2 f(\alpha^3) + \beta^2 f(\beta^3)]$

26. (4) $f(x) = x^5 + 5x - 1$

$f(1) = 5 \quad \Rightarrow f^{-1}(5) = 1$

$f(2) = 41 \quad \Rightarrow f^{-1}(41) = 2$

Put $f^{-1}(x) = t$

$\therefore \quad I = \int_5^{41} \frac{dx}{(f^{-1}(x))^5 + 5f^{-1}(x)}$

$= \int_1^2 \frac{f'(t)}{t^5 + 5t}\,dt$

$= \int_1^2 \frac{5t^4 + 5}{t^5 + 5t}\,dt$

$= \left[\log_e(t^5 + 5t)\right]_1^2$

$= \log_e 42 - \log_e 6 = \log_e 7$

27. (3) We have $\int_0^2 f'(2t)e^{f(2t)}dt = 5$

Put $e^{f(2t)} = y$

$\therefore \quad 2f'(2t)e^{f(2t)}dt = dy$

$\therefore \quad \frac{1}{2}\int_{e^{f(0)}}^{e^{f(4)}} dy = 5$

or $\int_{e^{f(0)}}^{e^{f(4)}} dy = 10$

or $e^{f(4)} - e^{f(0)} = 10$

or $e^{f(4)} = 10 + 1 = 11$

or $f(4) = \log 11$

28. (4) $f(x) = \cos(\tan^{-1}x)$

$\therefore \quad f'(x) = -\frac{\sin(\tan^{-1}x)}{1+x^2}$

$\therefore \quad I = \int_0^1 x\, f''(x)\, dx$

$= [x\, f'(x)]_0^1 - \int_0^1 f'(x)\, dx$ (Integrating by parts)

$= [f'(1)] - [f(x)]_0^1$

$= f'(1) - f(1) + f(0)$

Now, $f(0) = 1$; $f'(1) = -\frac{1}{2\sqrt{2}}$; $f(1) = \frac{1}{\sqrt{2}}$

$\therefore \quad I = 1 - \frac{3}{2\sqrt{2}}$

29. (1) Given $f'(1) = \tan \pi/6$, $f'(2) = \tan \pi/3$, $f'(3) = \tan \pi/4$

Now, $\int_2^3 f'(x)f''(x)dx + \int_1^3 f''(x)\, dx$

$= \left[\frac{(f'(x))^2}{2}\right]_2^3 + [f'(x)]_1^3$

$= \frac{(f'(3))^2 - (f'(2))^2}{2} + f'(3) - f'(1)$

$= \frac{(1)^2 - (\sqrt{3})^2}{2} + \left(1 - \frac{1}{\sqrt{3}}\right)$

$= \frac{1-3}{2} + 1 - \frac{1}{\sqrt{3}}$

$= \frac{-1}{\sqrt{3}}$

30. (2) $\int_1^e \left(\frac{\tan^{-1}x}{x} + \frac{\log x}{1+x^2}\right)dx$

$= \int_1^e \frac{\tan^{-1}x}{x}dx + \int_1^e \frac{\log x}{1+x^2}dx$

$= \int_1^e \frac{\tan^{-1}x}{x}dx + \left(\log x \tan^{-1}x\right)_1^e - \int_1^e \frac{\tan^{-1}x}{x}dx$

$= \tan^{-1} e$

31. (2) $\int_0^\pi [f(x) + f''(x)]\sin x dx$

$= \int_0^\pi f(x)\sin x dx + \int_0^\pi f''(x)\sin x dx$

$= (f(x)(-\cos x))_0^\pi + \int_0^\pi f'(x)\cos x dx$

$+ \sin x\, f'(x)|_0^\pi - \int_0^\pi \cos x\, f'(x)dx$

$= f(\pi) + f(0) = 5$ (given)

$\therefore \quad f(0) = 5 - f(\pi) = 5 - 2 = 3$

32. (2) $I_1 = \int_e^{e^4} \sqrt{\ln x}dx$

Putting $t = \sqrt{\ln x}$, i.e., $dt = \frac{dx}{2x\sqrt{\ln x}}$, we get

$dx = 2t\, e^{t^2}dt$

or $\int_e^{e^4} \sqrt{\ln x}dx = \int_1^2 2t^2 e^{t^2}\, dt$

$= t\, e^{t^2}\Big|_1^2 - \int_1^2 e^{t^2}\, dt$

$= 2e^4 - e - a$

33. (1) $\sum_{r=1}^{n}\int_0^1 f(r-1+x)\, dx$

$= \int_0^1 f(x)dx + \int_0^1 f(1+x)dx + \int_0^1 f(2+x)dx + \cdots$

$+ \int_0^1 f(n-1+x)dx$

$= \int_0^1 f(x)dx + \int_1^2 f(x)dx + \int_2^3 f(x)dx + \int_{r-1}^{2} f(x)dx + \cdots$

$+ \int_{n-1}^{n} f(x)dx.$

$= \int_0^n f(x)\, dx$

34. (3) $I = \int_0^{\pi/2} \sin|2x - \alpha|dx$

Put $2x - \alpha = t$

$\therefore \quad I = \frac{1}{2}\int_{-\alpha}^{(\pi-\alpha)} \sin|t|dt$

$= \frac{1}{2}\int_{-\alpha}^{0} -\sin t\, dt + \frac{1}{2}\int_0^{\pi-\alpha} \sin t\, dt$ ($\because \alpha \in [0, \pi]$)

$= \frac{1}{2}[1 - \cos\alpha] - \frac{1}{2}[-\cos\alpha - 1]$

$= 1$

35. (2) $\int_{-3}^{5} f(|x|)dx = \int_{-3}^{3} f(|x|)dx + \int_3^5 f(|x|)dx$

$= 2\int_0^3 f(x)dx + \int_3^5 f(x)dx$

$= 2\left(\int_0^1 f(x)dx + \int_1^2 f(x)dx + \int_2^3 f(x)dx\right)$

$+ \left(\int_3^4 f(x)dx + \int_4^5 f(x)dx\right)$

$= 2\left(0 + \frac{1}{2} + \frac{2^2}{2}\right) + \left(\frac{9}{2} + \frac{16}{2}\right) = \frac{35}{2}$

36. (3) $f(x)=\begin{cases}\int_{-1}^{x}-t\,dt, & -1\le x<0\\ \int_{-1}^{0}-t\,dt+\int_{0}^{x}t\,dt, & x\ge 0\end{cases}=\begin{cases}\frac{1}{2}(1-x^2), & -1\le x<0\\ \frac{1}{2}(1+x^2), & x\ge 0\end{cases}$

37. (2) $I=\int_{-a}^{a}\left(\cos^{-1}x-\sin^{-1}\sqrt{1-x^2}\right)dx$

$=\int_{-a}^{0}\cos^{-1}x\,dx+A-2\int_{0}^{a}\sin^{-1}\sqrt{1-x^2}\,dx$

$=\int_{0}^{a}\left(\pi-\cos^{-1}x\right)dx+A-2A$

$=a\pi-2A$ or $\lambda=2$

38. (2) Let $I=\int_{1}^{a}[x]f'(x)\,dx,\ a>1$

Let $a=k+h$, where $[a]=k$, and $0\le h<1$

$\therefore\quad \int_{1}^{a}[x]\,f'(x)\,dx=\int_{1}^{2}1\,f'(x)dx+\int_{2}^{3}2\,f'(x)\,dx$

$+\cdots+\int_{k-1}^{k}(k-1)f'(x)\,dx+\int_{k}^{k+h}k\,f'(x)\,dx$

$=[f(2)-f(1)]+2[f(3)-f(2)]+\cdots$

$+(k-1)\,[f(k)-f(k-1)]+k\,[f(k+h)-f(k)]$

$=-f(1)-f(2)-f(3)\cdots-f(k)+kf(k+h)$

$=[a]f(a)-[f(1)+f(2)+\cdots+f([a])]$

39. (1) When $e\le[x]<e^2,\quad 1\le\log[x]<2$

when $e^2\le[x]<e^3,\quad 2\le\log[x]<3$

$\therefore\quad \int_{3}^{8}1\,dx+\int_{8}^{10}2\,dx=9$

40. (2) $[x]=0\ \forall\ x\in[0,1)$

For $x\in[1,2)$, $[x]=1$

$\therefore\quad \frac{[x]}{1+x^2}=\frac{1}{1+x^2}<1\ \forall\ x\in[1,2)$ or $\left[\frac{[x]}{1+x^2}\right]=0$

For $x\in[-1,0)$, $[x]=-1$ or $\frac{[x]}{1+x^2}=-\frac{1}{1+x^2}$

Clearly, $2\ge 1+x^2>1\ \forall\ x\in[-1,0)$

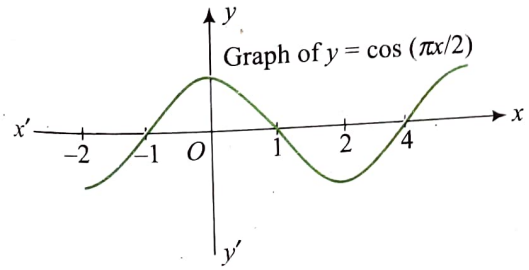
or $\quad\frac{1}{2}\le\frac{1}{1+x^2}<1$

or $\quad-\frac{1}{2}\ge-\frac{1}{1+x^2}>-1$

or $\quad\left[\frac{[x]}{1+x^2}\right]=-1\ \forall\ x\in[-1,0)$

Thus, the given integral $=-\int_{-1}^{0}dx=-1$.

41. (3)



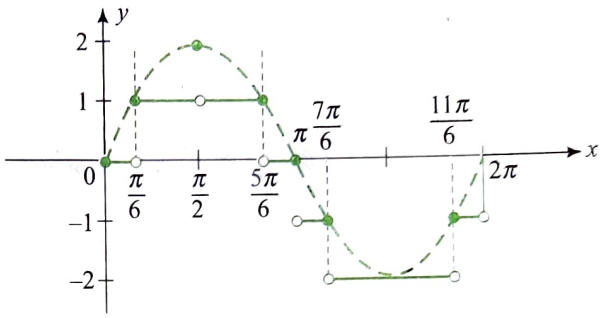
From graph, $\int_{-2}^{1}\left[x\left[1+\cos\frac{\pi x}{2}\right]+1\right]dx$

$=\int_{-2}^{-1}[x[1+(-1)]+1]dx+\int_{-1}^{1}[x[1+0]+1]dx$

$=(x)_{-2}^{-1}+\int_{-1}^{1}[x+1]dx$

$=(-1-(-2))+\int_{-1}^{0}0\,dx+\int_{0}^{1}1\,dx=2$

42. (2) $I=\int_{\pi}^{2\pi}[2\sin x]dx$



From the graph in the figure,

$I=\int_{\pi/6}^{5\pi/6}1dx+\int_{\pi}^{7\pi/6}-1\,dx+\int_{7\pi/6}^{11\pi/6}-2\,dx+\int_{11\pi/6}^{2\pi}-1\,dx$

$=\left(\frac{5\pi}{6}-\frac{\pi}{6}\right)+\left(-\frac{7\pi}{6}+\pi\right)+2\left(-\frac{11\pi}{6}+\frac{7\pi}{6}\right)$

$+\left(-2\pi+\frac{11\pi}{6}\right)$

$=\frac{2\pi}{3}-\frac{\pi}{6}-\frac{8\pi}{6}-\frac{\pi}{6}=-\pi$

43. (3) $I_1=\int_{0}^{\pi/2}\frac{\sin x-\cos x}{1+\sin x\cos x}dx$

$=\int_{0}^{\pi/2}\frac{\sin\left(\frac{\pi}{2}-x\right)-\cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}dx$

$=\int_{0}^{\pi/2}\frac{\cos x-\sin x}{1+\sin x\cos x}dx=-I_1$

or $\quad I_1=0$

$I_3=0$ as $\sin^3 x$ is odd.

$I_4=\int_{0}^{1}\ln\left(\frac{1-x}{x}\right)dx$

$=\int_{0}^{1}\ln\left(\frac{1-(1-x)}{1-x}\right)dx$

$=\int_{0}^{1}\ln\frac{x}{1-x}dx=-I_4$

or $\quad I_4=0$

$I_2=\int_{0}^{2\pi}\cos^6 x\,dx=2\int_{0}^{\pi}\cos^6 x\,dx\ne 0$

44. (3) $I=\int_{0}^{\pi/2}\frac{\sin x\,dx}{1+\sin x+\cos x}=\int_{0}^{\pi/2}\frac{\cos x\,dx}{1+\sin x+\cos x}$

or $\quad 2I=\int_{0}^{\pi/2}\frac{\sin x+\cos x+1-1}{\sin x+\cos x+1}dx$

$$2I = \frac{\pi}{2} - A$$

or $$I = \frac{\pi}{4} - \frac{1}{2}A$$

45. (2) $$I_1 = \int_{-100}^{101} \frac{dx}{(5+2x-2x^2)(1+e^{2-4x})}$$

$$= \int_{-100}^{101} \frac{dx}{(5+2(1-x)-2(1-x)^2)(1+e^{2-4(1-x)})}$$

or $$2I_1 = \int_{-100}^{101} \frac{dx}{5+2x-2x^2} = I_2$$

or $$\frac{I_1}{I_2} = \frac{1}{2}$$

46. (1) Put $x = \tan\theta$ and $dx = \sec^2\theta\, d\theta$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\tan\theta \sec^2\theta}{(1+\tan\theta)(\sec^2\theta)}\, d\theta \qquad (1)$$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\sin\theta\, d\theta}{\cos\theta + \sin\theta} = \frac{\pi}{4}$$

47. (3) $$I = \int_0^{\pi} e^{\cos^2 x} \cos^3(2n+1)x\, dx,\ n \in Z \qquad (1)$$

$$= \int_0^{\pi} e^{\cos^2(\pi-x)} \cos^3[(2n+1)(\pi-x)]\, dx$$

[Using $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$]

$$= \int_0^{\pi} e^{\cos^2 x} \cos^3[(2n+1)\pi - (2n+1)x]\, dx$$

$$= -\int_0^{\pi} -e^{\cos^2 x} \cos^3(2n+1)x\, dx$$

$$= -I$$

or $$I = 0$$

48. (3) Given f is a positive function, and

$$I_1 = \int_{1-k}^{k} x f[x(1-x)]\, dx$$

$$I_2 = \int_{1-k}^{k} f[x(1-x)]\, dx$$

Now, $$I_1 = \int_{1-k}^{k} xf[x(1-x)]\, dx \qquad (1)$$

$$= \int_{1-k}^{k} (1-x) f[(1-x)x]\, dx \qquad (2)$$

$$\left[\text{Using the property } \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right]$$

Adding (1) and (2), we get

$$2I_1 = \int_{1-k}^{k} f[x(1-x)]\, dx = I_2 \text{ or } \frac{I_1}{I_2} = \frac{1}{2}$$

49. (3) $$f(x) = \frac{e^x}{1+e^x}$$

$$\therefore \quad f(a) = \frac{e^a}{1+e^a}$$

and $$f(-a) = \frac{e^{-a}}{1+e^{-a}} = \frac{e^{-a}}{1+\frac{1}{e^a}} = \frac{1}{1+e^a}$$

$$\therefore \quad f(a) + f(-a) = \frac{e^a+1}{1+e^a} = 1$$

Let $f(-a) = \alpha$ or $f(a) = 1-\alpha$

Now, $$I_1 = \int_{\alpha}^{1-\alpha} xg(x(1-x))\, dx$$

$$= \int_{\alpha}^{1-\alpha} (1-x) g((1-x)(1-(1-x)))\, dx$$

$$= \int_{\alpha}^{1-\alpha} (1-x) g(x(1-x))\, dx$$

$$\therefore \quad 2I_1 = \int_{\alpha}^{1-\alpha} g(x(1-x))\, dx = I_2 \quad \text{or } \frac{I_2}{I_1} = 2$$

50. (1) $$\int_1^{\frac{1+\sqrt{5}}{2}} \frac{1+\frac{1}{x^2}}{x^2-1+\frac{1}{x^2}} \log\left(1+x-\frac{1}{x}\right) dx$$

$$= \int_1^{\frac{1+\sqrt{5}}{2}} \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+1} \log\left(1+x-\frac{1}{x}\right) dx$$

Put $$x - \frac{1}{x} = t$$

or $$\left(1+\frac{1}{x^2}\right)dx = dt$$

If $x = 1$, $t = 0$, and if $x = \frac{\sqrt{5}+1}{2}$, $t = 1$

$$\therefore \quad I = \int_0^1 \frac{\ln(1+t)\,dt}{1+t^2}$$

Put $t = \tan\theta$ and $dt = \sec^2\theta\, d\theta$

$$\therefore \quad I = \int_0^{\pi/4} \ln(1+\tan\theta)\, d\theta = \frac{\pi}{8}\log_e 2$$

51. (2) $$I = \int_0^{\pi/2} \sqrt{\tan x}\, dx \qquad (1)$$

or $$I = \int_0^{\pi/2} \sqrt{\cot x}\, dx \qquad (2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx$$

$$= \sqrt{2} \int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\, dx$$

$$= \sqrt{2} \int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{1-(\sin x - \cos x)^2}}\, dx$$

$$= \sqrt{2} \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} \qquad \text{(Putting } \sin x - \cos x = t\text{)}$$

$$= 2\sqrt{2} \int_0^1 \frac{dt}{\sqrt{1-t^2}}$$

$$= \sqrt{2}\,\pi$$

or $$I = \frac{\pi}{\sqrt{2}}$$

52. (3) $\int_0^a \frac{f(x)}{f(x)+f(a-x)} = \int_a^{2a} \frac{f(x)\,dx}{f(x)+f(3a-x)} = \ldots = \frac{a}{2}$

So, from given sum,

$$\lim_{n\to\infty}\left[\frac{a}{2}+\frac{a^2}{2}+\frac{a^3}{2}+\cdots+\frac{a^n}{2}\right]=\frac{7}{5}$$

or $\frac{a}{1-a}=\frac{14}{5}$ (sum of infinite G.P.)

or $5a = 14 - 14a$

or $a = \frac{14}{19}$

53. (2) $\int_0^1 \cot^{-1}(1-x+x^2)\,dx$

$$= \int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right)dx$$

$$= \int_0^1 \tan^{-1}\left(\frac{x+(1-x)}{1-x(1-x)}\right)dx$$

$$= \int_0^1 \tan^{-1}x\,dx + \int_0^1 \tan^{-1}(1-x)\,dx$$

$$= \int_0^1 \tan^{-1}x\,dx + \int_0^1 \tan^{-1}[1-(1-x)]\,dx$$

$$= 2\int_0^1 \tan^{-1}x\,dx \text{ or } \lambda = 2$$

54. (4) $I = \int_{-1}^{1} (1+x)^{1/2}(1-x)^{3/2}\,dx$

$\therefore \quad I = \int_{-1}^{1} (1-x)^{1/2}(1+x)^{3/2}\,dx$ (Replacing x by $-x$)

Adding, we get

$$2I = \int_{-1}^{1} (1+x)^{1/2}(1-x)^{1/2}\left[(1-x)+(1+x)\right]dx$$

or $I = \int_{-1}^{1} \sqrt{1-x^2}\,dx$

$\therefore \quad I = 2\int_0^1 \sqrt{1-x^2}\,dx$

$$= 2\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x\right]_0^1$$

$$= 2\left[0+\frac{1}{2}\cdot\frac{\pi}{2}-0-0\right] = \frac{\pi}{2}$$

55. (1) Let $I = \int_{-3\pi/4}^{5\pi/4} \frac{(\sin x+\cos x)}{e^{x-\pi/4}+1}\,dx$

$$= \int_{-3\pi/4}^{5\pi/4} \frac{\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)}{e^{x-\pi/4}+1}dx$$

Putting $x-\frac{\pi}{4}=t$ or $dx = dt$

$\therefore \quad I = \int_{-\pi}^{\pi} \frac{\sqrt{2}\cos t}{e^t+1}\,dt$ (1)

Replacing t by $\pi+(-\pi)-t$ or $-t$, we get

$$I = \int_{-\pi}^{\pi} \frac{\sqrt{2}\cos(-t)}{e^{-t}+1}\,dt = \int_{-\pi}^{\pi} \frac{e^t\sqrt{2}\cos t}{e^t+1}\,dt \quad (2)$$

Adding (1) and (2), we get

$$2I = \sqrt{2}\int_{-\pi}^{\pi} \cos t\,dt \text{ or } I = 0$$

56. (1) $I_2 = \int_{-\pi/4}^{\pi/4} \ln(\sin x+\cos x)\,dx$

$$= \int_0^{\pi/4} (\ln(\sin x+\cos x)+\ln(\sin(-x)+\cos(-x)))\,dx$$

$$= \int_0^{\pi/4} (\ln(\sin x+\cos x)+\ln(\cos x-\sin x))\,dx$$

$$= \int_0^{\pi/4} \ln(\cos^2 x-\sin^2 x)\,dx$$

$$= \int_0^{\pi/4} \ln(\cos 2x)\,dx$$

Putting $2x = t$, and $\frac{dt}{2} = dx$, we get

$$I_2 = \frac{1}{2}\int_0^{\pi/2} \ln(\cos t)\,dt$$

$$= \frac{1}{2}\int_0^{\pi/2} \ln\left(\cos\left(\frac{\pi}{2}-t\right)\right)dt$$

$$= \frac{1}{2}\int_0^{\pi/2} \ln(\sin t)dt = \frac{1}{2}I_1$$

or $I_1 = 2I_2$

57. (3) $I_1 = \int_0^{\pi/2} \frac{\cos^2 x}{1+\cos^2 x}\,dx$

$$= \int_0^{\pi/2} \frac{\cos^2(\pi/2-x)}{1+\cos^2(\pi/2-x)}\,dx$$

$$= \int_0^{\pi/2} \frac{\sin^2 x}{1+\sin^2 x}\,dx = I_2$$

Also $I_1+I_2 = \int_0^{\pi/2}\left(\frac{\sin^2 x}{1+\sin^2 x}+\frac{\cos^2 x}{1+\cos^2 x}\right)dx$

$$= \int_0^{\pi/2} \frac{\sin^2 x+\sin^2 x\cos^2 x+\cos^2 x+\sin^2 x\cos^2 x}{1+\sin^2 x+\cos^2 x+\sin^2 x\cos^2 x}\,dx$$

$$= \int \frac{1+2\sin^2 x\cos^2 x}{2+\sin^2 x\cos^2 x}\,dx = 2I_3$$

$\therefore \quad 2I_1 = 2I_3$ or $I_1 = I_3$ or $I_1 = I_2 = I_3$

58. (4) Let

$$I = \int_0^{\pi/2} \frac{x\sin x\cos x}{\cos^4 x+\sin^4 x}\,dx \quad \ldots(1)$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{(\pi/2-x)\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx \quad \ldots(2)$$

Adding (1) and (2), we get

$$I = \frac{\pi}{4}\int_0^{\pi/2} \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx$$

$$= \frac{\pi}{4}\int_0^{\pi/2} \frac{\sec^2 x\tan x}{\tan^4 x+1}\,dx \text{ (Dividing Nr. and Dr. by } \cos^4 x)$$

$$= \frac{\pi}{2\times 4}\int_0^{\pi/2} \frac{\left(\tan^2 x\right)'\,dx}{1+\left(\tan^2 x\right)^2}$$

$$= \frac{\pi}{8}\left[\tan^{-1}(\tan^2 x)\right]_0^{\pi/2}$$

$$= \frac{\pi}{8}\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{\pi^2}{16}$$

59. (2) $I_1 = \int_{\sin^2 t}^{1+\cos^2 t} xf(x(2-x))dx$

$$= \int_{\sin^2 t}^{1+\cos^2 t} (2-x)f(x(2-x))dx = 2I_2 - I_1$$

or $2I_1 = 2I_2$ or $\frac{I_1}{I_2} = 1$

60. (3) $I = \int_{-\pi/4}^{3\pi/4} \frac{dx}{\sqrt{2}(e^{x-\pi/4}+1)\cos\left(x-\frac{\pi}{4}\right)}$

Putting $x - \frac{\pi}{4} = t$, we get

$$I = \frac{1}{\sqrt{2}} \int_{-\pi/2}^{\pi/2} \frac{dt}{(e^t+1)\cos t} \quad (1)$$

$$= \frac{1}{\sqrt{2}} \int_{-\pi/2}^{\pi/2} \frac{e^t dt}{(e^t+1)\cos t} \quad (2)$$

Adding, (1) and (2) we get $2I = \frac{1}{\sqrt{2}} \int_{-\pi/2}^{\pi/2} \sec t dt$

$\therefore \quad I = \frac{1}{2\sqrt{2}} \int_{-\pi/2}^{\pi/2} \sec x dx$

$\therefore \quad k = \frac{1}{2\sqrt{2}}$

61. (2) $I = \int_2^4 (x(3-x)(4+x)(6-x)(10-x)+\sin x)dx \quad (1)$

$$= \int_2^4 ((6-x)(3-(6-x))(4+(6-x))(6-(6-x))$$

$$(10-(6-x)) + \sin(6-x))\,dx$$

$$= \int_2^4 ((6-x)(x-3)(10-x)x(4+x)+\sin(6-x))dx \quad (2)$$

Adding (1) and (2), we get

$$2I = \int_2^4 (\sin x + \sin(6-x))dx$$

$$= (-\cos x + \cos(6-x))_2^4$$

$$= -\cos 4 + \cos 2 + \cos 2 - \cos 4$$

$$= 2(\cos 2 - \cos 4)$$

or $I = \cos 2 - \cos 4$

62. (1) $\int_{-20\pi}^{20\pi} |\sin x|[\sin x]dx = \int_0^{20\pi} |\sin x|([\sin x]+[-\sin x])dx$

$$= -20\int_0^{\pi} (\sin x)dx$$

$$= -20(-\cos x)_0^{\pi}$$

$$= 20(2)$$

$$= -40$$

63. (1) $I = \int_0^1 f(x)[g(x) - g(1-x)]\,dx$

$$= -\int_0^1 f(1-x)\,[g(x) - g(1-x)]\,dx$$

or $2I = \int_0^1 [f(x) - f(1-x)]\,[g(x) - g(1-x)]\,dx$

$$= 2\int_0^{1/2} [f(x) - f(1-x)] \cdot [g(x) - g(1-x)]\,dx \le 0$$

64. (1) Let $I = \int_0^{\pi} \frac{x\tan x}{\sec x + \cos x}dx \quad (1)$

$$= \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \cos(\pi - x)}dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \cos x}dx \quad (2)$$

Adding (1) and (2) gives

$$2I = \pi \int_0^{\pi} \frac{\tan x}{\sec x + \cos x}\,dx$$

$$= \pi \int_0^{\pi} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \cos x}\,dx = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

Put $\cos x = z$. Therefore, $-\sin x\,dx = dz$.

When $x = 0$, $z = 1$, when $x = \pi$, $z = -1$.

$\therefore \quad 2I = \pi \int_1^{-1} \frac{-dz}{1+z^2} = \pi \int_{-1}^{1} \frac{dz}{1+z^2}$

$$= \pi\,|\tan^{-1} z\,|_{-1}^{1}$$

$$= \pi\,[\tan^{-1} 1 - \tan^{-1}(-1)]$$

$$= \pi\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \frac{2\pi^2}{4}$$

or $I = \frac{\pi^2}{4}$

65. (3) $f(x) = \int_0^{\pi} \frac{t\sin t}{\sqrt{1+\tan^2 x\sin^2 t}}dt \quad (1)$

Replacing t by $\pi - t$ and then adding $f(x)$ with (1), we get

$$f(x) = \frac{\pi}{2}\int_0^{\pi} \frac{\sin t}{\sqrt{1+\tan^2 x\sin^2 t}}dt$$

$$= \pi \int_0^{\pi/2} \frac{\sin t}{\sqrt{1+\tan^2 x(1-\cos^2 t)}}dt$$

$$= \pi \int_0^{\pi/2} \frac{\sin t}{\sqrt{\sec^2 x - \tan^2 x\cos^2 t}}dt$$

Let $y = \cos t$

$\therefore \quad dy = -\sin t\,dt$

$\therefore \quad f(x) = \pi\int_0^1 \frac{dy}{\sqrt{\sec^2 x - (\tan^2 x)y^2}}$

$$= \frac{\pi}{\tan x}\int_0^1 \frac{dy}{\sqrt{\operatorname{cosec}^2 x - y^2}}$$

$$= \frac{\pi}{\tan x}\left\{\sin^{-1}\frac{y}{\operatorname{cosec} x}\right\}_0^1$$

$$= \frac{\pi}{\tan x}\sin^{-1}(\sin x) = \frac{\pi x}{\tan x}$$

66. (2) $I = \int_{-3}^{3} x^8\{x^{11}\}dx$ (1)

Replacing x by $-x$, we have $I = \int_{-3}^{3} x^8\{-x^{11}\}dx$ (2)

Adding (1) and (2), we get

$$2I = \int_{-3}^{3} x^8\left(\{x^{11}\}+\{-x^{11}\}\right)dx = 2\int_0^3 x^8 dx$$

$$= 2\left(\frac{x^9}{9}\right)_0^3 = 2.3^7$$

or $I = 3^7$ [As $\{x\} + \{-x\} = 1$ for x is not an integer]

67. (3) $I = \int_0^{4\pi} \log_e\left|6\sin\left(x+\frac{\pi}{3}\right)\right|dx$

$$= \int_0^{4\pi} \log_e 6\,dx + \int_0^{4\pi} \log_e\left|\sin\left(x+\frac{\pi}{3}\right)\right|dx$$

$$= 4\pi\log_e 6 + 4\int_0^{\pi} \log_e|\sin x|dx$$

$$= 4\pi\log_e 6 + 8\int_0^{\pi/2} \log_e|\sin x|dx$$

$$= 4\pi\log_e 6 - 4\pi\log_e 2$$

$$= 4\pi\log_e 3$$

68. (2) Let $I = \int_0^{\pi} \frac{x\sin^2 x}{1+2|\cos x|\sin x}dx$ ...(1)

$$\therefore\quad I = \int_0^{\pi} \frac{(\pi-x)\sin^2 x}{1+2|\cos x|\sin x}dx \quad ...(2)$$

Adding (1) and (2), we get

$$2I = \pi\int_0^{\pi} \frac{\sin^2 x}{1+2|\cos x|\sin x}dx$$

$$\Rightarrow\quad I = \frac{\pi}{2}\times 2\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{1+2\cos x\sin x}dx$$

$$\therefore\quad I = \pi\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{1+\sin 2x}dx \quad ...(3)$$

$$\therefore\quad I = \pi\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{1+\sin 2x}dx \quad ...(4)$$

Adding (3) and (4), we get

$$2I = \pi\int_0^{\frac{\pi}{2}} \frac{dx}{(\cos x+\sin x)^2}$$

$$= \pi\int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{(1+\tan x)^2}dx$$

$$= \pi\int_0^{\infty} \frac{dt}{t^2}dx \qquad \text{(Putting } 1+\tan x = t\text{)}$$

$$= -\pi\left(\frac{1}{t}\right)_1^{\infty} = -\pi(0-1) = \pi$$

Hence, $I = \pi/2$

69. (1) $\int_{-\pi}^{\pi} \sin nx \sin mx\,dx = \int_0^{\pi} 2\sin mx \sin nx\,dx$

$$= \int_0^{\pi}[\cos(m-n)x - \cos(m+n)x]dx$$

$$= \left|\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right|_0^{\pi} = 0$$

70. (3) $I = \int_{\log\lambda}^{\log\frac{1}{\lambda}} \frac{f\left(x^2/4\right)[f(x)-f(-x)]}{g\left(x^2/4\right)[g(x)+g(-x)]}dx$

$$= \int_{\log\lambda}^{-\log\lambda} \frac{f\left(x^2/4\right)[f(x)-f(-x)]}{g\left(x^2/4\right)[g(x)+g(-x)]} = 0$$

(As the integrand is odd function)

71. (2) $I = \int_0^1 \frac{\tan^{-1}\left(\frac{x}{x+1}\right)}{\tan^{-1}\left(\frac{1+2x-2x^2}{2}\right)}dx$ ...(1)

$$= \int_0^1 \frac{\tan^{-1}\left(\frac{1-x}{(1-x)+1}\right)}{\tan^{-1}\left(\frac{1+2(1-x)-2(1-x)^2}{2}\right)}dx$$

$$= \int_0^1 \frac{\tan^{-1}\left(\frac{1-x}{2-x}\right)}{\tan^{-1}\left(\frac{1+2x-2x^2}{2}\right)}dx \quad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_0^1 \frac{\tan^{-1}\left(\frac{x}{x+1}\right)+\tan^{-1}\left(\frac{1-x}{2-x}\right)}{\tan^{-1}\left(\frac{1+2x-2x^2}{2}\right)}dx$$

$$= \int_0^1 \frac{\tan^{-1}\left(\frac{\frac{x}{x+1}+\frac{1-x}{2-x}}{1-\frac{x}{x+1}\cdot\frac{1-x}{2-x}}\right)}{\tan^{-1}\left(\frac{1+2x-2x^2}{2}\right)}dx$$

$$= \int_0^1 \frac{\tan^{-1}\left(\frac{1+2x-2x^2}{2}\right)}{\tan^{-1}\left(\frac{1+2x-2x^2}{2}\right)}dx = \int_0^1 1\,dx = 1$$

$$\Rightarrow\quad I = 1/2$$

72. (3) $I=\int_{-\pi/2}^{\pi/2}\frac{e^{|\sin x|}\cos x}{1+e^{\tan x}}dx$ ...(1)

$=\int_{-\pi/2}^{\pi/2}\frac{e^{|\sin(-x)|}\cos(-x)}{1+e^{\tan(-x)}}dx$

$\therefore \quad I=\int_{-\pi/2}^{\pi/2}\frac{e^{\tan x}e^{|\sin x|}\cos x}{e^{\tan x}+1}dx$ ...(2)

Adding (1) and (2), we get

$2I=\int_{-\pi/2}^{\pi/2}e^{|\sin x|}\cos x\,dx$

$=2\int_0^{\pi/2}e^{\sin x}\cos x\,dx$

$=2\left[e^{\sin x}\right]_0^{\pi/2}=2(e-1)$

$\therefore \quad I=e-1$

73. (2) $I=\int_{-\pi}^{\pi}\frac{2x\,(1+\sin x)}{1+\cos^2 x}dx$

$=\underbrace{\int_{-\pi}^{\pi}\frac{2x}{1+\cos^2 x}dx}_{\text{odd function}}+2\int_{-\pi}^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$

$=0+4\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$

$=4\int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}dx$

$=4\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx-4\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$

$\Rightarrow \quad 2I=4\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$

$\Rightarrow \quad I=2\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$

$=-2\pi\left[\tan^{-1}(\cos x)\right]_0^{\pi}$

$=-2\pi\times\left(-\frac{\pi}{2}\right)=\pi^2$

74. (1) $I=\int_{-\pi}^{\pi}\sum_{r=0}^{999}\cos rx\left(1+\sum_{r=1}^{999}\sin rx\right)dx$

$\therefore \quad I=\int_{-\pi}^{\pi}\sum_{r=0}^{999}\cos rx\left(1-\sum_{r=1}^{999}\sin rx\right)dx$ (Replacing x by $-x$)

Adding, we get

$2I=2\int_{-\pi}^{\pi}\sum_{r=0}^{999}\cos rx\,dx$

$\Rightarrow \quad I=\int_{-\pi}^{\pi}1\,dx+\int_{-\pi}^{\pi}\sum_{r=1}^{999}\cos rx\,dx$

$=2\pi+2\int_0^{\pi}\sum_{r=1}^{999}\cos rx\,dx$

$=2\pi+\left[\sum_{r=1}^{999}\sin rx\right]_0^{\pi}=2\pi+0=2\pi$

75. (3) Let $I_1=\int_3^{3+3T}f(2x)\,dx$.

Put $2x=y$, so that $I_1=\frac{1}{2}\int_6^{6+6T}f(y)\,dy$

$=\frac{1}{2}6\int_0^{T}f(y)\,dy \quad [\because f(x) \text{ has period } T]$

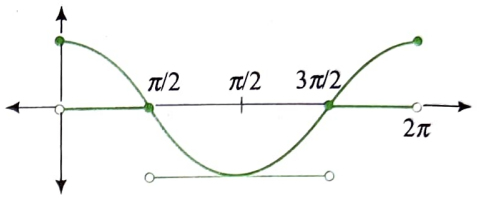
$=3I$

76. (3) Put $x-0.4=t$

$\therefore \quad \int_{0.6}^{3.6}\{t\}\,dt=\int_{0.6}^{0.6+3}\{t\}\,dt$

$=3\int_0^1(t-[t])\,dt=3\left(\frac{t^2}{2}\right)_0^1=\frac{3}{2}=1.5$

77. (3)



$I=\int_0^x[\cos t]dt$

$=\int_0^{2n\pi}[\cos t]dt+\int_{2n\pi}^{x}[\cos t]dt$

$=n\int_0^{2\pi}[\cos t]dt+\int_{2n\pi}^{2n\pi+\pi/2}[\cos t]dt+\int_{2n\pi+\frac{\pi}{2}}^{x}[\cos t]dt$

$=-n\pi+0+(x-(2n\pi+\pi/2))\,(-1)$

$=-n\pi+2n\pi+\pi/2-x$

$=(2n+1)\pi/2-x$

78. (3) $\int_0^x|\sin t|\,dt=\int_0^{2n\pi}|\sin t|\,dt+\int_{2n\pi}^{x}|\sin t|\,dt$

$=2n\int_0^{\pi}|\sin t|\,dt+\int_{2n\pi}^{x}\sin t\,dt$

(as x lies in either first or second quadrant)

$=2n(-\cos t)_0^{\pi}+(-\cos t)_{2n\pi}^{x}=4n-\cos x+1$

79. (1) Let $n\le x<n+1$, where $n\in I$

$I=\int_0^x\frac{2^t}{2^{[t]}}dt=\int_0^n 2^{\{t\}}dt+\int_n^x 2^{\{t\}}dt$

$=n\int_0^1 2^{\{t\}}dt+\int_n^x 2^{\{t\}}dt$

$=n\int_0^1 2^t\,dt+\int_n^x 2^{t-n}dt$

$=n\left.\frac{2^t}{\ln 2}\right|_0^1+\frac{1}{2^n}\left.\frac{2^t}{\ln 2}\right|_n^x$

$=\frac{n}{\ln 2}(2-1)+\frac{1}{2^n\ln 2}(2^x-2^n)$

$=\frac{n}{\ln 2}+\frac{1}{\ln 2}(2^{x-n}-1)$

$=\frac{[x]+2^{\{x\}}-1}{\ln 2}$

(3) $g(x) = \int_0^x f(t)dt$

$g(-x) = \int_0^{-x} f(t)dt = -\int_0^x f(-t)dt = \int_0^x f(t)dt$ as $f(-t) = -f(t)$

or $g(-x) = g(x)$

Thus, $g(x)$ is even.

Also, $g(x+2) = \int_0^{x+2} f(t)dt$

$= \int_0^2 f(t)dt + \int_2^{2+x} f(t)dt$

$= g(2) + \int_0^x f(t+2)dt$

$= g(2) + \int_0^x f(t)dt$

$= g(2) + g(x)$

Now, $g(2) = \int_0^2 f(t)dt = \int_0^1 f(t)dt + \int_1^2 f(t)dt$

$= \int_0^1 f(t)dt + \int_{-1}^0 f(t+2)dt$

$= \int_0^1 f(t)dt + \int_{-1}^0 f(t)dt$

$= \int_{-1}^1 f(t)dt = 0$ as $f(t)$ is odd

$g(2) = 0 \Rightarrow g(x+2) = g(x)$,

i.e., $g(x)$ is periodic with period 2.

$\therefore$ $g(4) = 0$ or $g(6) = 0$, $g(2n) = 0$, $n \in N$.

81. (2) $g\left(x + \frac{\pi n}{2}\right) = \int_0^{x+\frac{n\pi}{2}} (|\sin t| + |\cos t|)dt$

$= \int_0^x (|\sin t| + |\cos t|)dt + \int_x^{x+\frac{n\pi}{2}} (|\sin t| + |\cos t|)dt$

$= g(x) + n\int_0^{\pi/2} (|\sin t| + |\cos t|)dt$

(as $|\sin t| + |\cos t|$ has period $\pi/2$)

$= g(x) + ng\left(\frac{\pi}{2}\right)$

82. (3) $\frac{dx}{dt} = \sin^{-1}(\sin t)\cos t = t\cos t$

and $\frac{dy}{dt} = \frac{\sin t}{\sqrt{t}} \cdot \frac{1}{2\sqrt{t}} = \frac{\sin t}{2t}$ or $\frac{dy}{dx} = \frac{\sin t}{2t \cdot t\cos t} = \frac{\tan t}{2t^2}$.

83. (3) $f(x) = \int_2^x \frac{dt}{\sqrt{1+t^4}}$

$\Rightarrow$ $f'(x) = \frac{1}{\sqrt{1+x^4}} = \frac{dy}{dx}$

Now $g'(x) = \frac{dx}{dy} = \sqrt{1+x^4}$

when $y = 0$, i.e., $\int_2^x \frac{dt}{\sqrt{1+t^4}} = 0$ then $x = 2$

$\therefore$ $g'(0) = \sqrt{1+16} = \sqrt{17}$

84. (1) $\int_0^{t^2} xf(x)\,dx = \frac{2}{5}t^5$ (Here, $t > 0$)

Differentiating both sides w.r.t. t, we get

$t^2 f(t^2) \times 2t = \frac{2}{5} \times 5t^4$

or $f(t^2) = t$

Put $t = \frac{2}{5}$. Then $f\left(\frac{4}{25}\right) = \frac{2}{5}$.

85. (1) $f(x) = \cos x - \int_0^x (x-t)f(t)\,dt$

$= \cos x - x\int_0^x f(t)\,dt + \int_0^x tf(t)\,dt$

$\therefore$ $f'(x) = -\sin x - xf(x) - \int_0^x f(t)\,dt + xf(x)$

$= -\sin x - \int_0^x f(t)\,dt$

$\therefore$ $f''(x) = -\cos x - f(x)$

or $f''(x) + f(x) = -\cos x$

86. (3) $f^2(x) = \int_0^x f(t)\frac{\cos t}{2+\sin t}dt$

or $2f(x)f'(x) = f(x)\frac{\cos x}{2+\sin x}$ (Differentiating w.r.t. x using Leibniz rule)

or $2f'(x) = \frac{\cos x}{2+\sin x}$ [As $f(x)$ is not zero everywhere]

or $2\int f'(x)dx = \int \frac{\cos x}{2+\sin x}dx$

or $2f(x) = \log_e(2+\sin x) + \log C$.

Put $x = 0$. Then $2f(0) = \log 2 + \log C$, or $\log C = -\log 2$.

$\therefore$ $f(x) = \frac{1}{2}\ln\left(\frac{2+\sin x}{2}\right)$

87. (1) $\lim_{x\to 0}\frac{1}{x}\left[\int_y^a e^{\sin^2 t}dt + \int_a^{x+y} e^{\sin^2 t}dt\right]$

$= \lim_{x\to 0}\frac{1}{x}\int_y^{x+y} e^{\sin^2 t}dt$ $\left(\frac{0}{0}\text{ form}\right)$

Apply L'Hopital's rule, we get

$\lim_{x\to 0}\frac{e^{\sin^2(x+y)}\left(1+\frac{dy}{dx}\right) - e^{\sin^2 y}\frac{dy}{dx}}{1}$

$= e^{\sin^2 y}\left[1 + \frac{dy}{dx} - \frac{dy}{dx}\right] = e^{\sin^2 y}$

88. (1) $f(x) = \int_1^x \frac{e^t}{t}dt \Rightarrow f(1) = 0$ and $f'(x) = \frac{e^x}{x}$

Let $g(x) = f(x) - \ln(x)$, $x \in R^+$

$\therefore$ $g'(x) = f'(x) - \frac{1}{x} = \frac{e^x - 1}{x} > 0 \ \forall\ x \in R^+$

Thus, $g(x)$ is increasing for $x \in R^+$.

$g(1) = f(1) - \ln 1 = 0 - 0 = 0$

$\therefore \quad g(x) > 0 \ \forall \ x > 1$ and $g(x) \le 0 \ \forall \ x \in (0, 1]$

$\therefore \quad \ln x \ge f(x) \ \forall \ x \in (0, 1]$

89. (1) $\int_0^x f(t)dt = x + \int_x^1 tf(t)dt$

or $\quad \frac{d}{dx}\left(\int_0^x f(t)dt\right) = \frac{d}{dx}\left(x + \int_x^1 tf(t)dt\right)$ [Using Leibniz's rule]

or $\quad f(x) = 1 + 0 - xf(x)$

$= 1 - xf(x)$

$\therefore \quad f(1) = \frac{1}{1+x}$

$\therefore \quad f(1) = \frac{1}{2}$

90. (2) Given $xf(x) = x + \int_1^x f(t)dt$ ...(1)

$f(x) + xf'(x) = 1 + f(x)$

$\therefore \quad f'(x) = \frac{1}{x}$

or $\quad f(x) = \log|x| + c$

$f(1) = 1$ (from (1))

or $\quad f(x) = \log|x| + 1$

or $\quad f(e^{-1}) = 0$

91. (2) $\int_{\cos x}^1 t^2 f(t)dt = 1 - \cos x$

Differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}\int_{\cos x}^1 t^2 f(t)\,dt = \frac{d}{dx}(1 - \cos x)$$

or $\quad -\cos^2 x f(\cos x)(-\sin x) = \sin x$

or $\quad \cos^2 x f(\cos x)\sin x = \sin x$

or $\quad f(\cos x) = \frac{1}{\cos^2 x}$.

Now, $f\left(\frac{\sqrt{3}}{4}\right)$ is attained when $\cos x = \frac{\sqrt{3}}{4}$

$\therefore \quad f\left(\frac{\sqrt{3}}{4}\right) = \frac{16}{3} = 5.33$

$$\left[f\left(\frac{\sqrt{3}}{4}\right)\right] = 5$$

92. (4) $\int_0^x f(t)dt = \int_x^1 t^2 f(t)dt + \frac{x^{16}}{8} + \frac{x^6}{3} + a$ ...(1)

For $x = 1$, $\int_0^1 f(t)dt = 0 + \frac{1}{8} + \frac{1}{3} + a = \frac{11}{24} + a$ ...(2)

Differentiating both sides of (1) w.r.t. x,

We get, $f(x) = 0 - x^2 f(x) + 2x^{15} + 2x^5$

$\Rightarrow \quad f(x) = \frac{2(x^{15} + x^5)}{1 + x^2}$

$\Rightarrow \quad 2\int_0^1 \frac{x^{15} + x^5}{1 + x^2}dx = \frac{11}{24} + a$ (from (2))

$\Rightarrow \quad 2\int_0^1 (x^{13} - x^{11} + x^9 - x^7 + x^5)dx = \frac{11}{24} + a$

$\Rightarrow \quad 2\left(\frac{1}{14} - \frac{1}{12} + \frac{1}{10} - \frac{1}{8} + \frac{1}{6}\right) = \frac{11}{24} + a$

$\Rightarrow \quad a = -\frac{167}{840}$

93. (3) $f(x) = \int_{\frac{1}{e}}^{\tan x} \frac{tdt}{(1+t^2)} + \int_{\frac{1}{e}}^{\cot x} \frac{dt}{t(1+t^2)}$

$\therefore \quad f'(x) = \frac{\tan x}{1 + \tan^2 x}\sec^2 x + \frac{1}{\cot x(1 + \cot^2 x)}(-\text{cosec}^2 x)$

$= \tan x - \tan x = 0$

Thus, $f(x)$ is a constant function.

$$f\left(\frac{\pi}{4}\right) = \int_{\frac{1}{e}}^1 \frac{tdt}{(1+t^2)} + \int_{\frac{1}{e}}^1 \frac{dt}{t(1+t^2)}$$

$$= \int_{\frac{1}{e}}^1 \frac{1}{t}dt = \ln t\Big|_{1/e}^1 = 1$$

94. (1) $\lim_{x\to\infty} \frac{\int_0^x \tan^{-1} t\,dt}{\sqrt{x^2+1}}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

$$= \lim_{x\to\infty} \frac{\frac{\tan^{-1} x}{x}}{\sqrt{x^2+1}}$$

$$= \frac{\pi}{2}\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}$$

$$= \frac{\pi}{2}\lim_{x\to\infty} \sqrt{1 + \frac{1}{x^2}} = \frac{\pi}{2}$$

95. (3) $f(x) = \int_0^\pi \cos t\cos(x - t)dt$ ...(1)

$= \int_0^\pi \cos(\pi - t)\cos(x - (\pi - t))dt$

$\therefore \quad f(x) = \int_0^\pi \cos t \cdot \cos(x + t)dt$ ...(2)

From (1) + (2) gives

$$2f(x) = \int_0^\pi \cos t(2\cos x \cdot \cos t)dt$$

$\therefore \quad f(x) = \cos x\int_0^\pi \cos^2 t\,dt = 2\cos x\int_0^{\pi/2} \cos^2 t\,dt$

$\therefore \quad f(x) = \frac{\pi\cos x}{2}$.

Hence, minimum value of $f(x)$ is $\frac{-\pi}{2}$.

96. (2) $f(x) = \int_0^x \sqrt{1 - f^2(t)}dt + \frac{1}{2}$ ...(1)

$\Rightarrow \quad f'(x) = \sqrt{1 - f^2(x)}$

$\Rightarrow \quad \int \frac{f'(x)}{\sqrt{1 - f^2(x)}}dx = \int 1dx$

$\Rightarrow \quad \sin^{-1}(f(x)) = x + c$

From (1), $f(0) = \frac{1}{2}$,

$\therefore \quad c = \frac{\pi}{6}$

$\Rightarrow \quad f(x) = \sin\left(x + \frac{\pi}{6}\right)$

$\Rightarrow \quad f(\pi) = \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

97. (3) We have $\int_0^1 e^{x^2}(x-\alpha)\,dx = 0$

or $\int_0^1 e^{x^2} x\,dx = \int_0^1 e^{x^2}\alpha\,dx$

or $\frac{1}{2}\int_0^1 e^t\,dt = \alpha\int_0^1 e^{x^2}\,dx$, where $t = x^2$

or $\frac{1}{2}(e-1) = \alpha\int_0^1 e^{x^2}\,dx$ (1)

Since, e^{x^2} is an increasing function for $0 \le x \le 1$,

$1 \le e^{x^2} \le e$ when $0 \le x \le 1$

or $1(1-0) \le \int_0^1 e^{x^2}\,dx \le e(1-0)$

or $1 \le \int_0^1 e^{x^2}\,dx \le e$ (2)

From equations (1) and (2), we find that L.H.S. of equation (1) is positive and $\int_0^1 e^{x^2}\,dx$ lies between 1 and e. Therefore, α is a positive real number.

Now, from equation (1), $\alpha = \dfrac{\frac{1}{2}(e-1)}{\int_0^1 e^{x^2}\,dx}$ (3)

The denominator of equation (3) is greater than unity and the numerator lies between 0 and 1. Therefore, $0 < \alpha < 1$.

98. (3) Since e^{x^2} is an increasing function on (0, 1),
$m = e^0 = 1$, $M = e^1 = e$ [m and M are minimum and maximum values of $f(x) = e^{x^2}$ in the interval (0, 1), respectively]. Then

$1 < e^{x^2} < e$, for all $x \in (0, 1)$

or $1(1-0) < \int_0^1 e^{x^2}\,dx < e(1-0)$

or $1 < \int_0^1 e^{x^2}\,dx < e$

99. (2) $\left|\int_a^b f(x)\,dx - (b-a)f(a)\right| = \left|\int_a^b f(x)\,dx - \int_a^b f(a)\,dx\right|$

$= \left|\int_a^b (f(x) - f(a))\,dx\right|$

$\le \int_a^b |f(x) - f(a)|\,dx$

$\le \int_a^b |x-a|\,dx$

$= \int_a^b (x-a)\,dx = \frac{(b-a)^2}{2}$

100. (1) $I = \int_0^\infty \frac{x\log x\,dx}{(1+x^2)^2}$

Let $x = \frac{1}{t}$

$\therefore \quad I = \int_\infty^0 \frac{\left(\frac{1}{t}\right)\log\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)dt}{\left(1+\frac{1}{t^2}\right)^2}$

$= -\int_0^\infty \frac{t\log t}{(1+t^2)^2}\,dt = -I$

or $I = 0$

101. (1) $\int_0^\infty \left(\frac{\pi}{1+\pi^2x^2} - \frac{1}{1+x^2}\right)\log x\,dx$

$= \int_0^\infty \frac{\log\left(\frac{y}{\pi}\right)dy}{1+y^2} - \int_0^\infty \frac{\log x}{1+x^2}\,dx$

$= -\int_0^\infty \frac{\log\pi}{1+y^2}\,dy = -\frac{\pi}{2}\ln\pi$

102. (1) $I = \int_0^{\pi/2} \frac{\sin 2x}{x+1}\,dx$. Put $x = y/2$. Then,

$I = \int_0^\pi \frac{\sin y}{y+2}\,dy$

$= \left(\frac{-\cos y}{y+2}\right)_0^\pi - \int_0^\pi \frac{\cos y}{(y+2)^2}\,dy$ (Integrating by parts)

$= \frac{1}{\pi+2} + \frac{1}{2} - A$

103. (1) $I = \int_0^4 \frac{(y^2-4y+5)\sin(y-2)}{(2y^2-8y+1)}\,dy$. Put $y - 2 = z$. Then,

$I = \int_{-2}^2 \frac{z^2+1}{2z^2-7}\sin(z)\,dz = 0$

104. (1) Putting $x\tan\theta = z\sin\theta$ or $dx = \cos\theta\,dz$, we get

$I = \cos\theta \int_{\tan\theta}^1 f(z\sin\theta)\,dz$

$= -\cos\theta \int_1^{\tan\theta} f(x\sin\theta)\,dx$

105. (3) $I_1 = \int_0^1 \frac{e^x\,dx}{1+x}$, $I_2 = \int_0^1 \frac{x^2\,dx}{e^{x^3}(2-x^3)}$

In I_2, put $1 - x^3 = t$

$\therefore \quad I_2 = \frac{1}{3}\int_1^0 \frac{-dt}{e^{1-t}(1+t)} = \frac{1}{3e}\int_0^1 \frac{e^t\,dt}{1+t} = \frac{1}{3e}I_1$

$\therefore \quad \frac{I_1}{I_2} = 3e$

106. (3) In I_2, put $x + 1 = t$. Then

$I_2 = \int_{-2}^2 \frac{2t^2+11t+14}{t^4+2}\,dt = \int_{-2}^2 \frac{2x^2+11x+14}{x^4+2}\,dx$

$\therefore \quad I_1 + I_2 = \int_{-2}^2 \frac{x^6+3x^5+7x^4+2x^2+11x+14}{x^4+2}\,dx$

$$= \int_{-2}^{2} \frac{(x^2+3x+7)(x^4+2)+5x}{x^4+2}\,dx$$

$$= \int_{-2}^{2} (x^2+3x+7)\,dx + 5\int_{-2}^{2} \frac{x}{x^4+2}\,dx$$

$$= 2\int_{0}^{2} (x^2+7)\,dx = \frac{100}{3}$$

107. (2) $I_1 - I_2 = \int_0^{\pi/2} (\cos\theta - \sin 2\theta) f(\sin\theta + \cos^2\theta)\,d\theta$

Put $t = \sin\theta + \cos^2\theta$ and $dt = (\cos\theta - \sin 2\theta)\,d\theta$

or $I_1 - I_2 = \int_1^1 f(t)\,dt = 0$

108. (2) Put $x = a\cos^2\theta + b\sin^2\theta$ or $dx = 2(b-a)\sin\theta\cos\theta\,d\theta$. Then

$$\int_a^b (x-a)^3(b-x)^4\,dx$$

$$= 2(b-a)\int_0^{\pi/2} (a\cos^2\theta + b\sin^2\theta - a)^3 (b - a\cos^2\theta - b\sin^2\theta)^4 \sin\theta\cos\theta\,d\theta$$

$$= 2(b-a)^8 \int_0^{\pi/2} \sin^7\theta\cos^9\theta\,d\theta$$

$$= 2(b-a)^8 \int_0^{\pi/2} \sin^7\theta(1-\sin^2\theta)^4\cos\theta\,d\theta$$

$$= 2(b-a)^8 \int_0^1 x^7(1-x^2)^4\,dx$$

$$= 2(b-a)^8 \int_0^1 x^7(1-4x^2+6x^4-4x^6+x^8)\,dx$$

$$= 2(b-a)^8 \left[\frac{1}{8} - \frac{4}{10} + \frac{6}{12} - \frac{4}{14} + \frac{1}{16}\right] = \frac{(b-a)^8}{280}$$

109. (3) Putting $x = \frac{1}{1+y}$, $dx = -\frac{1}{(1+y)^2}\,dy$, we get

$$I(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \int_\infty^0 \frac{1}{(1+y)^{m-1}} \left(1 - \frac{1}{1+y}\right)^{n-1} \frac{(-1)}{(1+y)^2}\,dy$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

Since $I(m,n) = I(n,m)$, we have

$$I(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx.$$

110. (2) $\sin nx - \sin(n-2)x = 2\cos(n-1)x\sin x$

or $\int \frac{\sin nx}{\sin x}\,dx = \int 2\cos(n-1)\,dx + \int \frac{\sin(n-2)x}{\sin x}\,dx$

$$\therefore \int_0^{\pi/2} \frac{\sin 5x}{\sin x}\,dx = \int_0^{\pi/2} 2\cos 4x\,dx + \int_0^{\pi/2} \frac{\sin 3x}{\sin x}\,dx$$

$$= 0 + \int_0^{\pi/2} \frac{\sin 3x}{\sin x}\,dx$$

$$= \int_0^{\pi/2} dx$$

$$= \frac{\pi}{2}$$

111. (1) $I_3 = \int_0^\pi e^x(\sin x)^3\,dx$

$$= e^x(\sin x)^3\Big|_0^\pi - 3\int_0^\pi (\sin x)^2\cos x\, e^x\,dx$$

$$= 0 - 3(\sin x)^2\cos x\, e^x\Big|_0^\pi + 3\int_0^\pi (2\sin x\cos x\cos x - \sin x\sin^2 x)\, e^x\,dx$$

$$= 0 + 6\int_0^\pi \sin x\cos^2 x e^x\,dx - 3\int_0^\pi \sin^3 x\, e^x\,dx$$

$$= 6\int_0^\pi \sin x(1-\sin^2 x)e^x\,dx - 3\int_0^\pi \sin^3 x\, e^x\,dx$$

$$= 6\int_0^\pi \sin x e^x\,dx - 9\int_0^\pi \sin^3 x\, e^x\,dx = 6I_1 - 9I_3$$

or $10I_3 = 6I_1$

or $\frac{I_3}{I_1} = \frac{3}{5}$

112. (2) Differentiating, we get $f''(x) = f'(x)$

or $\int \frac{df'(x)}{f'(x)} = \int dx$

or $\ln f'(x) = x + c$ or $f'(x) = Ae^x$ (1)

or $\int f'(x)\,dx = \int Ae^x\,dx$ or $f(x) = Ae^x + B$ (2)

Now, $f(0) = 1$ or $A + B = 1$

$$\therefore f'(x) = f(x) + \int_0^1 (Ae^x + 1 - A)\,dx$$

$$Ae^x = (Ae^x + 1 - A) + |(Ae^x + (1-A)x)|_0^1$$

or $Ae^x = (Ae^x + 1 - A) + (Ae + (1-A) - A)$

or $1 - A + (Ae + 1 - A - A) = 0$

or $A(e-3) = -2$

or $A = \frac{2}{3-e}$ and $B = 1 - \frac{2}{3-e} = \frac{1-e}{3-e}$

or $f(\log_e 2) = \frac{4}{3-e} + \frac{1-e}{3-e} = \frac{5-e}{3-e}$

113. (3) $f'(x) \ge f^3(x) + \frac{1}{f(x)}$

or $\frac{f(x)f'(x)}{1+f^4(x)} \ge 1$

Integrating on the interval (a, b), we get

$$\int_a^b \frac{f(x)f'(x)}{1+f^4(x)}\,dx \ge \int_a^b dx$$

or $\left[\frac{1}{2}\tan^{-1} f^2(x)\right]_a^b \ge b - a$

or $b - a \le \frac{1}{2}\left[\lim_{x\to b^-} \tan^{-1} f^2(x) - \lim_{x\to a^+} \tan^{-1} f^2(x)\right]$

$$= \frac{1}{2}\left[\tan^{-1}\sqrt{3} - \tan^{-1}1\right]$$

$$= \frac{\pi}{24}$$

Multiple Correct Answers Type

1. (2), (3)

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=n+1}^{2n} f\left(\frac{r}{n}\right)=\int_1^2 f(x)\,dx$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r+n}{n}\right)=\int_0^1 f(1+x)\,dx=\int_1^2 f(t)\,dt=\int_1^2 f(x)dx$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)=\int_0^1 f(x)dx$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n} f\left(\frac{r}{n}\right)=\int_0^2 f(x)\,dx$$

2. (1), (3)

$$L=\lim_{n\to\infty}\frac{n^3\sum_{r=1}^{n} e^{r/n}}{(n+1)^m\sum_{r=1}^{n} r^{2m}}$$

$$=\lim_{n\to\infty}\frac{n^3\sum_{r=1}^{n} e^{r/n}\cdot\frac{1}{n}}{(n+1)^m n^{2m}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^{2m}\cdot\frac{1}{n}}$$

$$=\lim_{n\to\infty}\frac{n^3}{(n+1)^m n^{2m}}\cdot\frac{\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} e^{r/n}}{\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^{2m}}$$

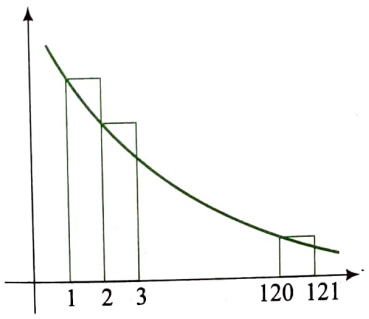
$$=\lim_{n\to\infty}\frac{n^3}{(n^3+n^2)^m}\cdot\frac{\int_0^1 e^x dx}{\int_0^1 x^{2m}dx}$$

For L to be non–zero finite, $m = 1$

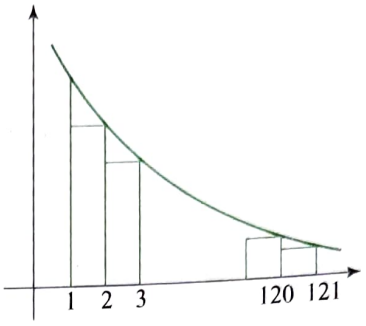
$$\therefore\quad L=\frac{\int_0^1 e^x dx}{\int_0^1 x^2 dx}=\frac{e-1}{1/3}=3(e-1)$$

3. (1), (2), (4)

Consider function $f(x)=\frac{1}{\sqrt{x}}$ for $1<x<121$



Clearly $p>\int_1^{121}\frac{1}{\sqrt{x}}dx=20$



and $q<\int_1^{121}\frac{1}{\sqrt{x}}dx=20$

$\therefore\quad p-20>20-q$

$\Rightarrow\quad p+q>40$

4. (1), (4)

$$S_n<\lim_{n\to\infty}S_n=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\,\frac{1}{1+k/n+(k/n)^2}$$

$$=\int_0^1\frac{dx}{1+x+x^2}$$

$$=\int_0^1\frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$=\left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right]_0^1=\frac{\pi}{3\sqrt{3}}$$

Now, $T_n>\frac{\pi}{3\sqrt{3}}$ as

$$h\sum_{k=0}^{n-1} f(k/n)>\int_0^1 f(x)\,dx>h\sum_{k=1}^{n} f(k/n)$$

5. (1), (3), (4)

$$I=\int_0^1\frac{2x^2+3x+3}{(x+1)\left(x^2+2x+2\right)}dx$$

$$=\int_0^1\frac{2(x^2+2x+2)-(x+1)}{(x+1)\left(x^2+2x+2\right)}dx$$

$$=\int_0^1\left(\frac{2}{x+1}-\frac{1}{x^2+2x+2}\right)dx$$

$$=\left[2\log(x+1)-\tan^{-1}(x+1)\right]_0^1$$

$$=2\log 2-\tan^{-1}2+\tan^{-1}1 \qquad (1)$$

$$=2\log 2-\tan^{-1}2+\frac{\pi}{4}$$

$$=\log 4-\left(\frac{\pi}{2}-\cot^{-1}2\right)+\frac{\pi}{4}$$

$$=-\frac{\pi}{4}+\log 4+\cot^{-1}2$$

From equation (1),

$$I = 2\log 2 - \tan^{-1}\left(\frac{2-1}{1+2\times 1}\right)$$

$$= 2\log 2 - \tan^{-1}\frac{1}{3}$$

$$= 2\log 2 - \cot^{-1} 3$$

6. (1), (4)

$$f'(x) = \frac{3^x}{1+x^2} > 0 \ \forall\, x > 0$$

or $$f'(x) = \frac{3^x}{1+x^2} > \frac{1}{1+x^2} \ \forall\, x \geq 1$$

$$\therefore \quad \int_1^x f'(x)\,dx > \int_1^x \frac{1}{1+x^2}\,dx$$

or $f(x) > \tan^{-1}x - \tan^{-1}1$ or $f(x) + \pi/4 > \tan^{-1}x$

7. (1), (2)

We know $\int_a^b |\sin x|\,dx$ represents the area under the curve from $x = a$ to $x = b$. We also know that area from $x = a$ to $x = a + \pi$ is 2.

$$\therefore \quad \int_a^b |\sin x|\,dx = 8 \text{ or } b - a = \frac{8\pi}{2} \qquad (1)$$

Similarly, $\int_0^{a+b} |\cos x|\,dx = 9$ or $a + b - 0 = \frac{9\pi}{2}$ (2)

From (1) and (2), $a = \frac{\pi}{4}$ and $b = \frac{17\pi}{4}$

$$\therefore \quad |a+b| = \frac{9\pi}{2},\ |a-b| = 4\pi,\ \frac{a}{b} = 17$$

Obviously, $\int_a^b \sec^2 x\,dx \neq 0$

8. (1), (2), (3)

$$g(x) = \int_0^x 2|t|\,dt$$

$$= \begin{cases} \int_0^x -2t\,dt, & x < 0 \\ \int_0^x 2t\,dt, & x \geq 0 \end{cases}$$

$$= \begin{cases} \left[-t^2\right]_0^x, & x < 0 \\ \left[t^2\right]_0^x, & x \geq 0 \end{cases}$$

$$= \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$$

$$= x|x|$$

Clearly, the function is continuous and differentiable at $x = 0$.

Also, $g'(x) = \begin{cases} -2x, & x < 0 \\ 2x, & x > 0 \end{cases} + 1$ which is non-differentiable at $x = 0$.

9. (1), (4)

$$A_{n+1} - A_n = \int_0^{\pi/2} \frac{\sin(2n+1)x - \sin(2n-1)x}{\sin x}\,dx$$

$$= \int_0^{\pi/2} 2\cos 2nx\,dx = 0$$

or $$A_{n+1} = A_n$$

$$B_{n+1} - B_n = \int_0^{\pi/2} \frac{\sin^2(n+1)x - \sin^2 nx}{\sin^2 x}\,dx$$

$$= \int_0^{\pi/2} \frac{\sin(2n+1)x}{\sin x}\,dx = A_{n+1}$$

10. (2), (3)

$$I = \int_0^\infty \frac{dx}{1+x^4} \qquad (1)$$

$$= \int_0^\infty \frac{x^2+1-x^2}{1+x^4}\,dx$$

$$= \int_0^\infty \frac{x^2}{1+x^4}\,dx + \int_0^\infty \frac{1-x^2}{1+x^4}\,dx = I_1 + I_2$$

$$I_2 = \int_0^\infty \frac{\frac{1}{x^2} - 1}{\frac{1}{x^2} + x^2}\,dx$$

Put $$x + \frac{1}{x} = y$$

$$\therefore \quad I_2 = \int_\infty^\infty \frac{-1}{y^2-2}\,dy = 0$$

$$\therefore \quad I = \int_0^\infty \frac{dx}{1+x^4} = \int_0^\infty \frac{x^2\,dx}{1+x^4} \qquad (2)$$

Adding (1) and (2), we get

$$2I = \int_0^\infty \frac{1+x^2}{1+x^4}\,dx = \int_0^\infty \frac{\frac{1}{x^2}+1}{\frac{1}{x^2}+x^2}\,dx \qquad \left(\text{Putting } x - \frac{1}{x} = y\right)$$

$$= \int_{-\infty}^\infty \frac{dy}{y^2+2}$$

$$= \left[\frac{1}{\sqrt{2}}\tan^{-1}\frac{y}{\sqrt{2}}\right]_{-\infty}^\infty = \frac{\pi}{\sqrt{2}}$$

or $$I = \frac{\pi}{2\sqrt{2}}$$

11. (1), (3)

$$\int_0^1 e^{x^2-x}\,dx$$

For $x \in (0, 1)$, $x^2 - x \in (-1/4, 0)$

$$\therefore \quad e^{-1/4} < e^{x^2-x} < e^0$$

or $$e^{-\frac{1}{4}} < \int_0^1 e^{x^2-x}\,dx < 1$$

12. (1), (2), (3)

Let $I = \int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx$

$$= \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(x)}\,dx$$

Adding (1) and (2), we get

$$2I = \int_a^b 1\,dx = b - a$$

or $I=\left(\frac{b-a}{2}\right)=10$ (given)

$\therefore \quad b-a=20$

13. (1), (2), (3)

For $a \le 0$, given equation becomes

$$\int_0^2 (x-a)dx \ge 1 \text{ or } a \le \frac{1}{2}$$

So, $a \le 0$

For $0 < a < 2$,

$$\int_0^2 |x-a|\,dx \ge 1 \text{ or } \int_0^a (a-x)dx + \int_a^2 (x-a)dx \ge 1$$

or $\frac{a^2}{2}+2-2a+\frac{a^2}{2} \ge 1$

or $a^2-2a+1 \ge 0$ or $(a-1)^2 \ge 0$

So, $0 < a < 2$

For $a \ge 2$,

$$\int_0^2 |x-a|\,dx \ge 1$$

or $\int_0^2 (a-x)dx \ge 1$

or $2a-2 \ge 1$ or $a \ge \frac{3}{2}$

So, $a \ge 2$

14. (1), (2), (4)

Given that $f(x)=\int_0^x |t-1|\,dt$

$$=\int_0^x (1-t)\,dt,\ 0 \le x \le 1$$

$$=x-\frac{x^2}{2}$$

Also, $f(x)=\int_0^1 (1-t)dt+\int_1^x (t-1)\,dt$, where $1 \le x \le 2$

$$=\frac{1}{2}+\frac{x^2}{2}-x+\frac{1}{2}=\frac{x^2}{2}-x+1$$

Thus, $f(x)=\begin{cases} x-\frac{x^2}{2}, & 0 \le x \le 1 \\ \frac{x^2}{2}-x+1, & 1 < x \le 2 \end{cases}$

$\therefore \quad f'(x)=\begin{cases} 1-x, & 0 \le x < 1 \\ x-1, & 1 < x < 2 \end{cases}$

Thus, $f(x)$ is continuous as well as differentiable at $x = 1$.

Also, $f(x) = \cos^{-1} x$ has one real root. Draw the graph and verify.

For range of $f(x)$:

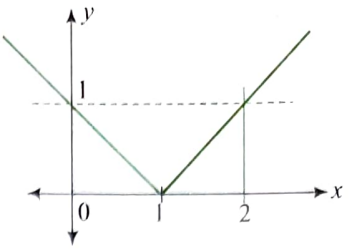
$f(x)=\int_0^x |t-1|\,dt$ is the value of area bounded by the curve

$y = |t - 1|$ and x-axis between the limits $t = 0$ and $t = x$.

Obviously, minimum area is obtained when $t = 0$ and $t = x$ coincide or $x = 0$.

Maximum value of area occurs when $t = 2$.

Hence, $f(2)$ = area of shaded region = 1.



15. (1), (2), (4)

$f(2-x)=f(2+x), f(4-x)=f(4+x)$

or $f(4+x)=f(4-x)=f(2+2-x)=f(2-(2-x))=f(x)$

Thus, the period of $f(x)$ is 4.

$$\int_0^{50} f(x)\,dx=\int_0^{48} f(x)dx+\int_{48}^{50} f(x)\,dx$$

$$=12\int_0^4 f(x)dx+\int_0^2 f(x)\,dx$$

[In second integral, replacing x by $x + 48$ and then using $f(x) = f(x + 48)$]

$$=12\left(\int_0^2 f(x)\,dx+\int_0^2 f(4-x)\,dx\right)+5$$

$$=12\left(\int_0^2 f(x)\,dx+\int_0^2 f(4+x)dx\right)+5$$

$$=24\int_0^2 f(x)\,dx+5=125$$

$$\int_{-4}^{46} f(x)dx=\int_{-4}^{-2} f(x)dx+\int_{-2}^{-2+48} f(x)dx$$

$$=\int_0^2 f(x+4)dx+12\int_0^4 f(x)dx$$

$$=\int_0^2 f(x)dx+24\int_0^2 f(x)dx$$

$$=125$$

Also, $\int_2^{52} f(x)dx=\int_2^4 f(x)dx+\int_4^{4+48} f(x)dx$

$$=\int_0^2 f(4-x)dx+12\int_0^4 f(x)dx$$

$$=\int_0^2 f(4+x)dx+24\int_0^2 f(x)dx$$

$$=\int_0^2 f(x)dx+24\int_0^2 f(x)dx$$

$$=125$$

$$\int_1^{51} f(x)dx=\int_1^3 f(x)dx+\int_3^{3+48} f(x)dx$$

$$=\int_1^3 f(x)dx+12\int_0^4 f(x)dx$$

$$=\int_0^2 f(x+1)dx+24\int_0^2 f(x)dx$$

$$\ne 125$$

16. (1), (4)

$$f(x+\pi)=\int_0^{x+\pi}(\cos(\sin t)+\cos(\cos t))\,dt$$

$$=\int_0^{\pi}(\cos(\sin t)+\cos(\cos t))\,dt + \int_{\pi}^{x+\pi}(\cos(\sin t)+\cos(\cos t))\,dt$$

$$=f(\pi)+\int_0^{x}(\cos(\sin t)+\cos(\cos t))\,dt$$

[$\because$ for $g(x)=\cos(\sin x)+\cos(\cos x)$, $f(x+\pi)=f(x)$]

$$=f(\pi)+f(x)$$

$$=f(\pi)+2f\left(\frac{\pi}{2}\right) \qquad [\because\ g(x) \text{ has period } \pi/2]$$

17. (2), (3), (4)

$$I_n=\int_0^{\pi/4}\tan^n x\,dx$$

$$=\int_0^{\pi/4}\tan^{n-2}x\tan^2 x\,dx$$

$$=\int_0^{\pi/4}\sec^2 x\tan^{n-2}x\,dx-\int_0^{\pi/4}\tan^{n-2}x\,dx$$

$$=\int_0^1 t^{n-2}\,dt-I_{n-2}, \qquad \text{where } t=\tan x$$

$$\therefore\quad I_n+I_{n-2}=\left(\frac{t^{n-1}}{n-1}\right)_0^1=\frac{1}{n-1}$$

Thus, I_2+I_4, I_4+I_6, ... are in H.P.

For $0<x<\pi/4$, we have $0<\tan^n x<\tan^{n-2}x$

So, $0<I_n<I_{n-2}$ or $I_n+I_{n+2}<2I_n<I_n+I_{n-2}$

or $$\frac{1}{n+1}<2I_n<\frac{1}{n-1} \text{ or } \frac{1}{2(n+1)}<I_n<\frac{1}{2(n-1)}$$

18. (1), (2), (4)

$$I_n=\int_0^1\frac{dx}{(1+x^2)^n}=\int_0^1(1+x^2)^{-n}\,dx$$

$$=\frac{x}{(1+x^2)^n}\Bigg|_0^1-\int_0^1(-n)(1+x^2)^{-n-1}2x\cdot x\,dx$$

$$=\frac{1}{2^n}+2n\int_0^1\frac{x^2\,dx}{(1+x^2)^{n+1}}$$

$$=\frac{1}{2^n}+2n\int_0^1\frac{1+x^2-1}{(1+x^2)^{n+1}}\,dx$$

$$=\frac{1}{2^n}+2nI_n-2nI_{n+1}$$

or $$2nI_{n+1}=2^{-n}+(2n-1)I_n$$

or $$2I_2=\frac{1}{2}+I_1=\frac{1}{2}+\tan^{-1}x\Big|_0^1$$

or $$I_2=\frac{1}{4}+\frac{\pi}{8}$$

Also, $4I_3=2^{-2}+3I_2=\frac{1}{4}+3\left(\frac{1}{4}+\frac{\pi}{8}\right)$. So, $I_3=\frac{1}{4}+\frac{3\pi}{32}$

19. (1), (2)

$$f(x)=x\int_1^x\frac{e^t}{t}\,dt-e^x$$

$$\therefore\quad f'(x)=x\frac{e^x}{x}+\int_1^x\frac{e^t}{t}\,dt-e^x$$

$$=\int_1^x\frac{e^t}{t}\,dt>0 \qquad [\because x\in[1,\infty)]$$

Thus, $f(x)$ is an increasing function.

20. (1), (2), (3)

$$f(x)=\int_a^x\frac{1}{f(x)}\,dx$$

or $$f'(x)=\frac{1}{f(x)}\cdot 1-0 \text{ or } f(x)f'(x)=1$$

or $$\int f(x)f'(x)dx=\int 1dx$$

or $$\frac{1}{2}[f(x)]^2=x+c \qquad (1)$$

Now, given that $\int_a^1[f(x)]^{-1}dx=\sqrt{2}$ or $f(1)=\sqrt{2}$

Thus, from (1), $\frac{1}{2}[f(1)]^2=1+c$ or $c=0$

or $$f(x)=\pm\sqrt{2x}$$

But $f(1)=\sqrt{2}$ or $f(x)=\sqrt{2x}$ or $f(2)=2$

Also, $f'(x)=\frac{1}{\sqrt{2x}}$ or $f'(2)=1/2$

$$\int_0^1 f(x)dx=\int_0^1\sqrt{2x}dx=\left[\frac{(2x)^{3/2}}{3}\right]_0^1=\frac{(2)^{3/2}}{3}$$

Also, $f^{-1}(x)=\frac{x^2}{2}$ or $f^{-1}(2)=2$

21. (1), (2)

$$f(x)=e^x+\int_0^1 e^x\,f(t)\,dt=e^x+k\,e^x, \text{ where } k=\int_0^1 f(t)dt$$

$$\therefore\quad k=\int_0^1(e^t+ke^t)dt=e+ke-1-k$$

$$\therefore\quad k=\frac{e-1}{2-e}$$

Thus, $f(x)=e^x\left(1+\frac{e-1}{2-e}\right)=\frac{e^x}{2-e}$

Obviously, $f(0)=\frac{1}{2-e}<0$

Also, $f'(x)=\frac{e^x}{2-e}<0$ for $\forall\ x\in R$.

Hence, $f(x)$ is a decreasing function.

Also, $$\int_0^1 f(x)\,dx=\int_0^1\frac{e^x}{2-e}\,dx$$

$$=\left[\frac{e^x}{2-e}\right]_0^1$$

$$=\frac{e-1}{2-e}<0$$

22. (1), (2)

$$\text{L.H.S.} = \int_0^x \left\{ \int_0^u f(t)\,dt \right\} du$$

Integrating by parts, choose 1 as the second function. Then,

$$\text{L.H.S.} = \left\{ u \int_0^u f(t)\,dt \right\}_0^x - \int_0^x f(u)\,u\,du$$

$$= x \int_0^x f(t)\,dt - \int_0^x f(u)\,u\,du$$

$$= x \int_0^x f(u)\,du - \int_0^x f(u)\,u\,du = \int_0^x f(u)\,(x-u)\,du$$

$$= \text{R.H.S.}$$

23. (1), (3), (4)

(1) The expression $f(x)\,f(c)\ \forall\ x \in (c-h, c+h)$ where $h \to 0^+$ is equivalent to $\lim\limits_{x \to c} f(x)\,f(c)$ which is equal to $(f(c))^2$ because $f(x)$ is continuous.

Therefore, $f(x)\,f(c) > 0\ \forall\ x \in (c-h, c+h)$ where $h \to 0^+$.

(2) We have $I = \lim\limits_{n \to \infty} \frac{1}{n} \ln\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right]$

$$= \lim_{n\to\infty} \frac{1}{n} \ln \prod_{k=1}^{n} \left(1+\frac{k}{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left(1+\frac{k}{n}\right)$$

$$= \int_0^1 \log_e(1+x)\,dx$$

$$= \int_1^2 \ln x\,dx = \left[x(\ln x - 1)\right]_1^2 = -1 + 2\ln 2$$

(3) Given $f(x) \geq 0$ or $\int_a^b f(x)dx \geq 0$.

But given $\int_a^b f(x)dx = 0$. So, this can be true only when $f(x) = 0$.

(4) $\int_a^b f(x)dx = 0$, i.e., $y = f(x)$ cuts x-axis at least once.

So, there exists at least one $c \in (a, b)$ for which $f(c) = 0$.

24. (1), (2)

$$\int_0^x [x]dx = \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx + \ldots + \int_{[x]-1}^{[x]} ([x]-1)\,dx + \int_{[x]}^{x} [x]\,dx$$

$$= 0 + 1 + 2 + 3 + \cdots + ([x]-1) + [x](x-[x])$$

$$= \frac{([x]-1)[x]}{2} + [x]\{x\} \qquad \ldots(1)$$

Now $\int_0^{[x]} x\,dx = \left[\frac{x^2}{2}\right]_0^{[x]} = \frac{[x]^2}{2}$ $\qquad \ldots(2)$

Comparing (1) and (2), we get

$$\frac{([x]-1)[x]}{2} + [x]\{x\} = \frac{[x]^2}{2}$$

$\therefore\ [x]^2 - [x] + 2[x]\{x\} = [x]^2$

$\Rightarrow\ [x]\,(2\{x\} - 1) = 0$

$\Rightarrow\ [x] = 0$ or $\{x\} = 1/2$

25. (1), (3)

$$f(\theta) = \int_0^1 \frac{\left|\sqrt{1-x^2} - \sin\theta\right|}{\sqrt{1-x^2}}\,dx$$

Put $x = \cos\phi$

$$\therefore\ f(\theta) = \int_0^{\frac{\pi}{2}} \left|\sin\phi - \sin\theta\right| d\phi$$

$$= \int_0^{\theta} (\sin\theta - \sin\phi)\,d\phi + \int_{\theta}^{\frac{\pi}{2}} (\sin\phi - \sin\theta)\,d\phi$$

$$= \left[\phi\sin\theta + \cos\phi\right]_0^{\theta} + \left[-\cos\phi - \phi\sin\theta\right]_{\theta}^{\pi/2}$$

$$= [\theta\sin\theta + \cos\theta - 0 - 1] - 0 - \frac{\pi}{2}\sin\theta + \cos\theta + \theta\sin\theta$$

$$= 2\,(\theta\sin\theta + \cos\theta) - \frac{\pi}{2}\sin\theta - 1$$

$$f'(\theta) = 2\,(\theta\cos\theta + \sin\theta - \sin\theta) - \frac{\pi}{2}\cos\theta$$

$$= 2\left(\theta - \frac{\pi}{4}\right)\cdot\cos\theta$$

$f'(\theta) = 0$

$\therefore\ \theta = \frac{\pi}{4}$, which is point of minima.

$\therefore\ f\left(\frac{\pi}{4}\right) = 2\left(\frac{\pi}{4}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - \frac{\pi}{2}\frac{1}{\sqrt{2}} - 1 = \sqrt{2} - 1$

$\therefore\ f(0) = 1$ and $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 1$

So, $f_{\text{min.}} = \sqrt{2} - 1$ and $f_{\text{max.}} = 1$

26. (1), (3)

$$\int_0^{\alpha} f(x)\,dx \qquad \ldots(1)$$

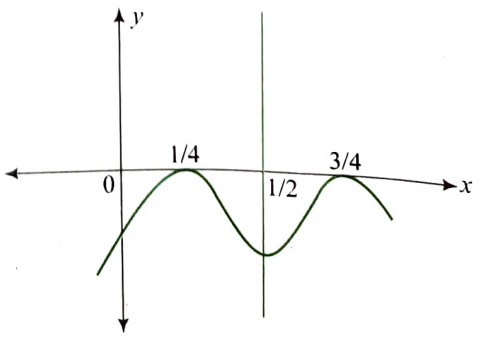
Put $x = \alpha t$

$$\therefore\ \int_0^{\alpha} f(x)\,dx = \alpha \int_0^1 f(\alpha t)\,dt \geq \alpha \int_0^1 f(t)\,dt \text{ as } \alpha \in (0, 1)$$

Also $\alpha \int_0^1 f(t)\,dt \geq \alpha^2 \int_0^1 f(t)\,dt$

$$\therefore\ \int_0^{\alpha} f(x)\,dx \geq \alpha^2 \int_0^1 f(x)\,dx$$

27. (1), (2), (3), (4)



$$f(x) = f(1-x)$$

Replacing x by $\frac{1}{2} + x$, we get

$$f\left(\frac{1}{2}+x\right) = f\left(\frac{1}{2}-x\right) \quad (1)$$

Hence, $f(x + 1/2)$ is an even function or $f(x + 1/2) \sin x$ is an odd function.

Also, $f'(x) = -f'(1-x)$ (2)

and for $x = 1/2$, we have $f'(1/2) = 0$.

Also, $\int_{1/2}^{1} f(1-t)e^{\sin \pi t}\, dt = -\int_{1/2}^{0} f(y)\, e^{\sin \pi y}\, dy$

(by putting, $1 - t = y$)

Since $f'(1/4) = 0$, $f'(3/4) = 0$ [from equation (2)].

Also, $f'(1/2) = 0$ [from equation (2)].

Thus, $f'(x) = 0$ at least twice in [0, 1] (by Rolle's theorem).

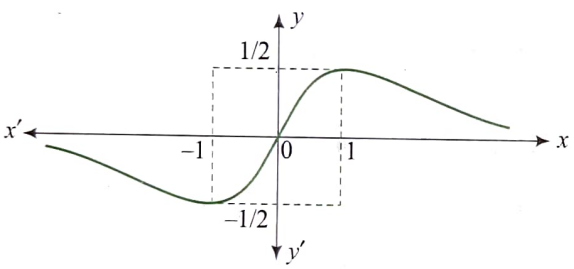
Linked Comprehension Type

For Problems 1–3

1. (4), 2. (1), 3. (3)

$$\int_2^x f(t)\,dt = \frac{x^2}{2} + \int_x^2 t^2 f(t)\,dt$$

Differentiating w.r.t. x, we get



$$f(x) = x + (-x^2 f(x))$$

or $f(x)\,[1 + x^2] = x$

Let $y = f(x) = \frac{x}{1+x^2}$

or $yx^2 - x + y = 0$

Since x is real, $D \geq 0$

or $1 - 4y^2 \geq 0$

or $y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

Also, $f(x)$ is an odd function. Hence, $\int_{-2}^{2} f(x)\,dx = 0$

$$f'(x) = \frac{1+x^2-2x^2}{1+x^2} = \frac{1-x^2}{1+x^2} \geq 0$$

or $x^2 - 1 \leq 0$

or $x \in [-1, 1]$

For Problems 4–6

4. (2), 5. (2), 6. (3)

$$f(x) = x^2 + \int_0^x e^{-t} f(x-t)\,dt \quad (1)$$

$$= x^2 + \int_0^x e^{-(x-t)} f(x-(x-t))\,dt$$

$$\left[\text{Using } \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right]$$

$$= x^2 + e^{-x}\int_0^x e^{t} f(t)\,dt \quad (2)$$

Differentiating w.r.t. x, we get

$$f'(x) = 2x - e^{-x}\int_0^x e^{t} f(t)\,dt + e^{-x}e^{x} f(x)$$

$$= 2x - e^{-x}\int_0^x e^{t} f(t)\,dt + f(x)$$

$$= 2x + x^2 \quad \text{[using equation (2)]}$$

$\therefore \quad f(x) = \frac{x^3}{3} + x^2 + c$

Also, $f(0) = 0$ [from equation (1)]

or $f(x) = \frac{x^3}{3} + x^2$

or $f'(x) = x^2 + 2x$

Thus, $f'(x) = 0$ has real roots. Hence, $f(x)$ is non-monotonic.

Hence, $f(x)$ is many-one, but range is R, and hence, is surjective.

$$\int_0^1 f(x)\,dx = \int_0^1 \left(\frac{x^3}{3} + x^2\right)dx$$

$$= \left[\frac{x^4}{12} + \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{12} + \frac{1}{3}$$

$$= \frac{5}{12}$$

For Problems 7–9

7. (3), 8. (4), 9. (3)

$$f(x) - \lambda \int_0^{\pi/2} \sin x \cos t\, f(t)\,dt = \sin x$$

or $f(x) - \lambda \sin x \int_0^{\pi/2} \cos t\, f(t)\,dt = \sin x$

or $f(x) - A \sin x = \sin x$

or $f(x) = (A + 1) \sin x$, where

$$A = \lambda \int_0^{\pi/2} \cos t\, f(t)\,dt$$

or $A = \lambda \int_0^{\pi/2} \cos t (A+1) \sin t\,dt$

$$= \frac{\lambda(A+1)}{2}\int_0^{\pi/2} \sin 2t\,dt$$

$$= \frac{\lambda(A+1)}{2}\left[\frac{-\cos 2t}{2}\right]_0^{\pi/2}$$

$$= \frac{\lambda(A+1)}{2}$$

$\therefore \quad A = \frac{\lambda}{2-\lambda}$

$\therefore \quad f(x) = \left(\frac{\lambda}{2-\lambda} + 1\right)\sin x = \left(\frac{2}{2-\lambda}\right)\sin x$

$\left(\frac{2}{2-\lambda}\right)\sin x = 2$

or $\sin x = (2 - \lambda)$

or $|2 - \lambda| \leq 1$

or $-1 \le \lambda - 2 \le 1$

or $1 \le \lambda \le 3$

$$\int_0^{\pi/2} f(x)\,dx = 3$$

or $$\int_0^{\pi/2} \frac{2}{2-\lambda}\sin x\,dx = 3$$

or $$-\left[\frac{2}{2-\lambda}\cos x\right]_0^{\pi/2} = 3$$

or $$\frac{2}{2-\lambda} = 3$$

or $\lambda = 4/3$

For Problems 10–13

10. (2) $f(x)$ is an odd function. Thus, $f(x) = -f(-x)$

$$\phi(-x) = \int_a^{-x} f(t)\,dt$$

Put $t = -y$

$$\therefore \quad \phi(-x) = \int_{-a}^{x} f(-t)(-dt)$$

$$= \int_{-a}^{x} f(t)\,dt = \int_{-a}^{a} f(t)\,dt + \int_a^x f(t)\,dt$$

$$= 0 + \int_a^x f(t)\,dt = \phi(x).$$

11. (4) If $f(x)$ is an even function, then

$$\phi(-x) = -\int_{-a}^{x} f(t)\,dt$$

$$= -\int_{-a}^{a} f(t)\,dt - \int_a^x f(t)\,dt$$

$$= -2\int_0^a f(t)\,dt - \int_a^x f(t)\,dt \qquad \text{[as } f(x) \text{ is an even function]}$$

Now, $$\int_0^a f(t)\,dt = \int_0^a f(a-t)\,dt$$

$$= \int_0^a -f(t)\,dt \qquad \text{[using } f(a-x) = -f(x)\text{]}$$

or $$\int_0^a f(t)\,dt = 0$$

or $$\phi(-x) = -\int_a^x f(t)\,dt = -\phi(x)$$

Thus, $\phi(x)$ is an odd function.

12. (4) $g(x+\alpha) + g(x) = 0$

or $g(x+2\alpha) + g(x+\alpha) = 0$

or $g(x+2\alpha) = g(x)$

Thus, $g(x)$ is periodic with period 2α.

$$\therefore \quad \int_b^{2k} g(t)\,dt = \int_b^{b+c} g(x)\,dx \qquad (\because\ b, k, c \text{ are in A.P.})$$

This is independent of b. Then c has least value 2α.

13. (4)
$$\int_{p+m\alpha}^{q+n\alpha} g(t)\,dt = \int_{p+m\alpha}^{p} g(x)\,dx + \int_p^q g(x)\,dx + \int_q^{q+n\alpha} g(x)\,dx$$

$$= -\frac{m}{2}\int_0^{2\alpha} g(x)\,dx + \int_p^q g(x)\,dx + \frac{n}{2}\int_0^{2\alpha} g(x)\,dx$$

$$= \int_p^q g(x)\,dx + \left(\frac{n-m}{2}\right)\int_0^{2\alpha} g(x)\,dx$$

For Problems 14–17

14. (2) Let $I(a) = \int_0^1 \frac{x^a - 1}{\log x}\,dx$ (1)

Differentiating w.r.t. a keeping x as constant, we get

$$\frac{dI(a)}{da} = \int_0^1 \frac{d}{da}\left(\frac{x^a - 1}{\log x}\right)dx$$

$$= \int_0^1 \frac{x^a \log x}{\log x}\,dx$$

$$= \int_0^1 x^a\,dx$$

$$= \frac{x^{a+1}}{a+1}\Big|_0^1 = \frac{1}{(a+1)}$$

Integrating both sides w.r.t. a, we get

$I(a) = \log(a+1) + c$

For $a = 0$, $I(0) = \log 1 + c$ [from equation (1)]

$0 = 0 + c$

$\therefore \quad I = \log(a+1)$

15. (3) Let $F(k) = \int_0^{\pi/2} \ln(\sin^2\theta + k^2\cos^2\theta)\,d\theta$

$$F'(k) = \int_0^{\pi/2} \frac{1}{\sin^2\theta + k^2\cos^2\theta}\,2k\cos^2\theta\,d\theta$$

$$= 2k\int_0^{\pi/2} \frac{\cos^2\theta}{\sin^2\theta + k^2\cos^2\theta}\,d\theta$$

$$= 2k\int_0^{\pi/2} \frac{d\theta}{\tan^2\theta + k^2}$$

$$= 2k\int_0^{\pi/2} \frac{\sec^2\theta - \tan^2\theta}{\tan^2\theta + k^2}\,d\theta$$

$$= 2k\int_0^{\infty} \frac{dt}{t^2 + k^2} - 2k\int_0^{\pi/2} d\theta + 2k^3\int_0^{\pi/2} \frac{d\theta}{\tan^2\theta + k^2}$$

(Putting $t = \tan\theta$)

$$= 2k\,\frac{1}{k}\tan^{-1}\frac{t}{k}\Big|_0^{\infty} - 2k\,\frac{\pi}{2} + k^2F'(k)$$

or $(1-k^2)F'(k) = \pi - k\pi = \pi(1-k)$

or $F'(k) = \frac{\pi}{1+k}$

or $F(k) = \pi\log(1+k) + c$

For $k = 1$, $F(1) = 0$ or $c = -\pi\log 2$

or $F(k) = \pi\log(1+k) - \pi\log 2$

16. (1) Let $I(a) = \int_0^{\pi/2} \log\left(\frac{1+a\sin x}{1-a\sin x}\right)\frac{dx}{\sin x}$

$$\frac{dI}{da} = \int_0^{\pi/2} \frac{2\sin x}{1-a^2\sin^2 x}\frac{dx}{\sin x}$$

$$= \int_0^{\pi/2} \frac{2\sec^2 x\,dx}{1+\tan^2 x - a^2\tan^2 x}$$

$$= \int_0^{\pi/2} \frac{2\sec^2 x\,dx}{1+(1-a^2)\tan^2 x}$$

$$= \int_0^{\infty} \frac{2\,dt}{1+(1-a^2)t^2} \qquad (\text{put } \tan x = t)$$

$$= \frac{2}{\sqrt{1-a^2}}\left[\tan^{-1}\left(t\sqrt{1-a^2}\right)\right]_0^{\infty} = \frac{\pi}{\sqrt{1-a^2}}$$

$\therefore \quad I = \pi\sin^{-1}a \qquad [\text{as } I(0) = 0]$

17. (3) $\int_0^{\pi} \frac{dx}{(a-\cos x)} = \frac{\pi}{\sqrt{a^2-1}}$

Differentiating both sides with respect to a, we get

$$-\int_0^{\pi} \frac{dx}{(a-\cos x)^2} = \frac{-\pi a}{(a^2-1)^{3/2}}$$

Again differentiating with respect to a, we get

$$2\int_0^{\pi} \frac{dx}{(a-\cos x)^3} = \frac{\pi(1+2a^2)}{(a^2-1)^{5/2}}$$

Putting $a = \sqrt{10}$, we get $\int_0^{\pi} \frac{dx}{(\sqrt{10}-\cos x)^3} = \frac{7\pi}{81}$

For Problems 18–20

18. (2), 19. (4), 20. (3)

$$f(x) = \sin x + \sin x \int_{-\pi/2}^{\pi/2} f(t)\,dt + \cos x \int_{-\pi/2}^{\pi/2} tf(t)\,dt$$

$$= \sin x\left(1 + \int_{-\pi/2}^{\pi/2} f(t)\,dt\right) + \cos x \int_{-\pi/2}^{\pi/2} tf(t)\,dt$$

$$= A\sin x + B\cos x$$

Thus, $A = 1 + \int_{-\pi/2}^{\pi/2} f(t)\,dt$

$$= 1 + \int_{-\pi/2}^{\pi/2} (A\sin t + B\cos t)\,dt$$

$$= 1 + 2B\int_0^{\pi/2} \cos t\,dt$$

$\therefore \quad A = 1 + 2B \qquad (1)$

$$B = \int_{-\pi/2}^{\pi/2} tf(t)\,dt$$

$$= \int_{-\pi/2}^{\pi/2} t(A\sin t + B\cos t)\,dt$$

$$= 2A\int_0^{\pi/2} t\sin t\,dt$$

$$= 2A\left[-t\cos t + \sin t\right]_0^{\pi/2}$$

$\therefore \quad B = 2A \qquad (2)$

From equations (1) and (2), we get

$A = -1/3, B = -2/3$

$\therefore \quad f(x) = -\frac{1}{3}(\sin x + 2\cos x)$

Thus, the range of $f(x)$ is $\left[-\frac{\sqrt{5}}{3}, \frac{\sqrt{5}}{3}\right]$

$$f(x) = -\frac{1}{3}(\sin x + 2\cos x)$$

$$= -\frac{\sqrt{5}}{3}\sin\left(x + \tan^{-1}2\right)$$

$$= -\frac{\sqrt{5}}{3}\cos\left(x - \tan^{-1}\frac{1}{2}\right)$$

$f(x)$ is invertible if $-\frac{\pi}{2} \le x + \tan^{-1}2 \le \frac{\pi}{2}$

or $\quad -\frac{\pi}{2} - \tan^{-1}2 \le x \le \frac{\pi}{2} - \tan^{-1}2$

or $\quad 0 \le x - \tan^{-1}\frac{1}{2} \le \pi$

or $\quad \tan^{-1}\frac{1}{2} \le x \le \pi + \tan^{-1}\frac{1}{2}$

or $\quad \pi \le x - \tan^{-1}\frac{1}{2} \le 2\pi$

or $\quad x \in \left[\pi + \cot^{-1}2,\ 2\pi + \cot^{-1}2\right]$

$$\int_0^{\pi/2} f(x)\,dx = -\frac{1}{3}\int_0^{\pi/2} (\sin x + 2\cos x)\,dx$$

$$= -\frac{1}{3}\left[-\cos x + 2\sin x\right]_0^{\pi/2} = -1$$

For Problems 21 and 22

21. (2), 22. (2)

$$u = \int_0^{\infty} \frac{dx}{x^4+7x^2+1} \text{ and } v = \int_0^{\infty} \frac{x^2\,dx}{x^4+7x^2+1}$$

$\therefore \quad u + v = \int_0^{\infty} \frac{1+x^2}{x^4+7x^2+1}\,dx$

$$= \int_0^{\infty} \frac{\frac{1}{x^2}+1}{\left(x-\frac{1}{x}\right)^2+9}\,dx$$

$$= \frac{1}{3}\left[\tan^{-1}\left(\frac{x-\frac{1}{x}}{3}\right)\right]_0^{\infty}$$

$$= \frac{1}{3}\left[\pi/2 + \pi/2\right] = \pi/3$$

$\therefore \quad u + v = \pi/3 \qquad (1)$

Now, $u - v = \int_0^{\infty} \frac{1-x^2}{x^4+7x^2+1}\,dx$

Let $x = \frac{1}{t}$ or $dx = -\frac{dt}{t^2}$

$$\therefore \quad u - v = \int_{\infty}^{0} \frac{1-\frac{1}{t^2}}{\frac{1}{t^4}+\frac{7}{t^2}+1}\left(-\frac{1}{t^2}\right)dt$$

$$= -\int_{0}^{\infty} \frac{1-t^2}{t^4+7t^2+1}dt$$

$$= -(u-v)$$

$$\therefore \quad u - v = 0 \qquad (2)$$

From (1) and (2), we get $u = v = \pi/6$

For Problems 23 and 24

23. (2), 24. (4)

$$f(x) = \int_0^1 \frac{dt}{1+|x-t|}$$

$$= \begin{cases} \int_0^1 \frac{dt}{1+|x-t|}, & x<0 \\ \int_0^1 \frac{dt}{1+|x-t|}, & 0\le x\le 1 \\ \int_0^1 \frac{dt}{1+|x-t|}, & x>1 \end{cases}$$

$$= \begin{cases} \int_0^1 \frac{dt}{1+t-x}, & x<0 \\ \int_0^x \frac{dt}{1+x-t} + \int_x^1 \frac{dt}{1+t-x}, & 0\le x\le 1 \\ \int_0^1 \frac{dt}{1+x-t}, & x>1 \end{cases}$$

$$= \begin{cases} [\log_e(1+t-x)]_0^1, & x<0 \\ [-\log_e(1+x-t)]_0^x + [\log_e(1+t-x)]_x^1, & 0\le x\le 1 \\ [-\log_e(1+x-t)]_0^1, & x>1 \end{cases}$$

$$= \begin{cases} \log_e\frac{2-x}{1-x}, & x<0 \\ \log_e(1+x)(2-x), & 0\le x\le 1 \\ \log_e\frac{1+x}{x}, & x>1 \end{cases}$$

For Problems 25 and 26

25. (2), 26. (2)

$$\int_0^{f(x)} f^{-1}(t)\,dt - \int_0^x (\cos t - f(t))dt = 0$$

Differentiating w.r.t. x,

$$f^{-1}(f(x))\cdot f'(x) - (\cos x - f(x)) = 0$$

$$\Rightarrow \quad x f'(x) + f(x) = \cos x$$

$$\Rightarrow \quad (x f(x))' = \cos x$$

$$\Rightarrow \quad x f(x) = \sin x + c$$

Given $f\left(\frac{\pi}{2}\right) = \frac{2}{\pi}$

$$\Rightarrow \quad c = 0$$

$$\Rightarrow \quad f(x) = \frac{\sin x}{x}$$

$$\int_0^{\pi/2} f(x)dx = \int_0^{\pi/2} \frac{\sin x}{x}dx$$

Now $\frac{2}{\pi} < \frac{\sin x}{x} < 1$

$$\therefore \quad 1 < \int_0^{\pi/2} \frac{\sin x}{x}dx < \frac{\pi}{2}$$

$$\lim_{x\to 0}\frac{\cos x}{f(x)} = \lim_{x\to 0}\frac{x}{\tan x} = 1$$

For Problems 27 and 28

27. (4), 28. (3)

$$U_{n+2} - U_{n+1} = \int_0^{\pi} \frac{\left(1-\cos(n+2)x\right) - \left(1-\cos(n+1)x\right)}{1-\cos x}.dx$$

$$= \int_0^{\pi} \frac{\cos(n+1)x - \cos(n+2)x}{1-\cos x}$$

$$= \int_0^{\pi} \frac{2\sin\left(n+\frac{3}{2}\right)x.\sin\frac{x}{2}}{2\sin^2 x/2}dx$$

$$\Rightarrow \quad U_{n+2} - U_{n+1} = \int_0^{\pi} \frac{\sin\left(n+\frac{3}{2}\right)x}{\sin\frac{x}{2}}dx \qquad \ldots(1)$$

$$\Rightarrow \quad U_{n+1} - U_n = \int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}}dx \qquad \ldots(2)$$

from (1) and (2) we get;

$$(U_{n+2} - U_{n+1}) - (U_{n+1} - U_n)$$

$$= \int_0^{\pi} \frac{\sin\left(n+\frac{3}{2}\right)x - \sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}}dx$$

$$\Rightarrow \quad U_{n+2} + U_n - 2U_{n+1}$$

$$= \int \frac{2\cos(n+1)x.\sin x/2}{\sin x/2}dx$$

$$= 2\int_0^{\pi} \cos(n+1)x\,dx$$

$$= 2\left(\frac{\sin(n+1)x}{n+1}\right)_0^{\pi} = 0$$

$$\Rightarrow \quad U_{n+2} + U_n = 2U_{n+1}$$

$$\Rightarrow \quad U_n, U_{n+1}, U_{n+2} \text{ are in AP}$$

$$U_0 = \int_0^{\pi} \frac{1-1}{1-\cos x}dx = 0,$$

$$U_1 = \int_0^{\pi} \frac{1-\cos x}{1-\cos x}dx = \pi$$

$$U_1 - U_0 = \pi \qquad \text{(common difference)}$$

$$\therefore \quad U_n = U_0 + n\pi = n\pi$$

Now, $I_n = \int_0^{\pi/2} \frac{\sin^2 n\theta}{\sin^2\theta}d\theta$

$$= \int_0^{\pi/2} \frac{1-\cos 2n\theta}{1-\cos 2\theta}d\theta = \frac{1}{2}\int_0^{\pi} \frac{1-\cos nx}{1-\cos x}dx$$

$$\Rightarrow \quad I_n = \frac{1}{2}n\pi$$

For Problems 29-31

29. (1) $\int_0^{\pi/2} \sin x\,dx = \frac{\left(\frac{\pi}{2}-0\right)}{4}\left(\sin 0 + \sin\frac{\pi}{2} + 2\sin\frac{\pi}{4}\right)$

$= \frac{\pi}{8}(1+\sqrt{2})$

30. (4) $\lim_{x\to a} \frac{\int_a^x f(x)\,dx - \left(\frac{x-a}{2}\right)(f(x)+f(a))}{(x-a)^3} = 0$

or $\lim_{h\to 0} \frac{\int_a^{a+h} f(x)\,dx - \frac{h}{2}(f(a+h)+f(a))}{h^3} = 0$

or $\lim_{h\to 0} \frac{f(a+h) - \frac{1}{2}[f(a)+f(a+h)] - \frac{h}{2}(f'(a+h))}{3h^2} = 0$

[Using L' Hospital's rule]

or $\lim_{h\to 0} \frac{\frac{1}{2}f(a+h) - \frac{1}{2}f(a) - \frac{h}{2}f'(a+h)}{3h^2} = 0$

or $\lim_{h\to 0} \frac{\frac{1}{2}f'(a+h) - \frac{1}{2}f'(a+h) - \frac{h}{2}f''(a+h)}{6h} = 0$

[Using L' Hospital's rule]

or $\lim_{h\to 0} \frac{-f''(a+h)}{12} = 0$

or $f''(a) = 0\ \forall\ a \in R$

Thus, $f(x)$ must be of maximum degree 1.

31. (2) $f''(x) < 0\ \forall\ x \in (a, b)$, for $c \in (a, b)$

$$F(c) = \frac{c-a}{2}(f(a)+f(c)) + \frac{b-c}{2}(f(b)+f(c))$$

$$= \frac{b-a}{2}f(c) + \frac{c-a}{2}f(a) + \frac{b-c}{2}f(b)$$

or $F'(c) = \frac{b-a}{2}f'(c) + \frac{1}{2}f(a) - \frac{1}{2}f(b)$

$$= \frac{1}{2}[(b-a)f'(c) + f(a) - f(b)]$$

$$F''(c) = \frac{1}{2}(b-a)f''(c) < 0$$

$[\because f''(x) < 0\ \forall\ x \in (a, b)$ and $b > a]$

Therefore, $F(c)$ is maximum at the point $(c, f(c))$ where

$$F'(c) = 0 \text{ or } f'(c) = 2\left(\frac{f(b)-f(a)}{b-a}\right).$$

Matrix Match Type

1. a → s; b → s; c → r; d → q.

a. $\int_{-1}^{1}[x+[x+[x]]]dx$ (use property $[x+n] = [x] + n$ if n is integer)

$$= \int_{-1}^{1} 3[x]dx = 3\int_{-1}^{1}[x]dx = 3\int_0^1([x]+[-x])dx$$

$= -3$ (as $[x] + [-x] = -1$)

b. $\int_2^5([x]+[-x])dx = \int_2^5 -1\,dx = -3$

c. $\text{sgn}(x-[x]) = \begin{cases}1, & \text{if } x \text{ is not an integer}\\ 0, & \text{if } x \text{ is an integer}\end{cases}$

Hence, $\int_{-1}^{3}\text{sgn}(x-[x])dx = 4(1-0) = 4.$

d. Let $I = 25\int_0^{\pi/4}\left(\tan^6(x-[x]) + \tan^4(x-[x])\right)dx$

$\left\{\because 0 < x \le \frac{\pi}{4} \Rightarrow [x] = 0\right\}$

$= 25\int_0^{\pi/4}(\tan^6 x + \tan^4 x)\,dx$

$= 25\int_0^{\pi/4}\tan^4 x(\tan^2 x + 1)dx$

$= 25\int_0^{\pi/4}\tan^4 x\sec^2 x\,dx$

$= 25\left(\frac{\tan^5 x}{5}\right)_0^{\pi/4} = 25\times\frac{1}{5} = 5$

2. a → r; b → p; c → s; d → q.

a. $\lim_{n\to\infty}\left[\frac{\int_0^2\left(1+\frac{t}{n+1}\right)^n dt}{n+1}\right] = \lim_{n\to\infty}\left[\left(1+\frac{t}{n+1}\right)^{n+1}\right]_0^2$

$= \lim_{n\to\infty}\left(1+\frac{2}{n+1}\right)^{n+1} - 1$

$= e^2 - 1$

b. $f'(x) = f(x) \Rightarrow f(x) = Ce^x$ and since $f(0) = 1$,

$1 = f(0) = C$

Therefore, $f(x) = e^x$ and, hence, $g(x) = x^2 - e^x$.

Thus, $\int_0^1 f(x)g(x)dx = \int_0^1(x^2e^x - e^{2x})dx$

$= x^2e^x\Big|_0^1 - 2\int_0^1 xe^x dx - \frac{e^{2x}}{2}\Big|_0^1$

$= (e-0) - 2xe^x\Big|_0^1 + 2e^x\Big|_0^1 - \frac{1}{2}(e^2-1)$

$= (e-0) - 2e + 2e - 2 - \frac{1}{2}(e^2-1)$

$= e - \frac{1}{2}e^2 - \frac{3}{2}$

c. $I = \int_0^1 e^{e^x}(1+xe^x)dx$

Let $e^x = t$

$\therefore \int_1^e e^t(1+t\log t)\frac{dt}{t} = \int_1^e e^t\left(\frac{1}{t}+\log t\right)dt$

$= \left[e^t\log t\right]_1^e = e^e$

d. $L = \lim_{k\to 0}\frac{\int_0^k(1+\sin 2x)^{\frac{1}{x}}dx}{k}$ $\left(\text{form } \frac{0}{0}\right)$

$= \lim_{k\to 0}(1+\sin 2k)^{\frac{1}{k}}$

$= e^{\lim_{k\to 0}\frac{1}{k}(\sin 2k)} = e^2$

3. **a → q; b → r, s; c → p; d → p.**

a. $I_1 = \int_{\pi/6}^{\pi/3} \sec^2\theta f(2\sin 2\theta)\,d\theta$

Applying property $\int_a^b f(a+b-x)\,dx = \int_a^b f(x)\,dx$,

$$I_1 = \int_{\pi/6}^{\pi/3} \sec^2\left(\frac{\pi}{2}-\theta\right) f\left(2\sin 2\left(\frac{\pi}{2}-\theta\right)\right)d\theta$$

$$= \int_{\pi/6}^{\pi/3} \operatorname{cosec}^2\theta f(2\sin 2\theta)\,d\theta = I_2$$

b. $f(x+1) = f(x+3)$ or $f(x) = f(x+2)$

Thus, $f(x)$ is periodic with period 2.

Then $\int_a^{a+b} f(x)\,dx$ is independent of a, for which b is multiple of 2.

Thus, $b = 2, 4, 6, \ldots$

c. Let $I = \int_1^4 \frac{\tan^{-1}[x^2]}{\tan^{-1}[x^2]+\tan^{-1}[25+x^2-10x]}$ (1)

Applying $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$, we get

$$I = \int_1^4 \frac{\tan[(5-x)^2]}{\tan^{-1}[(5-x)^2]+\tan^{-1}[x^2]}\,dx \quad (2)$$

Adding equations (1) and (2), we get

$$2I = \int_1^4 dx \text{ or } 2I = 3 \text{ or } I = 3/2$$

d. Let $y = \sqrt{x+\sqrt{x+\sqrt{x+\cdots}}} = \sqrt{x+y}$

or $y^2 - y - x = 0$

or $y = \frac{1 \pm \sqrt{1+4x}}{2.1} = \frac{1+\sqrt{1+4x}}{2}$ $(\because y > 1)$

$$\therefore \quad I = \int_0^2 \frac{1+\sqrt{1+4x}}{2}\,dx$$

$$= \left[\frac{x}{2} + \frac{(1+4x)^{3/2}}{\frac{3}{2}\times 2\times 4}\right]_0^2$$

$$= \left[\left(1+\frac{27}{12}\right) - \left(0+\frac{1}{12}\right)\right]$$

$$= 1 + \frac{26}{12} = \frac{19}{6}$$

$$\therefore \quad [I] = 3$$

4. **a → p; q; b → p, q; r. c → q, s; d → s.**

a. $I = \int_{-2}^{2} (\alpha x^3 + \beta x + \gamma)\,dx$

$\alpha x^3 + \beta x$ is an odd function

$I = 0 + 2\int_0^2 \gamma\,dx = 2\times 2\gamma = 4\gamma$

b. $I = \frac{1}{2}\int_0^1 2\sin\alpha x \sin\beta x\,dx$

$$= \frac{1}{2}\int_0^1 (\cos(\alpha-\beta)x - \cos(\alpha+\beta)x)\,dx$$

$$= \frac{1}{2}\left[\frac{\sin(\alpha-\beta)x}{\alpha-\beta} - \frac{\sin(\alpha+\beta)x}{\alpha+\beta}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{\sin(\alpha-\beta)}{\alpha-\beta} - \frac{\sin(\alpha+\beta)}{\alpha+\beta}\right] \quad (1)$$

Also, $2\alpha = \tan\alpha$ and $2\beta = \tan\beta$

$\therefore \quad 2(\alpha-\beta) = \tan\alpha - \tan\beta$ and $2(\alpha+\beta) = \tan\alpha + \tan\beta$

$$2(\alpha-\beta) = \frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta} \text{ and } 2(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$$

Substituting these values, we get

$I = (\cos\alpha\cos\beta) - (\cos\alpha\cos\beta) = 0.$

c. $f(x+\alpha) + f(x) = 0$

or $f(x+2\alpha) + f(x+\alpha) = 0$

or $f(x+2\alpha) = f(x)$

Thus, $f(x)$ is periodic with period 2α. Hence,

$$\int_\beta^{\beta+2\gamma\alpha} f(x)\,dx = \gamma\int_0^{2\alpha} f(x)\,dx$$

d. Let $I = \int_0^\alpha [\sin x]\,dx$, $\alpha \in [(2\beta+1)\pi, (2\beta+2)\pi]$, $\beta \in N$,

(where $[\cdot]$ denotes the greatest integer function.)

$$= \int_0^{2\beta\pi} [\sin x]\,dx + \int_{2\beta\pi}^{(2\beta+1)\pi} [\sin x]\,dx + \int_{(2\beta+1)\pi}^{\alpha} [\sin x]\,dx$$

$$= \beta\int_0^{2\pi} [\sin x]\,dx + 0 + \int_{(2\beta+1)\pi}^{\alpha} (-1)\,dx$$

$$= -\beta\pi + (2\beta+1)\pi - \alpha$$

$$= (\beta+1)\pi - \alpha$$

Thus, $\gamma\int_0^\alpha [\sin x]\,dx$ depends on α, β, and γ.

5. **(4)** **a.** $I = \int_0^\infty \frac{\sin 5x}{x}\,dx$

Let $5x = t$

$$\therefore \quad dx = \frac{1}{5}\,dt$$

$$\therefore \quad I = \int_0^\infty \frac{\sin(t)}{t/5}\cdot\frac{dt}{5} = \int_0^\infty \frac{\sin t}{t}\,dt = \alpha$$

b. $I = \int_0^\infty \frac{\sin^2 x}{x^2}\,dx$

$$= \int_0^\infty (\sin^2 x)\left(\frac{1}{x^2}\right)dx$$

$$= \left[\sin^2 x\left(\frac{-1}{x}\right)\right]_0^\infty - \int_0^\infty (2\sin x\cos x)\times\left(\frac{-1}{x}\right)dx$$

$$= \left[\frac{-\sin^2 x}{x}\right]_0^\infty + \int_0^\infty \frac{\sin 2x}{x}\,dx \text{ put } 2x = t$$

$$= \lim_{x\to\infty}\left[-\frac{\sin^2 x}{x}\right] - \lim_{x\to 0}\left[-\frac{\sin^2 x}{x}\right] + \alpha$$

$$= 0 - 0 + \alpha = \alpha$$

c. $I=\int_0^{\infty}\frac{\sin^3 x}{x}dx$

$$I=\frac{1}{4}\left[3\int_0^{\infty}\frac{\sin x}{x}dx-\int_0^{\infty}\frac{\sin 3x}{x}dx\right]$$

$$=\frac{1}{4}[3\alpha-\alpha]=\frac{\alpha}{2}$$

d. $\int_0^{\infty}\frac{\sin k_1x\cos k_2x}{x}dx$

$$=\frac{1}{2}\int_0^{\infty}\frac{2\sin k_1x\cos k_2x}{x}dx$$

$$=\frac{1}{2}\int_0^{\infty}\frac{\sin(k_1+k_2)x+\sin(k_1-k_2)x}{x}dx$$

$$=\frac{1}{2}(\alpha+\alpha)=\alpha$$

$$\therefore\quad\int_0^{\infty}\frac{\sin(k_1x).\cos k_2x}{x}dx-\alpha=0$$

6. (2) $I(m)=\int_0^{\pi}\log_e\left(1-2m\cos x+m^2\right)dx$

$$\Rightarrow\quad I(-m)=\int_0^{\pi}\log_e\left(1+2m\cos x+m^2\right)dx$$

$$=\int_0^{\pi}\log_e\left(1+2m\cos(\pi-x)+m^2\right)dx$$

$$=\int_0^{\pi}\log_e\left(1-2m\cos x+m^2\right)dx$$

$$=I(m)$$

$$\Rightarrow\quad I(-m)=I(m)$$

$$\Rightarrow\quad I(m)+I(-m)=\int_0^{\pi}\log_e\left(1-2m^2\cos 2x+m^4\right)dx$$

Put $2x=t$

$\Rightarrow\quad I(m)+(-m)=I(m^2)$

$\Rightarrow\quad 2I(m)=I(m^2)$

$$\Rightarrow\quad\frac{I(m^2)}{I(m)}=2$$

$$\Rightarrow\quad\frac{I(9)}{I(3)}=2\text{ and }\frac{I(25)}{I(5)}=2$$

$$\Rightarrow\quad\frac{I(81)}{I(9)}=2\text{ and }\frac{I(9)}{I(3)}=2$$

$$\Rightarrow\quad\frac{I(81)}{I(3)}=4$$

Numerical Value Type

1. (96) $I=\lim_{n\to\infty}\frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{6n}}{n\sqrt{n}}$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{6n}\sqrt{\frac{r}{n}}=\int_0^6\sqrt{x}\,dx=\left[\frac{2}{3}x^{3/2}\right]_0^6=\frac{2}{3}\cdot 6\sqrt{6}=\sqrt{96}$$

2. (2) $\lim_{n\to\infty}\frac{n}{2^n}\cdot\frac{x^{n+1}}{n+1}\Big]_0^2=\lim_{n\to\infty}\frac{n}{2^n}\cdot\frac{2^{n+1}}{n+1}=\lim_{n\to 0}\frac{2}{1+(1/n)}=2$

3. (5) We have $f(2x)=3f(x)$ (1)

and $\int_0^1 f(x)dx=1$ (2)

From equations (1) and (2), $\frac{1}{3}\int_0^1 f(2x)dx=1$

Put $2x=t$. Then $\frac{1}{6}\int_0^2 f(t)dt=1$

or $\int_0^2 f(t)dt=6$

or $\int_0^1 f(t)dt+\int_1^2 f(t)dt=6$

Hence, $\int_1^2 f(t)dt=6-\int_0^1 f(t)dt=6-1=5.$

4. (2). $\int_0^2|f'(x)|dx\geq\left|\int_0^2 f'(x)dx\right|$

or $\int_0^2|f'(x)|dx\geq|f(2)|=2$

5. (2) $I=\int_0^{3\pi/4}(\sin x+\cos x)dx+\int_0^{3\pi/4}\underbrace{x}_{\text{I}}\underbrace{(\sin x-\cos x)}_{\text{II}}dx$

$$=\int_0^{3\pi/4}(\sin x+\cos x)dx+\underbrace{x(-\cos x-\sin x)\Big|_0^{3\pi/4}}_{\text{zero}}+\int_0^{3\pi/4}(\sin x+\cos x)dx$$

$$=2\int_0^{3\pi/4}(\sin x+\cos x)dx=2(\sqrt{2}+1)$$

6. (7) $\sum_{r=1}^{100}\left(\int_0^1 f(r-1+x)dx\right)$

$$=\int_0^1 f(x)dx+\int_0^1 f(1+x)dx+\int_0^1 f(2+x)dx+\cdots+\int_0^1 f(99+x)dx$$

$$=\int_0^1 f(x)dx+\int_1^2 f(x)dx+\int_2^3 f(x)dx+\cdots+\int_{99}^{100} f(x)dx$$

$$=\int_0^{100} f(x)dx=7$$

7. (0) $\because$ Integrand is discontinuous at $\frac{\pi}{2}$, $\int_0^{\pi/2}0\cdot dx+\int_{\pi/2}^{3\pi/2}0\cdot dx=0$

$\because\quad 0<x<\frac{\pi}{2}$, $|\tan^{-1}\tan x|$

$=|\sin^{-1}\sin x|$ and $\frac{\pi}{2}<x<\frac{3\pi}{2}$ $|\tan^{-1}\tan x|=|\sin^{-1}\sin x|$

8. (4) Given $f(x)=x^3-\frac{3x^2}{2}+x+\frac{1}{4}=\frac{1}{4}(4x^3-6x^2+4x+1)$

$$=\frac{1}{4}(4x^3-6x^2+4x-1+2)$$

$$=\frac{1}{4}[x^4-(1-x)^4]+\frac{2}{4}$$

$$\therefore\quad f(1-x)=\frac{1}{4}[(1-x)^4-x^4]+\frac{2}{4}$$

$$\therefore\quad f(x)+f(1-x)=\frac{2}{4}+\frac{2}{4}=1$$ (1)

Replacing x by $f(x)$, we have

$$f[f(x)] + f[1 - f(x)] = 1 \quad (2)$$

Now, $I = \int_{1/4}^{3/4} f(f(x))dx$ (3)

Also, $I = \int_{1/4}^{3/4} f(f(1-x))dx = \int_{1/4}^{3/4} f(1-f(x))dx$ [using (1)] (4)

Adding (3) and (4), we get

$$2I = \int_{1/4}^{3/4} [f(f(x)) + f(1-f(x))]dx = \int_{1/4}^{3/4} dx = \frac{1}{2}$$

or $I = \frac{1}{4}$

$\therefore \quad I^{-1} = 4$

9. (0.5) $I = \int_0^1 \frac{\tan^{-1} x}{\cot^{-1}(1-x+x^2)}dx$

$$= \int_0^1 \frac{\tan^{-1} x}{\tan^{-1}\frac{(1-x)+x)}{1-(1-x)x}}dx$$

$$= \int_0^1 \frac{\tan^{-1} x}{\tan^{-1} x + \tan^{-1}(1-x)}dx \quad \ldots(1)$$

$$= \int_0^1 \frac{\tan^{-1}(1-x)}{\tan^{-1}(1-x) + \tan^{-1} x}dx \quad \ldots(2)$$

Adding (1) and (2), we get

$$2I = \int_0^1 1dx = 1$$

$\Rightarrow \quad I = \frac{1}{2}$

10. (0) $f(x)$ is symmetrical about line $x = 2$

$\therefore \quad f(2-x) = f(2+x)$

Replacing x by 2 – x, we get

$$f(x) = f(4-x)$$

$\therefore \quad f'(x) = -f'(4-x)$

$\therefore \quad I = \int_0^4 \cos(\pi x) f'(x)\,dx$

$$= \int_0^4 \cos(4\pi - \pi x) f'(4-x)dx$$

$$= -I$$

$\Rightarrow \quad 2I = 0$

$\Rightarrow \quad I = 0$

11. (6) Given $f^3(x) = \int_0^x t \cdot f^2(t)\,dt$

Differentiating, $3f^2(x)f'(x) = xf^2(x)$

$f(x) \neq 0$

$\therefore \quad f'(x) = \frac{x}{3} \quad \therefore \quad f(x) = \frac{x^2}{6} + C$

But $f(0) = 0 \Rightarrow C = 0$

$f(6) = 6$

12. (7) $F'(x) = (2x+3)\int_x^2 f(u)du$

$$\therefore \quad F''(x) = -(2x+3)f(x) + \left(\int_x^2 f(u)du\right)\cdot 2$$

$$F''(2) = -7f(2) + 0$$

13. (108) $I = \int_0^1 \frac{\sin^{-1}\sqrt{x}}{x^2 - x + 1}dx$ (1)

$$I = \int_0^1 \frac{\sin^{-1}\sqrt{1-x}}{x^2-x+1}dx = \int_0^1 \frac{\cos^{-1}\sqrt{x}}{x^2-x+1}dx \quad (2)$$

On adding (1) and (2), we get

$$2I = \int_0^1 \frac{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}{x^2-x+1}dx$$

$$= \frac{\pi}{2}\int_0^1 \frac{dx}{x^2-x+1}dx$$

$$= \frac{\pi}{2}\int_0^1 \frac{dx}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}dx$$

$$\therefore \quad 2I = \frac{\pi}{2}\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\left[\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)\right]_0^1 = \frac{\pi^2}{3\sqrt{3}}$$

Hence, $I = \frac{\pi^2}{6\sqrt{3}} = \frac{\pi^2}{\sqrt{108}} \equiv \frac{\pi^2}{\sqrt{n}}$.

14. (6) $y = f(x) \Rightarrow x = f^{-1}(y) \Rightarrow x = g(y)$

Given $y = f(x) = \int_0^x \frac{dt}{\sqrt{1+t^3}}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^3}} \text{ or } \frac{dx}{dy} = \sqrt{1+x^3}$$

$$g'(y) = \sqrt{1+g^3(y)}$$

$$g''(y) = \frac{3g^2(y)g'(y)}{2\sqrt{1+g^3(y)}}$$

$$\therefore \quad 2g''(y) = 3g^2(y)\frac{g'(y)}{\sqrt{1+g^3(y)}} = 3g^2(y)\frac{\sqrt{1+g^3(y)}}{\sqrt{1+g^3(y)}} = 3g^2(y)$$

or $2g''(y) = 3g^2(y)$

15. (2) We have $\int_{\sin t}^1 x^2 g(x)dx = (1 - \sin t)$ (1)

Differentiating both the sides of (1) with respect to t, we get

$0 - (\sin^2 t)\, g(\sin t)\, (\cos t) = -\cos t$

or $g(\sin t) = \frac{1}{\sin^2 t}$ (2)

Putting $t = \frac{\pi}{4}$ in (2), we get $g\left(\frac{1}{\sqrt{2}}\right) = 2$.

16. (6) $I=\int_0^{\infty}(x^2)^n\cdot x\,e^{-x^2}\,dx$

Put $x^2=t$ and $x\,dx=dt/2$

$\therefore \quad I=\frac{1}{2}\int_0^{\infty}t^n\,e^{-t}\,dt$

$=\frac{1}{2}\left[-t^n\,e^{-t}\right]_0^{\infty}+n\int_0^{\infty}t^{n-1}\,e^{-t}\,dt\Big]$

$=\frac{1}{2}\left[0+n\int_0^{\infty}t^{n-1}\,e^{-t}\,dt\right]$

$=\frac{n!}{2}=360$

$\Rightarrow \quad n=6$

17. (3) $f(x)\int_0^x e^t\sin(x-t)dt=\int_0^x e^{x-t}\sin(x-(x-t))dt=e^x\int_0^x e^{-t}\sin t\,dt$

$\therefore \quad f'(x)=e^xe^{-x}\sin x+e^x\int_0^x e^{-t}\sin t\,dt$

$=\sin x+e^x\int_0^x e^{-t}\sin t\,dt$

$\therefore \quad f''(x)=\cos x+e^xe^{-x}\sin x+e^x\int_0^x e^{-t}\sin t\,dt$

$=\cos x+\sin x+f(x)$

$\therefore \quad f''(x)-f(x)=\cos x+\sin x$

Range of $g(x)=f''(x)-f(x)$ is $[-\sqrt{2},\sqrt{2}]$.

Number of integers in the range is 3.

18. (32) $\frac{d}{dx}\int_4^x[4t^2-2F'(t)]dt=[4x^2-2F'(x)]\cdot 1-0$

or $\quad F'(x)=\frac{1}{x^2}[4x^2-2F'(x)]+\frac{-2}{x^3}\int_4^x[4t^2-2F'(t)]dt$

or $\quad F'(4)=\frac{1}{16}[64-2F'(4)]-\frac{1}{32}\int_4^4 g(x)dx$

or $\quad \left(1+\frac{1}{8}\right)F'(4)=4$

or $\quad F'(4)=\frac{32}{9}$

19. (208) Let $I=\int_0^1{}^{207}C_7\cdot\underbrace{x^{200}}_{\text{II}}\cdot\underbrace{(1-x)^7}_{\text{I}}\,dx$

$={}^{207}C_7\left[(1-x)^7\cdot\frac{x^{201}}{201}\Big|_0^1+\frac{7}{201}\int_0^1(1-x)^6\cdot x^{201}\,dx\right]$

$={}^{207}C_7\cdot\frac{7}{201}\int_0^1(1-x)^6\cdot x^{201}\,dx$

Integrating by parts again six times more, we get

$$I={}^{207}C_7\cdot\frac{7!}{201\cdot202\cdot203\cdot204\cdot205\cdot206\cdot207}\int_0^1x^{207}\,dx$$

$$=\frac{(207)!}{7!(200)!}\cdot\frac{7!}{201\cdot202\cdots207}\cdot\frac{1}{208}$$

$$=\frac{(207)!}{(207)!\,7!}\cdot\frac{7!}{208}=\frac{1}{208}=\frac{1}{k}$$

or $\quad k=208$

20. (8) $I_{11}=\int_0^1\underbrace{(1-x^5)^{11}}_{\text{I}}\cdot\underbrace{1}_{\text{II}}\,dx$

$=(1-x^5)^{11}\cdot x\Big]_0^1+11\int_0^1(1-x^5)^{10}5x^4\cdot x\,dx$

$=0-55\int_0^1(1-x^5)^{10}(1-x^5-1)\,dx$

$=-55\int_0^1(1-x^5)^{11}\,dx+55I_{10}$

or $\quad 56I_{11}=55I_{10}$

or $\quad \frac{I_{10}}{I_{11}}=\frac{56}{55}$

21. (5051)

$\int_0^1(1-x^{50})^{101}\,dx$

$=\left[x(1-x^{50})^{101}\right]_0^1-\int_0^1x\cdot101(1-x^{50})^{100}(-50x^{49})dx$

$=0+5050\int_0^1x^{50}(1-x^{50})^{100}\,dx$

$=-5050\int_0^1(1-x^{50}-1)(1-x^{50})^{100}\,dx$

$=-5050\int_0^1(1-x^{50})^{101}\,dx+5050\int_0^1(1-x^{50})^{100}\,dx$

$\Rightarrow \quad 5051\int_0^1(1-x^{50})^{101}\,dx=5050\int_0^1(1-x^{50})^{100}\,dx$

$\Rightarrow \quad 5050\frac{\int_0^1(1-x^{50})^{100}\,dx}{\int_0^1(1-x^{50})^{101}\,dx}=5051$

22. (0) We have $J=\int_{-5}^{-4}(3-x^2)\tan(3-x^2)dx$

Put $(x+5)=t$. Then

$J=\int_0^1(3-(t-5)^2)\tan(3-(t-5)^2)\,dt$

$=\int_0^1(-22+10t-t^2)\tan(-22+10t-t^2)dt$

Now, $K=\int_{-2}^{-1}(6-6x+x^2)\tan(6x-x^2-6)\,dx.$

Put $(x+2)=z$. Then

$K=\int_0^1(6-6(z-2)+(z-2)^2)\tan(6(z-2)-(z-2)^2-6)\,dz$

$=\int_0^1(22-10z+z^2)\tan(-22+10z-z^2)dz$

Hence, $(J+K)=0$.

23. (2) $I=\int_{\sqrt{2}-1}^{\sqrt{2}+1}\frac{(x^2+1)^2-(x^2-1)}{(x^2+1)^2}dx=\int_{\sqrt{2}-1}^{\sqrt{2}+1}\left(1-\frac{(x^2-1)}{(x^2+1)^2}\right)dx$

$$=2-\underbrace{\int_{\sqrt{2}-1}^{\sqrt{2}+1}\frac{(x^2-1)}{(x^2+1)^2}dx}_{I_1}$$

$$I_1=\int_{1/a}^{a}\frac{(x^2-1)}{(x^2+1)^2}dx, \text{ where } a=\sqrt{2}+1$$

Put $x=\frac{1}{t}$ and $dx=-\frac{1}{t^2}dt$

or $I_1=\int_{a}^{1/a}\frac{\frac{1}{t^2}-1}{\left(\frac{1}{t^2}+1\right)^2}\cdot\left(-\frac{1}{t^2}\right)dt=-\int_{a}^{1/a}\frac{(1-t^2)t^4}{t^4(1+t^2)^2}dt$

$$=-\int_{a}^{1/a}\frac{(1-t^2)}{(1+t^2)^2}dt=\int_{a}^{1/a}\frac{t^2-1}{(t^2+1)^2}dt$$

$$=-\int_{1/a}^{a}\frac{t^2-1}{(t^2+1)^2}dt=-I_1$$

or $2I_1=0$

or $I_1=0$

$\therefore$ $I=2$

24. (3) We have

$$f(x)=\sin x+\int_{-\pi/2}^{\pi/2}(\sin x+t f(t))dt=\sin x+\pi\sin x+\int_{-\pi/2}^{\pi/2}t f(t)dt$$

$\therefore$ $f(x)=(\pi+1)\sin x+A$ (1)

Now, $A=\int_{-\pi/2}^{\pi/2}t\left((\pi+1)\sin t+A\right)dt=2(\pi+1)\left(\underbrace{\int_{0}^{\pi/2}t\sin t\,dt}_{\text{(I) (II) (By part)}}\right)$

$\therefore$ $A=2(\pi+1)$

Hence, $f(x)=(\pi+1)\sin x+2(\pi+1)$.

Therefore, $f_{\max.}=3(\pi+1)=M$

and $f_{\min.}=(\pi+1)=m$.

$\therefore$ $\frac{M}{m}=3$

25. (9) $f(x)=x+x\int_0^1 t f(t)dt+\int_0^1 t^2 f(t)dt$

$\therefore$ $f(x)=x(1+A)+B$, where

$A=\int_0^1 t f(t)dt$ and $B=\int_0^1 t^2 f(t)dt$

Now, $A=\int_0^1 t\left[t(1+A)+B\right]dt=\frac{t^3}{3}(1+A)\Big|_0^1+\frac{B}{2}t^2\Big|_0^1=\frac{1+A}{3}+\frac{B}{2}$ (1)

or $4A-3B=2$

Again, $B=\int_0^1 t^2\left[t(1+A)+B\right]dt=\frac{t^4(1+A)}{4}+\frac{Bt^3}{3}\Big]_0^1$

$$=\frac{1+A}{4}+\frac{B}{3}$$

or $8B-3A=3$ (2)

Solving equations (1) and (2), we have $B=\frac{18}{23}=f(0)$.

26. (56) $f(1)=0$ and $f(2)=30$

$\therefore$ $\int_0^2 f(x)dx+\int_0^{30}f^{-1}(y)dy$

$$=\int_0^1 f(x)dx+\int_1^2 f(x)dx+\int_0^{30}f^{-1}(y)dy$$

$$=\int_0^1(4x^3+2x-6)dx+\int_1^2 f(x)dx+\int_1^2 tf'(t)dt$$

$$=[x^4+x^2-6x]_0^1+\int_1^2 f(x)dx+[tf(t)]_1^2-\int_1^2 f(t)dt$$

$$=-4+2f(2)-f(1)$$

$$=-4+2\times 30-f(1)$$

$$=56$$

27. (8) $I=\int_1^3\sqrt{1+(x-1)^3}dx+\int_1^3\left((x^2-1)^{\frac{1}{3}}+1\right)dx$

$$=\int_1^3 f(x)dx+\int_1^3 f^{-1}(x)dx$$

In second integral put $f^{-1}(x)=y$

$\therefore$ $x=f(y)$

$\therefore$ $dx=f'(y)$

Also, if $1=\sqrt{1+(x-1)^3}$, $x=1$

and if $3=\sqrt{1+(x-1)^3}$, $x=3$

$\therefore$ $I=\int_1^3 f(x)dx+\int_1^3 yf'(y)dy$

$$=\int_1^3 f(x)dx+[yf(y)]_1^3-\int_1^3 f(y)dy$$

$$=3\times f(3)-1\times f(1)$$

$$=9-1$$

$$=8$$

28. (2) $I=\int_0^1\cos^{-1}\left((x-x^2)-\sqrt{(1-x^2)(2x-x^2)}\right)dx$

$$=\int_0^1\cos^{-1}\left(x(1-x)-\left(\sqrt{1-x^2}\right)\left(\sqrt{1-(1-x)^2}\right)\right)dx$$

$$=\int_0^1\left(\cos^{-1}(x)+\cos^{-1}(1-x)\right)dx$$

$$=2\int_0^1\cos^{-1}x\,dx$$

Put $x=\cos\theta$

$\therefore$ $I=2\int_0^{\pi/2}\theta\sin\theta\,d\theta$

$$=2[-\theta\cos\theta+\sin\theta]_0^{\pi/2}$$

$$=2$$

Archives

JEE MAIN

Single Correct Answer Type

1. (4) Let $I = \int_0^{\pi} [\cot x]\, dx$ (i)

$= \int_0^{\pi} [\cot(\pi - x)]\, dx = \int_0^{\pi} [-\cot x]\, dx$ (ii)

Adding (i) and (ii), we get

$2I = \int_0^{\pi} [\cot x]\, dx + \int_0^{\pi} [-\cot x]\, dx = \int_0^{\pi} (-1)\, dx$

[Since $[x] + [-x]$ is equal to -1 if $x \notin Z$, and is equal to 0 if $x \in Z$]

$= [-x]_0^{\pi} = -\pi$

$\therefore \quad I = -\frac{\pi}{2}$

2. (3) $p'(x) = p'(1-x)$

$\Rightarrow \quad p(x) = -p(1-x) + c$

At $x = 0$,

$p(0) = -p(1) + c \quad \Rightarrow \quad 42 = c$

$\therefore \quad p(x) = -p(1-x) + 42$

$\Rightarrow \quad p(x) + p(1-x) = 42$

$I = \int_0^1 p(x)\, dx = \int_0^1 p(1-x)\, dx$

$\therefore \quad 2I = \int_0^1 (p(x) + p(1-x))dx = \int_0^1 (42)\, dx$

$\Rightarrow \quad I = 21$

3. (2) We have

$$I = \int_0^1 \frac{8\log(1+x)}{1+x^2} dx$$

Put $x = \tan\theta$

$$\Rightarrow \quad I = \int_0^{\frac{\pi}{4}} 8 \cdot \frac{\log(1+\tan\theta)}{\sec^2\theta} \sec^2\theta\, d\theta$$

$$= 8\int_0^{\pi/4} \log(1+\tan\theta) d\theta$$

$$= 8\int_0^{\pi/4} \log(1+\tan(\pi/4-\theta))\, d\theta$$

$$= 8\int_0^{\pi/4} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right) d\theta$$

$$= 8\int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta$$

$$= 8(\log 2)\cdot\frac{\pi}{4} - I$$

$\Rightarrow \quad 2I = 2\pi\log 2$

$\Rightarrow \quad I = \pi\log 2$

4. (1) We have

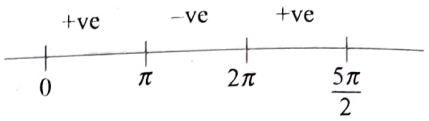
$$f(x) = \int_0^x \sqrt{t}\sin t\, dt$$

$\Rightarrow \quad f'(x) = \sqrt{x}\sin x$

For maximum of minimum value of $f(x)$, $f'(x) = 0$

$\Rightarrow \quad x = 2n\pi,\ n \in Z$

Sign scheme of $f'(x)$ for $x \in \left(0, \frac{5\pi}{2}\right)$ is



$f'(x)$ changes its sign from +ve to −ve in the neighborhood of π and from − to + in the neighborhood of 2π.

Hence $f(x)$ has local maximum at $x = \pi$ and local minima at $x = 2\pi$.

5. (2) $g(x+\pi) = \int_0^{x+\pi} \cos^4 t\, dt$

$= \int_0^x \cos^4 t\, dt + \int_x^{x+\pi} \cos^4 t\, dt$ ($\because \cos^4 t$ has period π)

$= g(x) + \int_0^{\pi} \cos^4 t\, dt$

$= g(x) + g(\pi)$

6. (4) $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$ (i)

$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}} = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}\, dx}{\sqrt{\tan x}+1}$ (ii)

Adding (i) and (ii)

$\Rightarrow \quad 2I = \int_{\pi/6}^{\pi/3} 1\, dx \quad \Rightarrow \quad 2I = \frac{\pi}{3} - \frac{\pi}{6}$

$\Rightarrow \quad 2I = \frac{\pi}{6} \quad \Rightarrow \quad I = \frac{\pi}{12}$

7. (1) $y = \int_0^x |t|\, dt$

Case I : If $x > 0$

$$y = \int_0^x t\, dt = \left[\frac{t^2}{2}\right]_0^x = \frac{x^2}{2} \Rightarrow \frac{dy}{dx} = x = 2 \text{ (given)}$$

$\Rightarrow \quad x = 2$ and $y = 2$

$\therefore \quad$ equation of tangent is $(y-2) = 2(x-2)$

or $\quad y - 2x + 2 = 0$

Hence, x intercept $= 1$.

Case II : $x < 0$

$$y = \int_0^x -t\, dt = \left[\frac{-t^2}{2}\right]_0^x = \frac{-x^2}{2}$$

$\therefore \quad \frac{dy}{dx} = -x = 2$

$\therefore \quad x = -2, \quad \therefore y = -2,$

$\therefore$ equation of tangent is $y+2=2(x+2)$

or $2x-y+2=0$

$\therefore$ x intercept $=-1$

8. (4) $I=\int_0^{\pi}\sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,dx$

$=\int_0^{\pi}\left|1-2\sin\frac{x}{2}\right|dx$

$=\int_0^{\pi/3}\left(1-2\sin\frac{x}{2}\right)dx+\int_{\pi/3}^{\pi}\left(2\sin\frac{x}{2}-1\right)dx$

$=\left(x+4\cos\frac{x}{2}\right)\Big|_0^{\pi/3}+\left(-4\cos\frac{x}{2}-x\right)\Big|_{\pi/3}^{\pi}$

$=4\sqrt{3}-4-\frac{\pi}{3}$

9. (3) $I=\int_2^4\frac{\log x^2}{\log x^2+\log(36-12x+x^2)}dx$

$I=\frac{2}{2}\int_2^4\frac{\log|x|}{\log|x|+\log|6-x|}dx$ (i)

$I=\int_2^4\frac{\log|6-x|}{\log|6-x|+\log|x|}dx$ $\left\{\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx\right\}$ (ii)

Adding (i) and (ii)

$$2I=\int_2^4\frac{\log|x|+\log|6-x|}{\log|x|+\log|6-x|}dx=\int_2^4 dx=2$$

Hence, $I=1$

10. (1) $L=\lim_{n\to\infty}\left(\frac{(n+1)(n+2)\ldots\ldots(n+2n)}{n^{2n}}\right)^{1/n}$

$\therefore$ $\log_e L=\frac{1}{n}\left(\lim_{n\to\infty}\sum_{r=1}^{2n}\log\left(1+\frac{r}{n}\right)\right)$

$\therefore$ $\log_e L=\int_0^2\log(1+x)\,dx$

$\therefore$ $\log_e L=(x\log(1+x))_0^2-\int_0^2\frac{x}{1+x}dx$

$\therefore$ $\log_e L=2\log_e 3-\int_0^2\left(1-\frac{1}{1+x}\right)dx$

$\therefore$ $\log_e L=2\log 3-(x-\log(1+x))_0^2$

$\therefore$ $\log_e L=2\log 3-(2-\log 3)$

$\therefore$ $\log_e L=3\log 3-2=\log\frac{27}{e^2}$

$\therefore$ $L=\frac{27}{e^2}$

11. (3) $I=\int_{\pi/4}^{3\pi/4}\frac{dx}{1+\cos x}$ ...(i)

$\Rightarrow$ $I=\int_{\pi/4}^{3\pi/4}\frac{dx}{1-\cos x}$ ...(ii)

Adding (i) and (ii), we get

$$2I=\int_{\pi/4}^{3\pi/4}\frac{2}{\sin^2 x}dx$$

$\Rightarrow$ $I=\int_{\pi/4}^{3\pi/4}\operatorname{cosec}^2 dx$

$\Rightarrow$ $I=-(\cot x)_{\pi/4}^{3\pi/4}=2$

12. (1) $I=\int_{-\pi/2}^{\pi/2}\frac{\sin^2 x}{1+2^x}dx$ ...(i)

or $I=\int_{-\pi/2}^{\pi/2}\frac{\sin^2(-x)}{1+2^{-x}}dx$

or $I=\int_{-\pi/2}^{\pi/2}\frac{2^x\cdot\sin^2 x}{2^x+1}dx$ ...(ii)

Adding (i) and (ii), we get

$$2I=\int_{-\pi/2}^{\pi/2}\sin^2 x\,dx=2\int_0^{\pi/2}\sin^2 x\,dx$$

$$=\int_0^{\pi/2}(1-\cos 2x)dx$$

$$=\left[x-\frac{\sin 2x}{2}\right]_0^{\pi/2}=\frac{\pi}{2}$$

$\therefore$ $I=\frac{\pi}{4}$

JEE ADVANCED

Single Correct Answer Type

1. (3) $f'=\pm\sqrt{1-f^2}$

or $f(x)=\sin x$ or $f'(x)=-\sin x$ (not possible)

$\therefore$ $f(x)=\sin x$

Also, $x>\sin x\ \forall\ x>0$.

2. (1) $\int_0^1\left(x^6-4x^5+5x^4-4x^2+4-\frac{4}{1+x^2}\right)dx$

$$=\left[\frac{x^7}{7}-\frac{2x^6}{3}+x^5-\frac{4x^3}{3}+4x\right]_0^1-\pi$$

$$=\frac{1}{7}-\frac{2}{3}+1-\frac{4}{3}+4-\pi=\frac{22}{7}-\pi$$

3. (2) $e^{-x}f(x) = 2 + \int_0^x \sqrt{t^4+1}\,dt$ (1)

Now, $f(f^{-1}(x)) = x$

$\therefore \quad f'(f^{-1}(x))(f^{-1}(x))' = 1$

or $\quad (f^{-1})'(2) = \dfrac{1}{f'(f^{-1}(2))}$

From (1),

$f(0) = 2$ or $f^{-1}(2) = 0$

or $\quad (f^{-1})'(2) = \dfrac{1}{f'(0)}$

$e^{-x}f(x) = 2 + \int_0^x \sqrt{t^4+1}\,dt$

Differentiating w.r.t. x,

or $\quad e^{-x}(f'(x) - f(x)) = \sqrt{x^4+1}$

Put $x = 0$

$\therefore \quad f'(0) - 2 = 1$

or $\quad f'(0) = 3$

$(f^{-1})'(2) = 1/3$

4. (1) Put $x^2 = t$ and $2x\,dx = dt$

$$\therefore \quad I = \frac{1}{2}\int_{\ln 2}^{\ln 3} \frac{\sin t}{\sin t + \sin(\ln 6 - t)}\,dt$$

$$= \frac{1}{2}\int_{\ln 2}^{\ln 3} \frac{\sin(\ln 6 - t)}{\sin(\ln 6 - t) + \sin t}\,dt$$

$$\therefore \quad 2I = \frac{1}{2}\int_{\ln 2}^{\ln 3} 1\,dt \quad \text{or} \quad I = \frac{1}{4}\ln\frac{3}{2}.$$

5. (3) $R_1 = \int_{-1}^{2} xf(x)\,dx = \int_{-1}^{2} (2-1-x)f(2-1-x)\,dx$

$$= \int_{-1}^{2} (1-x)f(1-x)\,dx$$

$$= \int_{-1}^{2} (1-x)f(x)\,dx$$

Hence, $2R_1 = \int_{-1}^{2} f(x)\,dx = R_2.$

6. (4) Given $f'(x) - 2f(x) < 0$

or $\quad f'(x)e^{-2x} - 2e^{-2x}f(x) < 0$

or $\quad \dfrac{d}{dx}(f(x)e^{-2x}) < 0$

Thus, $g(x) = f(x)e^{-2x}$ is decreasing function.

Also, $f(1/2) = 1$.

Now, $g(x) < g(1/2)$ or $f(x)e^{-2x} < f(1/2)e^{-1}$ or $f(x) < e^{2x-1}$

or $\quad 0 < \int_{1/2}^{1} f(x)\,dx < \int_{1/2}^{1} e^{2x-1}\,dx$ or $0 < \int_{1/2}^{1} f(x)\,dx < \dfrac{e-1}{2}$

7. (2) $F(x) = \int_0^{x^2} f(\sqrt{t})\,dt$

$F(0) = 0$

$F'(x) = 2x\,f(x) = f'(x)$

$\Rightarrow \quad \int \dfrac{f'(x)}{f(x)}\,dx = \int 2x\,dx$

$\Rightarrow \quad \log_e f(x) = x^2 + c$

$\Rightarrow \quad f(x) = e^{x^2+c}$

$\Rightarrow \quad f(x) = e^{x^2}$ $(\because f(0) = 1)$

$\Rightarrow \quad F(x) = \int_0^{x^2} e^t\,dt$

$\Rightarrow \quad F(x) = e^{x^2} - 1$ $(\because F(0) = 0)$

$\Rightarrow \quad F(2) = e^4 - 1$

8. (1) $\int_{\pi/4}^{\pi/2} (2\operatorname{cosec} x)^{17}\,dx$

Let $e^u + e^{-u} = 2\operatorname{cosec} x$,

For $x = \dfrac{\pi}{4}$, $u = \ln(1+\sqrt{2})$

For $x = \dfrac{\pi}{2}$, $u = 0$

Also, $\operatorname{cosec} x + \cot x = e^u$ and $\operatorname{cosec} x - \cot x = e^{-u}$

$\Rightarrow \quad \cot x = \dfrac{e^u - e^{-u}}{2}$

Also $(e^u - e^{-u})\,du = -2\operatorname{cosec} x \cot x\,dx$

$$\Rightarrow \quad I = -\int_{\ln(1+\sqrt{2})}^{0} (e^u + e^{-u})^{17} \frac{(e^u - e^{-u})}{2\operatorname{cosec} x \cot x}\,du$$

$$= -2\int_{\ln(1+\sqrt{2})}^{0} (e^u + e^{-u})^{16}\,du$$

$$= \int_0^{\ln(1+\sqrt{2})} 2(e^u + e^{-u})^{16}\,du$$

9. (4) $f'(x) = \dfrac{192x^3}{2 + \sin^4(\pi x)}\ \forall\ x \in R;\ f\left(\dfrac{1}{2}\right) = 0$

Now, $64x^3 \le f'(x) \le 96x^3\ \forall\ x \in \left[\dfrac{1}{2}, 1\right]$

$$\therefore \quad \int_{1/2}^{x} 64x^3\,dx \le \int_{1/2}^{x} f'(x)\,dx \le \int_{1/2}^{x} 96x^3\,dx$$

So, $\quad 16x^4 - 1 \le f(x) \le 24x^4 - \dfrac{3}{2}\ \forall\ x \in \left[\dfrac{1}{2}, 1\right]$

$$\therefore \quad \int_{1/2}^{1} (16x^4 - 1)\,dx \le \int_{1/2}^{1} f(x)\,dx \le \int_{1/2}^{1} \left(24x^4 - \frac{3}{2}\right)dx$$

$$\therefore \quad \frac{16}{5}\cdot\frac{31}{32} - \frac{1}{2} \le \int_{1/2}^{1} f(x)\,dx \le \frac{24}{5}\cdot\frac{31}{32} - \frac{3}{4}$$

$$\Rightarrow \quad \frac{26}{10} \le \int_{1/2}^{1} f(x)\,dx \le \frac{78}{20}$$

(I) $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \cos x}{1+e^x} dx$ ... (1)

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \cos x}{1+\frac{1}{e^x}} dx$ (Replacing x by $-x$)

$\therefore \quad I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \cos x \cdot e^x}{1+e^x} dx$ ... (2)

Adding (1) and (2), we get

$$2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \cos x \, dx$$

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} x^2 \cos x \, dx$$

$$= x^2 \cdot \sin x \Big|_0^{\pi/2} - \int_0^{\frac{\pi}{2}} 2x \sin x dx$$

$$= \frac{\pi^2}{4} - 2\left[(-x\cos x)_0^{\pi/2} - \int_0^{\pi/2} (-\cos x) dx\right]$$

$$= \frac{\pi^2}{4} - 2\left[0 + \sin x \Big|_0^{\pi/2}\right]$$

$$= \frac{\pi^2}{4} - 2[1] = \frac{\pi^2}{4} - 2$$

Multiple Correct Answers Type

1. (1), (2), (3)

$$I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x} dx$$

$$= \int_0^{\pi} \left(\frac{\sin nx}{(1+\pi^x)\sin x} + \frac{\pi^x \sin nx}{(1+\pi^x)\sin x}\right) dx = \int_0^{\pi} \frac{\sin nx}{\sin x} dx$$

Now, $I_{n+2} - I_n = \int_0^{\pi} \frac{\sin(n+2)x - \sin nx}{\sin x} dx$

$$= \int_0^{\pi} \frac{2\cos(n+1)x \sin x}{\sin x} dx = 0$$

$\therefore \quad I_1 = \pi, I_2 = \int_0^{\pi} 2\cos x \, dx = 0$

2. (2), (3)

$$f(x) = \ln x + \int_0^x \sqrt{1+\sin t} \, dt$$

$$f'(x) = \frac{1}{x} + \sqrt{1+\sin x}$$

$$f''(x) = -\frac{1}{x^2} + \frac{\cos x}{2\sqrt{1+\sin x}}$$

(1) f'' is not defined for $x = -\frac{\pi}{2} + 2n\pi, n \in Z$ so (1) is wrong

(2) $f'(x)$ always exist for $x > 0$

(3) $|f'(x)| < |f(x)|$

Since $f' > 0$ and $f > 0$

we have $f'(x) < f(x)$

$$\Rightarrow \quad \frac{1}{x} + \sqrt{1+\sin x} < \ln x + \int_0^x \sqrt{1+\sin x} \, dx$$

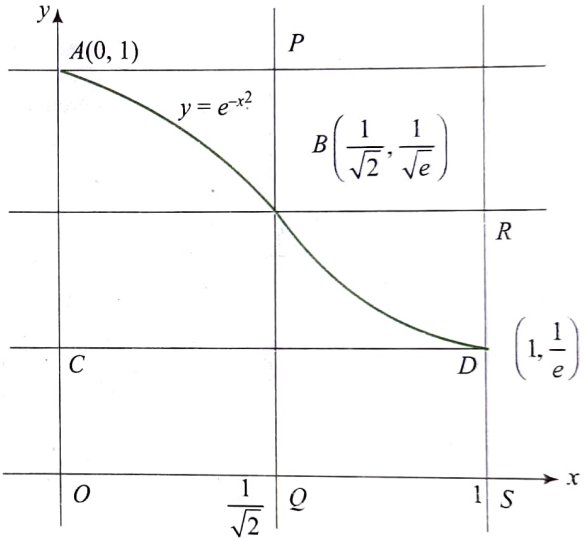
Now, LHS is bounded but R.H.S. RHS is Increasing with value approaching to infinity.

So there exist some α beyond which RHS is greater than LHS

(4) $|f(x)| + |f'(x)| \le \beta$ is wrong as f is increasing and its range is not bound while β is finite.

3. (1), (2), (4)

$S > \frac{1}{e}$ (As area of rectangle $OCDS = 1/e$)



Since $e^{-x^2} \ge e^{-x} \; \forall \; x \in [0,1]$, we have

$$S > \int_0^1 e^{-x} dx = \left(1 - \frac{1}{e}\right)$$

Area of rectangle $OAPQ$ + Area of rectangle $QBRS > S$

or $\quad S < \frac{1}{\sqrt{2}}(1) + \left(1 - \frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{e}}\right)$

Since $\frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right) < 1 - \frac{1}{e}$

Option (3) is incorrect.

4. (2), (4)

$$\text{Given limit} = \frac{\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \left(\frac{r}{n}\right)^a}{\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{a-1} \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \left(a + \frac{r}{n}\right)} = \frac{\int_0^1 x^a dx}{\int_0^1 (a+x) dx}$$

$$= \frac{2}{(2a+1)(a+1)} = \frac{2}{120}$$

$\therefore \quad a = 7 \text{ or } -\frac{17}{2}$

5. (1), (3)

For continuity at $x = a$

$$\lim_{x \to a^-} g(x) = 0$$

$$g(a) = \int_a^a f(t)dt = 0$$

$$\lim_{x \to a^+} g(x) = \lim_{x \to a^+} \int_a^x f(t)\,dt = 0$$

Hence, $g(x)$ is continuous at $x = a$.

For continuity at $x = b$

$$\lim_{x \to b^-} g(x) = \lim_{x \to b^-} \int_a^x f(t)dt = \int_a^b f(t)dt$$

$$= \lim_{x \to b^+} g(x) = g(b)$$

Thus, $g(x)$ is continuous at $x = b$.

$$g'(x) = \begin{cases} 0, & x < a \\ f(x) & a < x < b \\ 0, & x > b \end{cases}$$

Since $f(x) \geq 1$ for $x \in [a, b]$, $g(x)$ is non-differentiable at $x = a$ and $x = b$.

6. (1), (3), (4)

$$f(x) = \int_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t}$$

$$\Rightarrow \quad f'(x) = \frac{e^{-\left(x+\frac{1}{x}\right)}}{x} - \frac{e^{-\left(x+\frac{1}{x}\right)}}{\frac{1}{x}}\left(-\frac{1}{x^2}\right)$$

$$= \frac{2e^{-\left(x+\frac{1}{x}\right)}}{x} > 0 \text{ for } x \in [1, \infty)$$

Therefore, $f(x)$ is increasing in $[1, \infty)$.

$f'(x) > 0$ for $x \in (0, 1)$.

Hence, $f(x)$ is increasing.

Also, $f(x) + f\left(\frac{1}{x}\right) = \int_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t} + \int_{x}^{1/x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t} = 0$

$$g(x) = f(2^x) = \int_{2^{-x}}^{2^x} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t}\,dt$$

$$\therefore \quad g(-x) = \int_{2^x}^{2^{-x}} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t}\,dt = -g(x)$$

Hence, $f(2^x)$ is an odd function.

7. (1), (3)

Let $\int_0^{\pi} e^t \left(\sin^6 at + \cos^4 at\right) dt = I_1$

and $I_2 = \int_{\pi}^{2\pi} e^t \left(\sin^6 at + \cos^4 at\right) dt$

Put $t = x + \pi$ in I_2

$\therefore \quad dt = dx$

For $a = 2$ and $a = 4$

$$\therefore \quad \int_0^{\pi} e^{x+\pi} \left(\sin^6 ax + \cos^4 ax\right) dt = e^{\pi} I_1$$

Similarly, $\int_{2\pi}^{3\pi} e^t \left(\sin^6 at + \cos^4 at\right) dt = e^{2\pi} I_1$

and $\int_{3\pi}^{4\pi} e^t \left(\sin^6 at + \cos^4 at\right) dt = e^{3\pi} I_1$

$$\therefore \quad \frac{\int_0^{4\pi} e^t \left(\sin^6 at + \cos^4 at\right) dt}{\int_0^{\pi} e^t \left(\sin^6 at + \cos^4 at\right) dt}$$

$$= \frac{\int_0^{\pi} e^t \left(\sin^6 at + \cos^4 at\right) dt + \int_{\pi}^{2\pi} e^t \left(\sin^6 at + \cos^4 at\right) dt + \int_{2\pi}^{3\pi} e^t \left(\sin^6 at + \cos^4 at\right) dt + \int_{3\pi}^{4\pi} e^t \left(\sin^6 at + \cos^4 at\right) dt}{\int_0^{\pi} e^t \left(\sin^6 at + \cos^4 at\right) dt}$$

$$= 1 + e^{\pi} + e^{2\pi} + e^{3\pi}$$

$$= \frac{e^{4\pi} - 1}{e^{\pi} - 1}$$

8. (1), (2)

$$f(x) = (7\tan^6 x - 3\tan^2 x) \cdot \sec^2 x$$

$$\therefore \quad \int_0^{\frac{\pi}{4}} f(x)\,dx = \int_0^1 \left(7t^6 - 3t^2\right) dt = \left(t^7 - t^3\right)_0^1 = 0$$

Now, $\int_0^{\frac{\pi}{4}} xf(x)dx = \int_0^1 \left(7t^6 - 3t^2\right) \tan^{-1} t\,dt$

$$= \left(\tan^{-1} t \cdot \left(t^7 - t^3\right)\right)_0^1 - \int_0^1 \left(t^7 - t^3\right) \frac{1}{1+t^2}\,dt$$

$$= \int_0^1 \frac{t^3(1-t^4)}{1+t^2}\,dt = \int_0^1 t^3(1-t^2)dt = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

9. (2), (3)

$$f(x) = \lim_{n \to \infty} \left\{ \frac{n^{2n}\left(\frac{x}{n}+1\right)\cdots\left(\frac{x}{n}+\frac{1}{n}\right)}{n!\,n^{2n}\left(\frac{x^2}{n^2}+1\right)\left(\frac{x^2}{n^2}+\frac{1}{2^2}\right)\cdots\left(\frac{x^2}{n^2}+\frac{1}{n^2}\right)} \right\}^{x/n}$$

$$\Rightarrow \quad \ln f(x) = \lim_{n \to \infty} \frac{x}{n} \left\{ \sum_{r=1}^{n} \ln\left(\frac{x}{n}+\frac{1}{r}\right) - \sum_{r=1}^{n} \ln\left(\frac{rx^2}{n^2}+\frac{1}{r}\right) \right\}$$

$$= \lim_{n \to \infty} \frac{x}{n} \left\{ \sum_{r=1}^{n} \ln\left(1+\frac{rx}{n}\right) - \sum_{r=1}^{n} \ln\left(1+\frac{r^2x^2}{n^2}\right) \right\}$$

10.

$$= x\int_0^1 \ln(\ln(1+xy))\,dy - x\int_0^1 \ln(1+x^2y^2)\,dy$$

Let $xy = t$

$$\Rightarrow \quad \ln(f(x)) = \int_0^x \ln(1+t)\,dt - \int_0^x \ln(1+t^2)\,dt$$

$$\Rightarrow \quad \frac{f'(x)}{f(x)} = \ln\left(\frac{1+x}{1+x^2}\right)$$

$$\Rightarrow \quad \frac{f'(2)}{f(2)} = \ln\frac{3}{5} (< 0)$$

$$\Rightarrow \quad f'(2) < 0$$

$$\frac{f'(3)}{f(3)} = \ln\left(\frac{4}{10}\right) = \ln\left(\frac{2}{5}\right)$$

$$\Rightarrow \quad \frac{f'(2)}{f(2)} \geq \frac{f'(3)}{f(3)}$$

Now $\frac{f'(x)}{f(x)} > 0$ in $(0, 1)$ and $\frac{f'(x)}{f(x)} < 0$ in $(1, \infty)$

$f(x)$ is increasing in $(0, 1)$ and decreasing in $[1, \infty)$ (as $f(x)$ is positive).

Hence $f(1) \geq f\left(\frac{1}{2}\right)$ and $f\left(\frac{1}{3}\right) \leq f\left(\frac{2}{3}\right)$

10. (2), (4)

For option (1),

Let $g(x) = e^x - \int_0^x f(t)\sin t\,dt$

$\therefore \quad g'(x) = e^x - (f(x)\cdot \sin x) > 0 \ \forall\, x \in (0, 1)$

$[\because f(x) \in (0, 1)$ and $e^x > 1$ for $x \in (0, 1)]$

So, g(x) is strictly increasing function.

Also, $g(0) = 1$

$\Rightarrow \quad g(x) > 1 \ \forall\, x \in (0, 1)$

Thus, option (1) is not possible.

For option (2),

Let $h(x) = x^9 - f(x)$

Now, $h(0) = -f(0) < 0$ $(\because f \in (0, 1))$

Also, $h(1) = 1 - f(1) > 0$ $(\because f \in (0, 1))$

$\Rightarrow \quad h(0)\cdot h(1) < 0$

So, option (2) is correct (using Intermediate Value Theorem).

For option (3),

Let $G(x) = f(x) + \int_0^{\frac{\pi}{2}} f(t).\sin t\,dt$

$\Rightarrow \quad G(x) > 0 \ \forall\, x \in (0, 1)$ $(\because f \in (0, 1))$

So, option (3) is not possible.

For option (4),

Let $H(x) = x - \int_0^{\frac{\pi}{2}-x} f(t)\cos t\,dt$

$$\therefore \quad H(0) = 0 - \int_0^{\frac{\pi}{2}-1} f(t).\cos t\,dt < 0$$

Also, $H(1) = 1 - \int_0^{\frac{\pi}{2}-1} f(t).\cos t\,dt > 0$

$\Rightarrow \quad H(0).\ H(1) < 0$

So, option (4) is correct (using Intermediate Value Theorem).

11. (2), (4)

We have $\sum_{k=1}^{98}\int_k^{k+1}\frac{1}{x+1}dx < \sum_{k=1}^{98}\int_k^{k+1}\frac{k+1}{x(x+1)}dx < \sum_{k=1}^{98}\int_k^{k+1}\frac{dx}{x}$

$$\Rightarrow \quad \sum_{k=1}^{98}(\log_e(k+2) - \log_e(k+1)) < I < \sum_{k=1}^{98}(\log_e(k+1) - \log_e k)$$

$\Rightarrow \quad \log_e 50 < I < \log_e 99$

$\Rightarrow \quad \frac{49}{50} < I < \log_e 99$

Alternative method:

Put $x - k = p$

$$\Rightarrow \quad I = \sum_{k=1}^{98}\int_0^1 \frac{k+1}{(k+p)(k+p+1)}dp$$

Now, $I > \sum_{k=1}^{98}\int_0^1 \frac{k+1}{(k+p+1)^2}dp$

$$\Rightarrow \quad I > \sum_{k=1}^{98}(k+1)\left(\frac{-1}{k+p+1}\right)_0^1$$

$$\Rightarrow \quad I > \sum_{k=1}^{98}(k+1)\left(\frac{1}{k+1} - \frac{1}{k+2}\right)$$

$$\Rightarrow \quad I > \sum_{k=1}^{98}\frac{1}{k+2} = \frac{1}{3} + \ldots + \frac{1}{100}$$

$$\Rightarrow \quad I > \frac{1}{100} + \ldots + \frac{1}{100} = \frac{98}{100}$$

$$\Rightarrow \quad I > \frac{49}{50}$$

Now $\frac{k+1}{x(x+1)} < \frac{k+1}{x(k+1)}$ $(\because$ least value of $x + 1$ is $k + 1)$

$$\Rightarrow \quad \frac{k+1}{x(x+1)} < \frac{1}{x}$$

$$\Rightarrow \quad I < \sum_{k=1}^{98}\int_k^{k+1}\frac{1}{x}dx$$

$$\Rightarrow \quad I < \sum_{k=1}^{98}\log_e(k+1) - \log_e k$$

$\Rightarrow \quad 1 < \log_e 99$

12. (None)

$$g(x) = \int_{\sin x}^{\sin(2x)} \sin^{-1}(t)dt$$

$\Rightarrow \quad g'(x) = 2(\cos 2x)\sin^{-1}(\sin 2x) - (\cos x)\sin^{-1}(\sin x)$

$$\Rightarrow \quad g'\left(\frac{\pi}{2}\right) = 2(-1)(0) - \cos\left(\frac{\pi}{2}\right)(1) = 0$$

Also, $g'\left(-\frac{\pi}{2}\right) = 0$

No option is matching.

Linked Comprehension Type

1. (1) $g\left(\frac{1}{2}\right) = \lim_{h \to 0^+} \int_h^{1-h} t^{-1/2}(1-t)^{-1/2}\,dt$

$$= \int_0^1 \frac{dt}{\sqrt{t-t^2}} = \int_0^1 \frac{dt}{\sqrt{\frac{1}{4} - \left(t - \frac{1}{2}\right)^2}} = \sin^{-1}\left(\frac{t-\frac{1}{2}}{\frac{1}{2}}\right)\Bigg|_0^1$$

$$= \sin^{-1}1 - \sin^{-1}(-1) = \pi$$

2. (4) $g(a) = \lim_{h \to 0^+} \int_h^{1-h} t^{-a}(1-t)^{a-1}\,dt$

$$\Rightarrow \quad g(1-a) = \lim_{h \to 0^+} \int_h^{1-h} t^{-(1-a)}(1-t)^{(1-a)-1}\,dt$$

$$= \lim_{h \to 0^+} \int_h^{1-h} t^{a-1}(1-t)^{-a}\,dt$$

$$= \lim_{h \to 0^+} \int_h^{1-h} (1-t)^{a-1}(1-(1-t))^{-a}\,dt$$

$$= \lim_{h \to 0^+} \int_h^{1-h} (1-t)^{a-1}\,t^{-a}\,dt$$

Thus, $g(a) = g(1-a)$

$\Rightarrow \quad g'(a) = -g'(1-a)$

$\Rightarrow \quad g'(1/2) = -g'(1-1/2)$

$\Rightarrow \quad g'(1/2) = 0$

3. (1), (2), (3)

$f(x) = x\,F(x)$ (1)

$\therefore \quad f'(x) = xF'(x) + F(x)$ (2)

$\Rightarrow \quad f'(1) = F'(1) + F(1) = F'(1) < 0$

$F(1) = 0$ and $F(3) = -4$

Also, $F'(x) < 0$ for all $x \in (1/2, 3)$.

So, $F(x)$ is decreasing and hence $F(2) < 0$.

$\therefore \quad f(2) = 2F(2) < 0$

Also, for $x \in (1, 3)$,

$f'(x) = xF'(x) + F(x) < 0$

4. (3), (4)

$$\int_1^3 x^3 F''(x)\,dx = 40$$

$$\Rightarrow \quad \left[x^3F'(x)\right]_1^3 - \int_1^3 3x^2F'(x)\,dx = 40$$

$\Rightarrow \quad [x^2f'(x) - xf(x)]_1^3 - 3(-12) = 40$ (Using (1) and (2))

$\Rightarrow \quad 9f'(3) - 3f(3) - f'(1) + f(1) = 4$

$\Rightarrow \quad 9f'(3) + 36 - f'(1) + 0 = 4$ ($\because F(1), \therefore f(1) = 0$)

$\Rightarrow \quad 9f'(3) - f'(1) + 32 = 0$

$$\int_1^3 xF(x)\,dx = \left[\frac{x^2}{2}F(x)\right]_1^3 - \frac{1}{2}\int_1^3 x^2F'(x)\,dx$$

$$= \frac{9}{2}F(3) - \frac{1}{2}F(1) + 6$$

$$= -18 + 6 = -12$$

Matrix Match Type

1. (4) a. Let $f(x) = ax^2 + bx$, ($\because f(0) = 0$)

Given $\int_0^1 f(x)\,dx = 1$

$\Rightarrow \quad 2a + 3b = 6$

$\Rightarrow \quad (a, b) \equiv (0, 2)$ and $(3, 0)$

b. $f(x) = \sin(x^2) + \cos(x^2)$

$$= \sqrt{2}\cos\left(x^2 - \frac{\pi}{4}\right)$$

For maximum value, $x^2 - \frac{\pi}{4} = 2n\pi,\ n \in Z$

$\Rightarrow \quad x^2 = 2n\pi + \frac{\pi}{4},\ n \in Z$

$\Rightarrow \quad x = \pm\sqrt{\frac{x}{4}}, \pm\sqrt{\frac{9\pi}{4}}$ as $x \in \left[-\sqrt{13}, \sqrt{13}\right]$

c. $I = \int_{-2}^2 \frac{3x^2}{(1+e^x)}\,dx$ (1)

$$= \int_{-2}^2 \frac{3(-x)^2}{1+e^{-x}}\,dx$$

$\therefore \quad I = \int_{-2}^2 \frac{e^x(3x)^2}{e^x+1}\,dx$ (2)

Adding (1) and (2)

$$\Rightarrow \quad I + I = \int_{-2}^2 \frac{3x^2}{(1+e^x)}\,dx + \int_{-2}^2 \frac{e^x(3x^2)}{e^x+1}\,dx$$

$$= \int_{-2}^2 3x^2\,dx = 2\int_0^2 3x^2\,dx = 16$$

$\Rightarrow \quad I = 8$

d. We have $I = \dfrac{\int_{-1/2}^{1/2} \cos 2x \cdot \log\left(\frac{1+x}{1-x}\right)dx}{\int_0^{1/2} \cos 2x \cdot \log\left(\frac{1+x}{1-x}\right)dx}$

Let, $f(x) = \cos 2x \ln\left(\frac{1+x}{1-x}\right)$

$\therefore \quad f(-x) = \cos(-2x) \ln\left(\frac{1-x}{1+x}\right)$

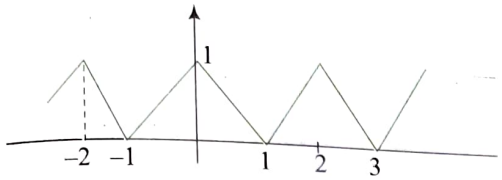
$$= -\cos(2x) \ln\left(\frac{1+x}{1-x}\right) = -f(x)$$

Thus, $f(x)$ is an odd function.

$\Rightarrow \quad I = 0$

Numerical Value Type

1. (4) $f(x) = \begin{cases} x-1, & 1 \le x < 2 \\ 1-x, & 0 \le x < 1 \end{cases}$



$f(x)$ is periodic with period 2 and even.

$$\therefore \quad I = \int_{-10}^{10} f(x)\cos \pi x\, dx$$

$$= 2\int_0^{10} f(x)\cos \pi x\, dx$$

$$= 2\times 5\int_0^{2} f(x)\cos \pi x\, dx$$

$$= 10\left[\int_0^1 (1-x)\cos \pi x\, dx + \int_1^2 (x-1)\cos \pi x\, dx\right] = 10(I_1 + I_2)$$

$$I_2 = \int_1^2 (x-1)\cos \pi x\, dx \qquad (\text{put } x-1 = t)$$

$$I_2 = -\int_0^1 t\cos \pi t\, dt$$

$$I_1 = \int_0^1 (1-x)\cos \pi x\, dx = -\int_0^1 x\cos(\pi x)dx$$

$$\therefore \quad I = 10\left[-2\int_0^1 x\cos \pi x\, dx\right] = -20\left[x\frac{\sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2}\right]_0^1$$

$$= -20\left[-\frac{1}{\pi^2} - \frac{1}{\pi^2}\right] = \frac{40}{\pi^2}$$

$$\therefore \quad \frac{\pi^2}{10} I = 4$$

2. (0) $y'(x) + y(x)\, g'(x) = g(x)\, g'(x)$

or $e^{g(x)}\, y'(x) + e^{g(x)}\, g'(x)\, y(x) = e^{g(x)}\, g(x)\, g'(x)$

or $\dfrac{d}{dx}(y(x)\, e^{g(x)}) = e^{g(x)}\, g(x)\, g'(x)$

$\therefore \quad y(x)\, e^{g(x)} = \int e^{g(x)} g(x) g'(x) dx$

$= \int e^t\, t\, dt$, where $g(x) = t$

$= (t-1)\, e^t + c$

$\therefore \quad y(x)\, e^{g(x)} = (g(x) - 1)\, e^{g(x)} + c$

$x = 0 \Rightarrow 0 = (0-1).\, 1 + c$ or $c = 1$

$x = 2 \Rightarrow y(2)\cdot 1 = (0-1)\cdot(1) + 1$

$\therefore \quad y(2) = 0.$

3. (2) $\displaystyle\int_0^1 4x^3 \left\{\frac{d^2}{dx^2}(1-x^2)^5\right\} dx$

$$= \left[4x^3 \frac{d}{dx}(1-x^2)^5\right]_0^1 - \int_0^1 12x^2 \frac{d}{dx}(1-x^2)^5\, dx$$

(Integrating using by parts)

$$= \left[4x^3 \times 5(1-x^2)^4(-2x)\right]_0^1 - 12\left[\left[x^2(1-x^2)^5\right]_0^1 - \int_0^1 2x(1-x^2)^5\, dx\right]$$

$$= 0 - 0 - 12\,[0-0] + 12\int_0^1 2x(1-x^2)^5\, dx$$

$$= 12\left[-\frac{(1-x^2)^6}{6}\right]_0^1 = 12\left[0 + \frac{1}{6}\right] = 2$$

4. (7) $f(x)$ is continuous odd function and vanishes exactly at one point.

$\therefore \quad f(0) = 0$

$$F(x) = \int_{-1}^{x} f(t)dt = \int_{-1}^{1} f(t)dt + \int_{1}^{x} f(t)dt$$

$$= 0 + \int_1^x f(t)dt \qquad (\text{as } f(t) \text{ is an odd function})$$

$f(t)$ is odd function

$\therefore \quad f(f(t))$ is also odd function

$\therefore \quad |f(f(t))|$ is an even function

$\therefore \quad t|f(f(t))|$ is an odd function

$$\therefore \quad G(x) = \int_1^x t\left|f(f(t))\right| dt = \int_1^x t\left|f(f(t))\right| dt$$

Now, $\displaystyle\lim_{x\to 1}\frac{F(x)}{G(x)}$ $\qquad \left(\frac{0}{0}\ \text{form}\right)$

$$= \lim_{x\to 1}\frac{f(x)}{x\left|f(f(x))\right|} \qquad (\text{Using L'Hospital's Rule})$$

$$= \frac{\frac{1}{2}}{1\left|f\left(\frac{1}{2}\right)\right|} = \frac{1}{14} \quad (\text{Given})$$

$$\Rightarrow \quad f\left(\frac{1}{2}\right) = 7$$

5. (9) $\displaystyle\alpha = \int_0^1 e^{9x + 3\tan^{-1}x}\left(\frac{12+9x^2}{1+x^2}\right) dx$

Let $9x + 3\tan^{-1}x = t$

$$\Rightarrow \quad \left(9 + \frac{3}{1+x^2}\right) dx = dt$$

$$\Rightarrow \quad \left(\frac{12+9x^2}{1+x^2}\right) dx = dt$$

$$\Rightarrow \quad \alpha = \int_0^{9+\frac{3\pi}{4}} e^t dt = e^{9+\frac{3\pi}{4}} - 1$$

$$\Rightarrow \quad \log_e(1+\alpha) = 9 + \frac{3\pi}{4}$$

6. (3) $F(x)=\int_{x}^{x^2+\frac{\pi}{6}} 2\cos^2 t\,dt$

$\therefore \quad F'(x)=2\left(\cos\left(x^2+\frac{\pi}{6}\right)\right)^2 2x-2\cos^2 x$

According to the question,

$\therefore \quad F'(a)+2=\int_0^a f(x)dx$

$\Rightarrow \quad 2\left(\cos\left(a^2+\frac{\pi}{6}\right)\right)^2 2a-2\cos^2 a+2=\int_0^a f(x)dx$

Differentiating w.r.t. a, we get

$$4\cos^2\left(a^2+\frac{\pi}{6}\right)+4a\times 2\cos\left(a^2+\frac{\pi}{6}\right)\left(-\sin\left(a^2+\frac{\pi}{6}\right)\right)\times 2a+4\cos a\sin a=f(a)$$

$\therefore \quad f(0)=4\left(\frac{\sqrt{3}}{2}\right)^2=3$

7. (0) $I=\int_{-1}^{2}\frac{x[x^2]}{2+[x+1]}dx=\int_{-1}^{2}\frac{x[x^2]}{3+[x]}dx$

$=\int_{-1}^{0}\frac{0}{3-1}dx+\int_{0}^{1}\frac{0}{3+1}dx+\int_{1}^{\sqrt{2}}\frac{x\cdot 1}{3+1}dx+0$

$=\frac{1}{4}\left[\frac{x^2}{2}\right]_1^{\sqrt{2}}=\frac{2-1}{8}=\frac{1}{4}$

$\therefore \quad 4I-1=0$

8. (1) Let $f(x)=\int_0^x\frac{t^2dt}{1+t^4}-2x+1$

$\Rightarrow \quad f'(x)=\frac{x^2}{1+x^4}-2=\frac{-2x^4+x^2-2}{x^4+1}<0,\ \forall x\in R$

$\therefore$ $f(x)$ is decreasing function.

$f(0)=1$ (from (1))

Now $0<\frac{t^2}{1+t^4}<1$

$\therefore \quad 0<\int_0^1\frac{t^2dt}{1+t^4}<1$

$\therefore \quad f(1)=\int_0^1\frac{t^2dt}{1+t^4}-1<0$

$\therefore$ Graph of $y=f(x)$ cuts x-axis exactly once for $x\in(0,1)$.

Hence only one solution

9. (2) $g(x)=\int_x^{\frac{\pi}{2}}(f'(t)\operatorname{cosec}t-f(t)\operatorname{cosec}t\cot t)dt$

$=\int_X^{\frac{\pi}{2}}\frac{d}{dt}(f(t)\operatorname{cosec}t)dt$

$=f\left(\frac{\pi}{2}\right)\operatorname{cosec}\left(\frac{\pi}{2}\right)-\frac{f(x)}{\sin x}=3-\frac{f(x)}{\sin x}$

$\therefore \quad \lim_{x\to 0}g(x)=3-\log\lim_{x\to 0}\frac{f(x)}{\sin x}$

$=\lim_{x\to 0}g(x)$

$=3-\log\lim_{x\to 0}\frac{f'(x)}{\cos x}$ (Using L'Hopital Rule)

$=3-1$ ($\because f'(0)=1$)

$=2$

10. (1) $y_n=\left[\frac{(n+1)(n+2)\ldots(n+n)}{n^n}\right]^{1/n}$

$\Rightarrow \quad y_n=\left(\frac{n+1}{n}\cdot\frac{n+2}{n}\cdot\ldots\cdot\frac{n+n}{n}\right)^{\frac{1}{n}}$

$\therefore \quad \log y_n=\log\left(\frac{n+1}{n}\cdot\frac{n+2}{n}\cdot\ldots\cdot\frac{n+n}{n}\right)^{\frac{1}{n}}$

$=\frac{1}{n}\sum_{r=1}^{n}\log\frac{n+r}{n}$

$=\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r}{n}\right)$

$\therefore \quad \lim_{n\to\infty}\log y_n=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r}{n}\right)$

$\therefore \quad \log\left(\lim_{n\to\infty}y_n\right)=\int_0^1\log(1+x)dx$

$\therefore \quad \log L=\int_1^2\log x\,dx$

$=|x\log x-x|_1^2=2\log 2-1$

$=\log\frac{4}{e}$

$\Rightarrow \quad L=\frac{4}{e}$

$\Rightarrow \quad [L]=1$

11. (2) $I=\int_0^{1/2}\frac{1+\sqrt{3}}{((x+1)^2(1-x)^6)^{1/4}}dx$

$=\int_0^{1/2}\frac{(1+\sqrt{3})dx}{(1+x)^2\left[\frac{(1-x)^6}{(1+x)^6}\right]^{1/4}}$

Put $\frac{1-x}{1+x}=t$

$\Rightarrow \quad \frac{-2dx}{(1+x)^2}=dt$.

$\therefore \quad I=\int_1^{1/3}\frac{(1+\sqrt{3})dt}{-2t^{3/2}}=\frac{-(1+\sqrt{3})}{2}\times\left|\frac{-2}{\sqrt{t}}\right|_1^{1/3}$

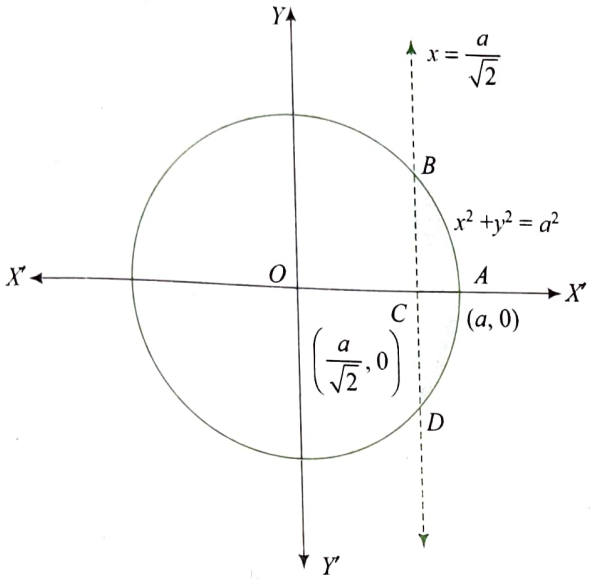
$=(1+\sqrt{3})(\sqrt{3}-1)=2$

Chapter 9

Concept Application Exercises

Exercise 9.1

1. The area of the smaller part of the circle, $x^2 + y^2 = a^2$, cut off by the line, $x = \frac{a}{\sqrt{2}}$, is the area $ABCDA$.



Solving $x^2 + y^2 = a^2$ and $x = \frac{a}{\sqrt{2}}$ for their points of intersection, we get

$$\frac{a^2}{2} + y^2 = a^2$$

or $y^2 = \frac{a^2}{2}$ or $y = \pm\frac{a}{\sqrt{2}}$

$\therefore$ Area $ABCD = 2 \times$ Area ABC

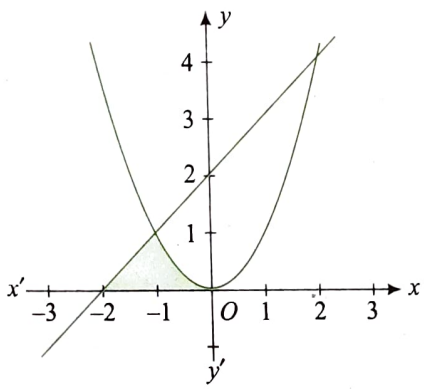
$$= 2\int_{\frac{a}{\sqrt{2}}}^{a} \sqrt{a^2 - x^2}\, dx = 2\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{\frac{a}{\sqrt{2}}}^{a}$$

$$= 2\left[\frac{a^2}{2}\left(\frac{\pi}{2}\right) - \frac{a}{2\sqrt{2}}\sqrt{a^2 - \frac{a^2}{2}} - \frac{a^2}{2}\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right]$$

$$= 2\left[\frac{a^2\pi}{4} - \frac{a^2}{4} - \frac{a^2\pi}{8}\right]$$

$$= \frac{a^2}{2}\left(\frac{\pi}{2} - 1\right) \text{ sq. units.}$$

2.



Required area $= \int_{-2}^{-1}(x+2)\, dx + \int_{-1}^{0} x^2\, dx$

$$= \left|\frac{x^2}{2} + 2x\right|_{-2}^{-1} + \left|\frac{x^3}{3}\right|_{-1}^{0}$$

$$= \left(\frac{1}{2} - 2\right) - (2 - 4) + \left(0 + \frac{1}{3}\right)$$

$$= \frac{5}{6} \text{ sq. units.}$$

3. The given curve is

$$y = \begin{cases} \sqrt{(4 - x^2)}, & 0 \le x < 1 \\ \sqrt{(3x)}, & 1 \le x \le 3 \end{cases}$$

Obviously, the curve is the arc of the circle $x^2 + y^2 = 4$ (1)
between $0 \le x < 1$ and the arc of parabola $y^2 = 3x$ (2)
between $1 \le x \le 3$

Required area = Shaded area

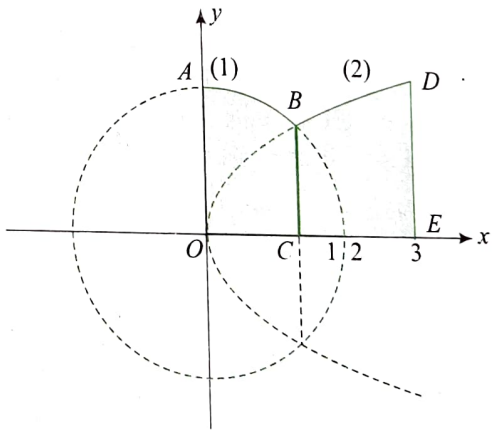
= Area $OABCO$ + Area $CBDEC$

$$= \left|\int_0^1 \sqrt{(4 - x^2)}\, dx\right| + \left|\int_1^3 \sqrt{3x}\, dx\right|$$

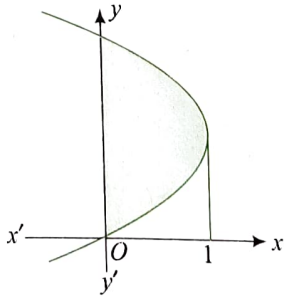
$$= \left|\left[\frac{1}{2}x\sqrt{(4 - x^2)} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1\right| + \left|\sqrt{3}\left[\frac{2}{3}x^{3/2}\right]_1^3\right|$$

$$= \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right) + \frac{2}{3}(9 - \sqrt{3})$$

$$= \frac{1}{6}(2\pi - \sqrt{3} + 36) \text{ sq. units.}$$



4.



$$A = \int_0^2 x\, dy = \int_0^2 \left(2y - y^2\right) dy = \left[y^2 - \frac{y^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$$

5. Given $y = 1 + \frac{8}{x^2}$.

Here y is always positive; hence, curve is lying above the x-axis.

$\therefore$ Req. area $= \int_2^4 y\,dx = \int_2^4 \left(1 + \frac{8}{x^2}\right) dx$

$$= \left[x - \frac{8}{x}\right]_2^4 = 4.$$

If $x = a$ bisects the area, then we have

$$\int_2^a \left(1 + \frac{8}{x^2}\right) dx = \left[x - \frac{8}{x}\right]_2^a = \left[a - \frac{8}{a} - 2 + 4\right] = \frac{4}{2}.$$

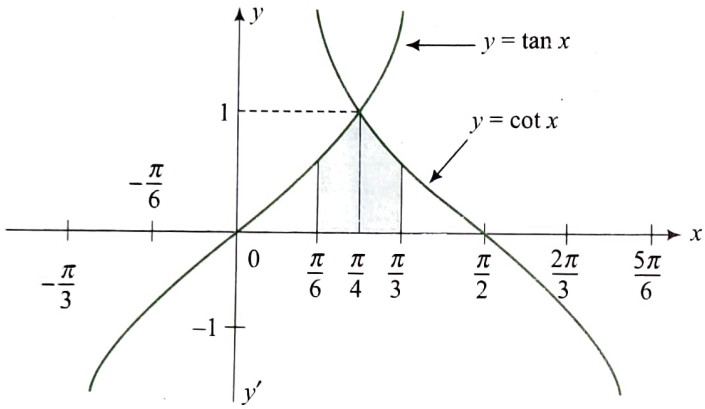
or $\quad a - \frac{8}{a} = 0$

or $\quad a^2 = 8$

or $\quad a = \pm 2\sqrt{2}$

Since $a > 2$, $a = 2\sqrt{2}$.

6.



The two curves are

$y = \tan x$, where $-\pi/3 \le x \le \pi/3$ (1)

$y = \cot x$, where $\pi/6 \le x \le 3\pi/2$ (2)

At the point of intersection of the two curves

$\tan x = \cot x$ or $\tan^2 x = 1$ or $\tan x = \pm 1$, $x = \pm\pi/4$

Thus, the curves intersect at $x = \pi/4$

The required area is the shaded area. Therefore,

$$A = \int_{\pi/6}^{\pi/4} \tan x\,dx + \int_{\pi/4}^{\pi/3} \cot x\,dx$$

$$= [\log \sec x]_{\pi/6}^{\pi/4} + [\log \sin x]_{\pi/4}^{\pi/3}$$

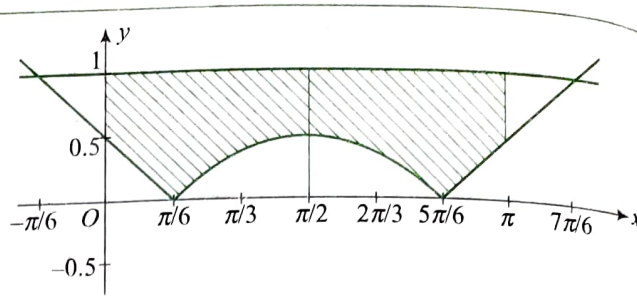
$$= \left(\log \sqrt{2} - \log \frac{2}{\sqrt{3}}\right) + \left(\log \frac{\sqrt{3}}{2} - \log \frac{1}{\sqrt{2}}\right)$$

$$= \log \sqrt{2} + \log \frac{\sqrt{3}}{2} + \log \frac{\sqrt{3}}{2} + \log \sqrt{2}$$

$$= 2\left(\log \sqrt{2}\,\frac{\sqrt{3}}{2}\right)$$

$$= 2 \log \sqrt{\frac{3}{2}} = \log 3/2 \text{ sq. units.}$$

7. $y = \left|\sin x - \frac{1}{2}\right| = \begin{cases} \frac{1}{2} - \sin x, & 0 \le x \le \frac{\pi}{6} \\ \sin x - \frac{1}{2}, & \frac{\pi}{6} < x < \frac{5\pi}{6} \\ \frac{1}{2} - \sin x, & \frac{5\pi}{6} \le x \le \pi \end{cases}$



$\therefore$ Required area is

$$A = \pi - 2\left(\int_0^{\frac{\pi}{6}} \left(\frac{1}{2} - \sin x\right) dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\sin x - \frac{1}{2}\right) dx\right)$$

$$= \pi - 2\left(\frac{1}{2}x + \cos x\right)_0^{\pi/6} + 2\left(-\cos x - \frac{1}{2}x\right)_{\pi/6}^{\pi/2}$$

$$= \frac{\pi}{2} + 2$$

8. $y = xe^{-ax}$

$\therefore \quad y' = e^{-ax} - xae^{-ax}$

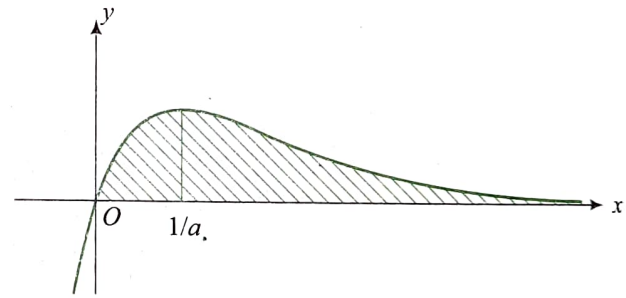
$= e^{-ax}(1 - ax)$

Let $\quad y' = 0$

$\therefore \quad x = \frac{1}{a}$, which is point of maxima.

$$\lim_{x\to\infty} x \cdot e^{-ax} = \lim_{x\to\infty} \frac{x}{e^{ax}} = \lim_{x\to\infty} \frac{1}{ae^{ax}} = 0$$

$$\lim_{x\to-\infty} x \cdot e^{-ax} = -\infty$$



$$\text{Area} = \int_0^\infty xe^{-ax}dx = \left[\frac{x \cdot e^{-ax}}{-a}\right]_0^\infty + \frac{1}{a}\int_0^\infty e^{-ax}\,dx$$

$$= (0) - \left[\frac{1}{a^2}e^{-ax}\right]_0^\infty$$

$$= -\frac{1}{a^2}[0 - 1] = \frac{1}{a^2}$$

Given that $\frac{1}{a^2} = \frac{1}{9}$

$\Rightarrow \quad a = 3$

9. We have curve $xy^2 = 4(2 - x)$

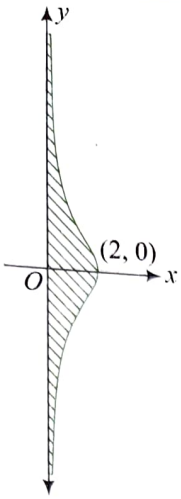
$\therefore \quad x = \frac{8}{y^2 + 4}$

The given curve is symmetrical about x-axis and meets it at $(2, 0)$.

Also, when $y \to \pm\infty$, $x \to 0$.

So, y-axis is asymptote to the curve.

The graph of the function is as shown in the following figure.



The line $x = 0$, i.e., y-axis is asymptote.

Area of the shaded region $= 2\int_0^\infty \frac{8}{y^2+4}dx$

$= 16\left[\frac{1}{2}\tan^{-1}\frac{y}{2}\right]_0^\infty = 8\times\frac{\pi}{2}$

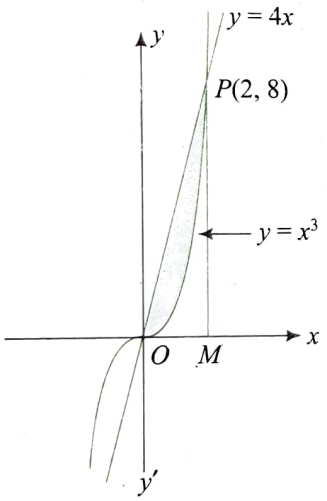
$= 4\pi$ sq. units

Exercise 9.2

1. The line $y = 4x$ meets $y = x^3$ at $4x = x^3$.

$\therefore \quad x = 0, 2, -2 \Rightarrow y = 0, 8, -8$

$\Rightarrow \quad A = \int_0^2 (4x - x^3) = \left(2x^2 - \frac{x^4}{4}\right)_0^2 = 4$ sq. units



2. Given curves $x^2 = 4y$ and $x = 4y - 2$ intersect, when

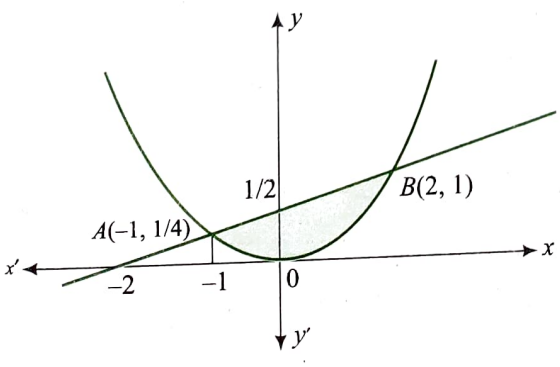
$x^2 = x + 2$

or $\quad x^2 - x - 2 = 0$

or $\quad x = 2, -1$

$\Rightarrow \quad y = 1, 1/4$

Hence, points of intersection are $A\ (-1, 1/4)$, $B(2, 1)$



Required area = Shaded region in the figure

$= \int_{-1}^{2}\left(\frac{x+2}{4} - \frac{x^2}{4}\right)dx$

$= \frac{1}{4}\left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$

$= \frac{1}{4}\left[\left(2+4-\frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)\right]$

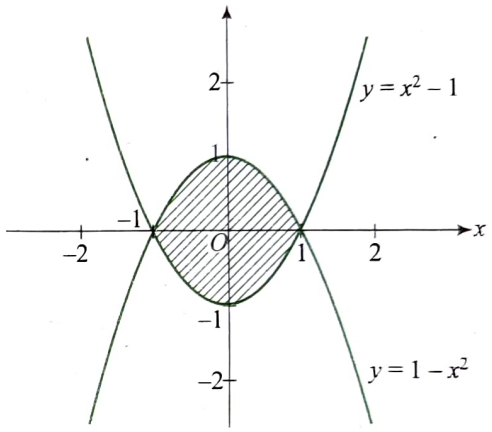
$= \frac{1}{4}\left[\frac{10}{3} - \left(\frac{-7}{6}\right)\right] = \frac{1}{4}\left[\frac{27}{6}\right] = 9/8$ sq. units.

3. $x^4 + 1 = 2x^2 + y^2$

$\therefore \quad x^4 - 2x^2 + 1 = y^2$

$\therefore \quad (x^2 - 1)^2 = y^2$

$\therefore \quad y = \pm(x^2 - 1)$, which are two parabolas.



Required area $A = 4\int_0^1 (1 - x^2)\,dx$

$= 4\left[x - \frac{x^3}{3}\right]_0^1 = \frac{8}{3}$ sq. units

4. Both the given curves are parabola.

$y = 4x - x^2$ (1)

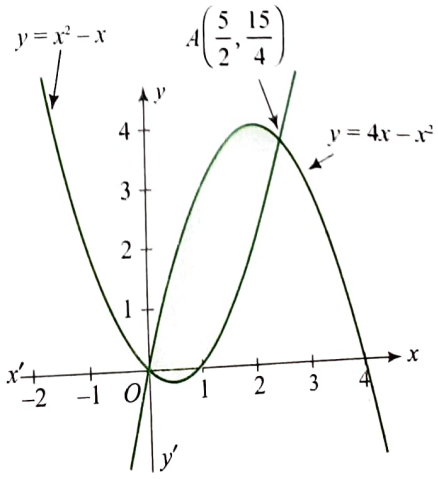
and $y = x^2 - x$ (2)

Solving (1) and (2), we get

$4x - x^2 = x^2 - x$

or $\quad x = 0, x = \frac{5}{2}$.

Thus, two curves intersect at $O(0, 0)$ and $A\left(\frac{5}{2}, \frac{15}{4}\right)$



Here the area below x-axis,

$$A_1 = \int_0^1 (x - x^2)\,dx$$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \text{ sq. units}$$

Area above x-axis,

$$A_2 = \int_0^{5/2} (4x - x^2)\,dx - \int_1^{5/2} (x^2 - x)\,dx$$

$$= \left(2x^2 - \frac{x^3}{3}\right)_0^{5/2} - \left(\frac{x^3}{3} - \frac{x^2}{2}\right)_1^{5/2}$$

$$= \left(\frac{25}{2} - \frac{125}{24}\right) - \left[\left(\frac{125}{24} - \frac{25}{8}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)\right]$$

$$= \frac{25}{2} - \frac{125}{12} + \frac{25}{8} - \frac{1}{6}$$

$$= \frac{300 - 250 + 75 - 4}{24} = \frac{121}{24} \text{ sq. units.}$$

Thus, the ratio of area above x-axis to area below x-axis is

$$A_2 : A_1 = \frac{121}{24} : \frac{1}{6} = \frac{121}{4} = 121 : 4.$$

5. The given curves are

$$x^2 + y^2 = 16 \quad \ldots(i)$$

$$y^2 = 6x \quad \ldots(ii)$$

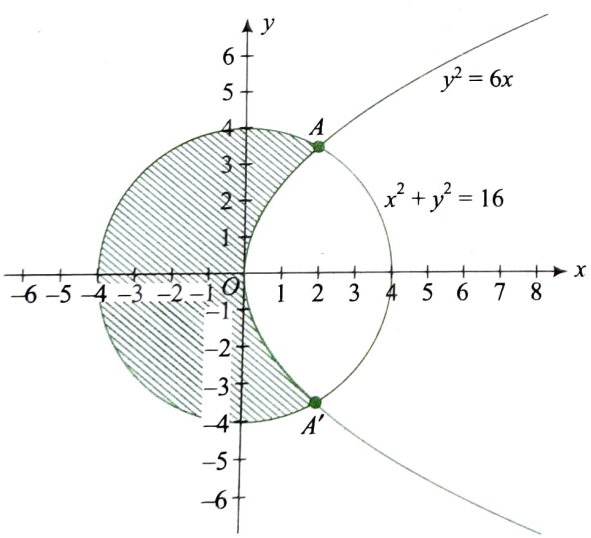
Solving $x^2 + 6x = 16$

or $(x - 2)(x + 8) = 0$

$\therefore$ $x = 2$ (as $x = -8$ is not possible)

Thus, the points of intersection are $A(2, 2\sqrt{3})$ and $A'(2, -2\sqrt{3})$.

The graphs of the curves is as shown in the following figure.



From the figure, required area is

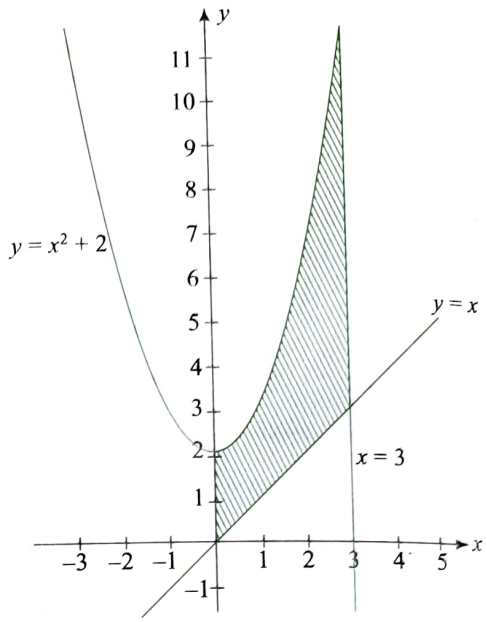
$$= \text{Area of semicircle} + 2\int_0^2 \left(\sqrt{16 - x^2} - \sqrt{6x}\right)dx$$

$$= 8\pi + 2\left[\frac{x}{2}\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4} - \sqrt{6}\,\frac{2x^{3/2}}{3}\right]_0^2$$

$$= 8\pi + 2\left[2\sqrt{3} + \frac{4\pi}{3} - \sqrt{6}\,\frac{2\cdot 2^{3/2}}{3}\right]$$

$$= \frac{4}{3}(8\pi - \sqrt{3}) \text{ sq. units}$$

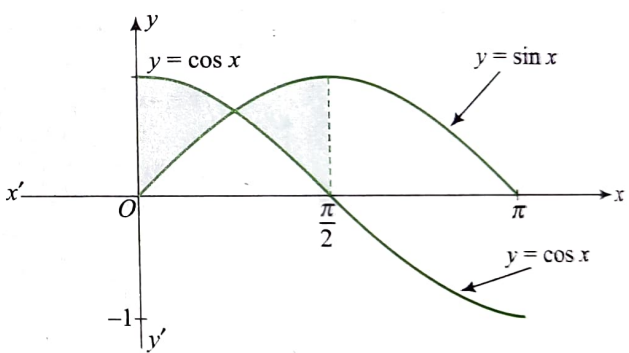
6. $y = x^2 + 2$ is parabola having vertex at $(0, 2)$ and having concavity upward.



Then, required area $= \int_0^3 ((x^2 + 2) - x)\,dx = \left[\frac{x^3}{3} + 2x\right]_0^3 - \left[\frac{x^2}{2}\right]_0^3$

$$= [9 + 6] - \left[\frac{9}{2}\right] = \frac{21}{2} \text{ sq. units.}$$

7.



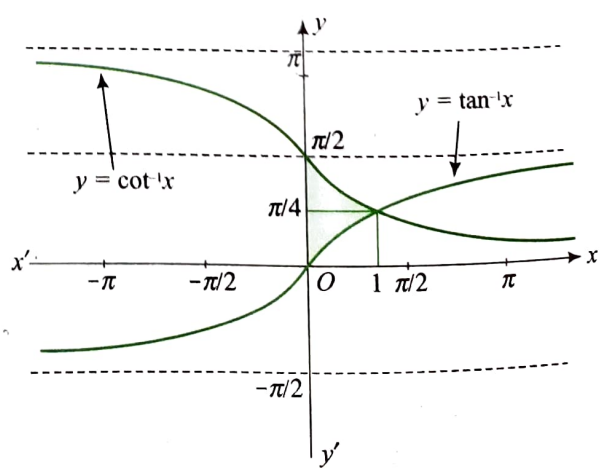
Required area $= \int_0^{\pi/2} |\sin x - \cos x|\,dx$

$$= 2\int_0^{\pi/4} (\cos x - \sin x)\,dx$$

$$= 2\,|\sin x + \cos x|_0^{\pi/4}$$

$$= 2\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1\right) = 2(\sqrt{2} - 1) \text{ sq. units}$$

8.



Integrating along x-axis, we get

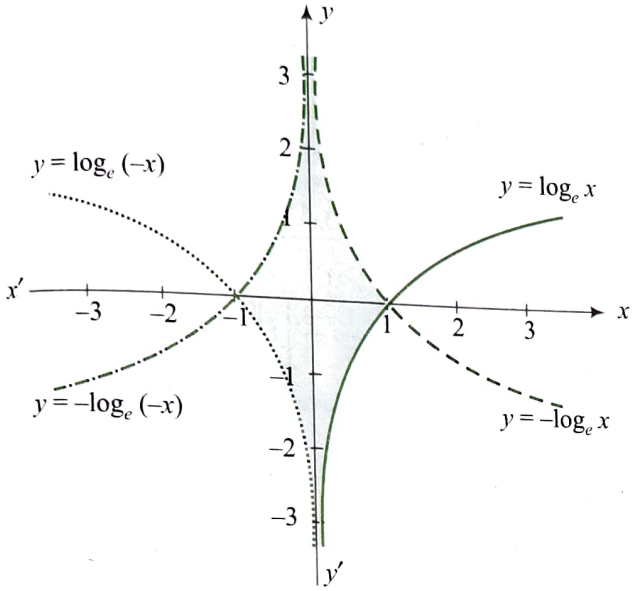
$$A = \int_0^1 \left(\cot^{-1} x - \tan^{-1} x\right) dx = \int_0^1 \left(\frac{\pi}{2} - 2\tan^{-1} x\right) dx$$

Integrating along y-axis, we get

$$A = 2\int_0^{\pi/4} x\,dy = 2.\int_0^{\pi/4} \tan y\,dy = \left[\log(\sec y)\right]_0^{\pi/4}$$

$$= \log \sqrt{2} \text{ sq. units}$$

9.



From the figure, required area = area of shaded region

$$= 1 + 1 + 1 + 1 = 4 \text{ sq. units.}$$

10. Given inequalities are

$$y^2 \le 4x \quad (1)$$

and $4x^2 + 4y^2 \le 9$ (2)

Points satisfying (1) lies on or in interior to parabola $y^2 = 4x$ and those of (2) lies on or inside circle $4x^2 + 4y^2 = 9$

Solving we get,

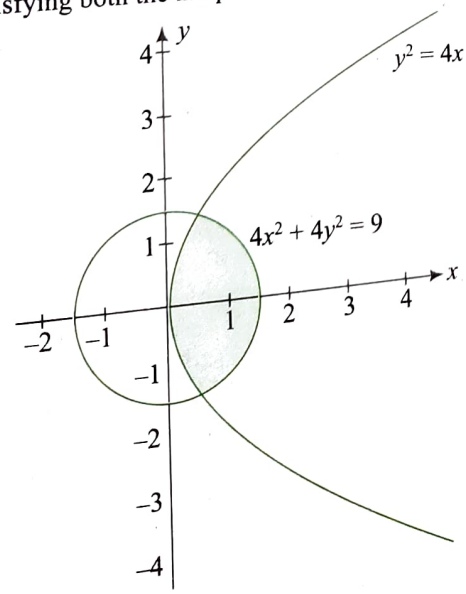
$4x^2 + 16x = 9$

or $(2x - 1)(2x + 9) = 0$

or $x = 1/2$ (as $x = -9$ not possible)

Therefore, the points of intersection of both curves are $\left(\frac{1}{2}, \sqrt{2}\right)$ and $\left(\frac{1}{2}, -\sqrt{2}\right)$.

The graph of these two curves and the common region of the points satisfying both the inequalities is as shown in the figure.



From the figure, required area is given by

$$A = 2\int_1^{\frac{1}{2}} 2\sqrt{x}\ dx + 2\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{2}\sqrt{9 - 4x^2}\ dx$$

Putting $2x = t$ in the second integral, we get

$$dx = \frac{dt}{2}$$

$$\therefore \quad A = 2\left[\int_0^{\frac{1}{2}} 2\sqrt{x}\,dx + \frac{1}{4}\int_1^3 \sqrt{(3)^2 - (t)^2}\,dt\right]$$

$$= 2\left(\left[2\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{\frac{1}{2}} + \frac{1}{4}\left[\frac{t}{2}\sqrt{9 - t^2} + \frac{9}{2}\sin^{-1}\left(\frac{t}{3}\right)\right]_1^3\right)$$

$$= 2\left(\frac{2}{3\sqrt{2}} + \frac{1}{4}\left[\left\{0 + \frac{9}{2}\sin^{-1}(1)\right\} - \left\{\frac{1}{2}\sqrt{8} + \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right\}\right]\right)$$

$$= 2\left(\frac{9\pi}{16} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right) + \frac{\sqrt{2}}{12}\right)$$

11. The given curves are

$$y = \sqrt{5 - x^2} \quad (1)$$

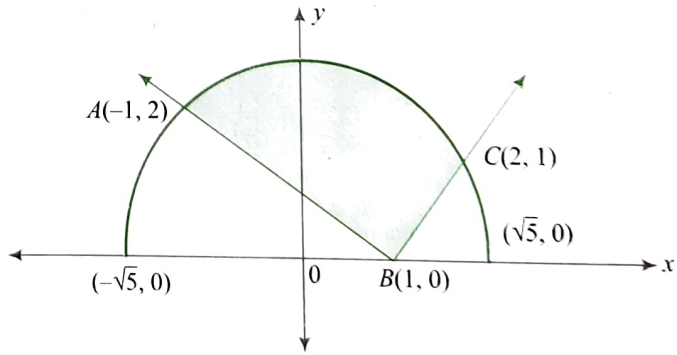
$$y = |x - 1| \quad (2)$$

We can clearly see that on squaring the both sides of (1), equation (2) represents a circle.

But as y is + ve square root, (1) represents the upper-half of the circle with centre (0, 0) and radius $\sqrt{5}$.

Equation (2) represents the curve

$$y = \begin{cases} -x + 1 \text{ if } x < 1 \\ x - 1 \quad \text{if } x \ge 1 \end{cases}$$



Graph of these curves are as shown in the figure with point of intersection of $y = \sqrt{5 - x^2}$ and $y = -x + 1$ as $A(-1, 2)$

and of $y = \sqrt{5 - x^2}$ and $y = x - 1$ as $C(2, 1)$

$\therefore$ Required area = Shaded area

$$= \int_{-1}^{2} \sqrt{5 - x^2}\,dx - \int_{-1}^{2} |x - 1|\,dx$$

$$= \left[\frac{x}{2}\sqrt{5 - x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)\right]_{-1}^{2} - \int_{-1}^{1} -(x - 1)dx$$

$$- \int_1^2 (x - 1)\,dx$$

$$= \left(\frac{2}{2}\sqrt{5-4} + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}}\right) - \left(\frac{-1}{2}\sqrt{5-1}\right.$$

$$\left. + \frac{5}{2}\sin^{-1}\left(\frac{-1}{\sqrt{5}}\right)\right) - \left(\frac{-x^2}{2} + x\right)_{-1}^{1} - \left(\frac{x^2}{2} - x\right)_{1}^{2}$$

$$= 1 + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}} + 1 + \frac{5}{2}\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

$$- \left[\left(\frac{-1}{2}+1\right) - \left(\frac{-1}{2}-1\right)\right] - \left[(2-2) - \left(\frac{1}{2}-1\right)\right]$$

$$= 2 + \frac{5}{2}\left[\sin^{-1}\frac{2}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{5}}\right] - 2 - \frac{1}{2}$$

$$= \frac{5}{2}\left[\sin^{-1}\frac{2}{\sqrt{5}} + \cos^{-1}\frac{2}{\sqrt{5}}\right] - \frac{1}{2} = \frac{5}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}$$

$$= \frac{5\pi - 2}{4} \text{ sq. units.}$$

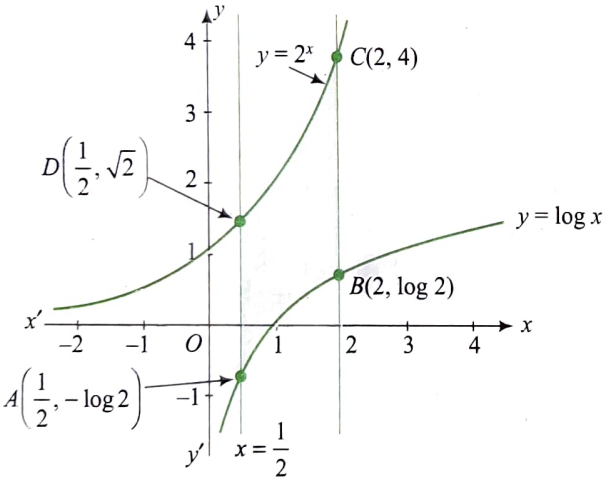
12. The given curves are

$$x = \frac{1}{2} \quad (1)$$

$$x = 2 \quad (2)$$

$$y = \log_e x \quad (3)$$

$$y = 2^x \quad (4)$$



Required area = Area of $ABCDA$

$$= \int_{1/2}^{2}\left(2^x - \log x\right)dx$$

$$= \left[\frac{2^x}{\log 2} - (x\log x - x)\right]_{1/2}^{2}$$

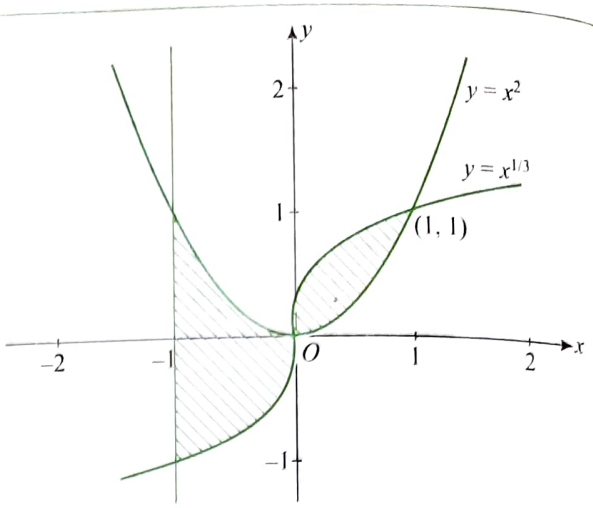
$$= \left(\frac{4}{\log 2} - 2\log 2 + 2\right) - \left(\frac{\sqrt{2}}{\log 2} - \frac{1}{2}\log\frac{1}{2} + \frac{1}{2}\right)$$

$$= \left(\frac{4-\sqrt{2}}{\log 2} - \frac{5}{2}\log 2 + \frac{3}{2}\right) \text{ sq. units}$$

13. Solving given curves we get $x^2 = x^{1/3}$.

$\therefore$ $x = 0$ or $x = 1$

Required area, $A = \int_{-1}^{0}[x^2 - x^{1/3}]dx + \int_{0}^{1}[x^{1/3} - x^2]dx$



$$= \left[\frac{x^3}{3} - \frac{3x^{4/3}}{4}\right]_{-1}^{0} + \left[\frac{3x^{4/3}}{4} - \frac{x^3}{3}\right]_{0}^{1}$$

$$= -\left[-\frac{1}{3} - \frac{3}{4}\right] + \left[\frac{3}{4} - \frac{1}{3}\right] = \frac{3}{2} \text{ sq. units.}$$

14. Solving given curves, we have $x - \sin x = x + \cos x$

$\therefore$ $\tan x = -1$

For smallest bounded area we must consider two consecutive solutions,

i.e., $x = -\frac{\pi}{4}, \frac{3\pi}{4}$

$\therefore$ Required area $= \left|\int_{-\pi/4}^{3\pi/4}[(x+\cos x) - (x - \sin x)]dx\right|$

$$= \left|\int_{-\pi/4}^{3\pi/4}[\cos x + \sin x]dx\right|$$

$$= [\sin x - \cos x]_{-\pi/4}^{3\pi/4}$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$$

Exercise 9.3

1. According to the given conditions

$$\int_0^t\left[f(x) - (x^4 - 4x^2)\right]dx = k\int_0^t\left[(2x^2 - x^3) - f(x)\right]dx$$

Differentiating both sides w.r.t. t, we get

$$f(t) - (t^4 - 4t^2) = k(2t^2 - t^3 - f(t))$$

or $(1 + k)f(t) = k2t^2 - kt^3 + t^4 - 4t^2$

$$\Rightarrow \quad f(t) = \frac{1}{k+1}[t^4 - kt^3 + (2k-4)t^2]$$

Hence, required f is given by $f(x) = \frac{1}{k+1}(x^4 - kx^3 + 2(k-2)x^2)$.

2. Area $= \int_1^b f(x)\,dx = \sqrt{b^2+1} - \sqrt{2}$

Differentiating both sides w.r.t. b, we get

$$f(b) = \frac{b}{\sqrt{b^2+1}}$$

$$\therefore \quad f(x) = \frac{x}{\sqrt{x^2+1}}$$

3. According to the question $\int_0^a f(x)\,dx = \frac{a^2}{2} + \frac{a}{2}\sin a + \frac{\pi}{2}\cos a$

$\therefore \quad f(a) = \frac{d}{da}\left(\frac{a^2}{2} + \frac{a}{2}\sin a + \frac{\pi}{2}\cos a\right)$

$= a + \frac{a}{2}\cos a + \frac{1}{2}\sin a - \frac{\pi}{2}\sin a$

$\therefore \quad f\left(\frac{\pi}{2}\right) = \frac{1}{2}$

4. According to the question $\int_0^x f(x)dx = \lambda\{f(x)\}^{n+1}$

Where λ is constant of proportionality

Differentiating both sides w.r.t x,

$f(x) = \lambda(n+1)(f(x))^n f'(x)$

or $(f(x)^{n-1})f'(x) = \frac{1}{\lambda(n+1)}$

Integrating both sides w.r.t x, $\frac{(f(x))^n}{n} = \frac{x}{\lambda(n+1)} + C$

$f(0) = 0, \therefore C = 0$

$(f(x))^n = \frac{nx}{\lambda(n+1)}$

$f(1) = 1$

$\therefore \quad \frac{n}{\lambda(n+1)} = 1$

$\therefore \quad f(x) = x^{1/n}$

5. $R_1 : |x+y| + |x-y| \le 4$

Considering $x + y + x - y \le 4$, we get $x \le 2$

Similarly, for other cases, we get $x \ge -2$, $y \le 2$ and $y \ge -2$

$\therefore$ R_1 is the square bounded by lines $x = \pm 2$ and $y = \pm 2$

$R_2 : |x| \le 1$

$\therefore \quad -1 \le x \le 1$

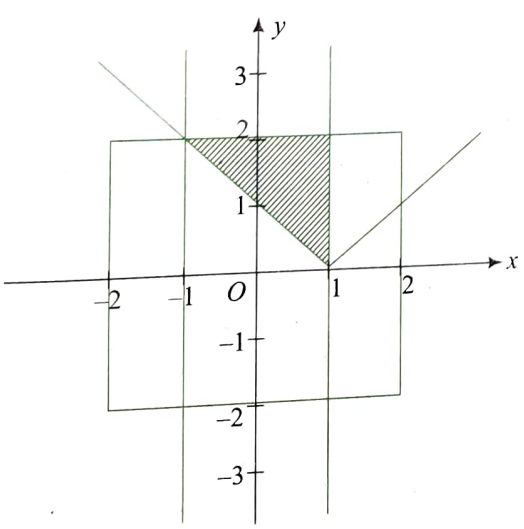
$\therefore$ R_2 is the region of all points between $x = \pm 1$

$y \ge \sqrt{x^2 - 2x + 1}$

$R_3 : y \ge |x - 1|$

$\therefore$ R_3 is the open triangular region above x-axis between the lines $y = 1 - x$ and $y = x - 1$

$\therefore$ Required common region is as shown in the following figure.



Required Area = Area of the shaded portion

$= \frac{1}{2} \times 2 \times 2$ sq. units = 2 sq. units

6. R_1: points $P(x, y)$ are nearer to $(1, 0)$ than to $x = -1$

$\therefore \quad \sqrt{(x-1)^2 + y^2} < |x+1|$

$\Rightarrow \quad y^2 < 4x$

$\Rightarrow$ point P lie inside parabola $y^2 = 4x$

R_2: Points $P(x, y)$ are nearer to $(0, 0)$ than to $(8, 0)$

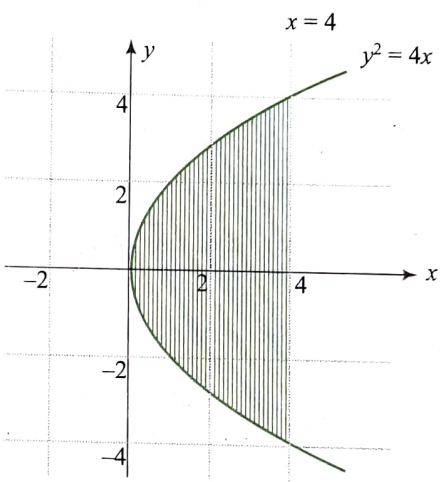
$\therefore \quad |x| < |x - 8|$

$\Rightarrow \quad x^2 < x^2 - 16x + 64$

$\Rightarrow \quad x < 4$

$\Rightarrow$ point P lie to the left side of line $x = 4$

The area of common region of R_1 and R_2 is area bounded by $x = 4$ and $y^2 = 4x$.



$\therefore$ Required area $= 2\int_0^4 \left(4 - \frac{y^2}{4}\right)dy$ (Integrating along x-axis)

$= 2\left[4y - \frac{y^3}{12}\right]_0^4$

$= 2\left[16 - \frac{64}{12}\right]$

$= \frac{64}{3}$ sq. units

7. Required area, $A = 2\int_0^{1/2} f^{-1}(x)\,dx$

Let $f^{-1}(x) = t$

$\Rightarrow \quad A = 2\int_0^1 tf'(t)dt$

$= 2\left[tf(t)\right]_0^1 - 2\int_0^1 f(t)dt$

$= 2f(1) - \int_0^1 \frac{2t}{1+t^2}dt$

$= 1 - [\log(1 + t^2)]_0^1$

$= 1 - \log 2$

Exercises

Single Correct Answer Type

1. (1) Clearly t can be any real number

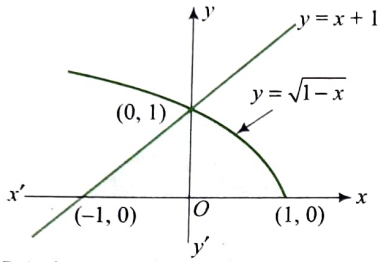
Let $t = \tan\theta \Rightarrow x = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

$\Rightarrow \quad x = \cos 2\theta$ and $y = \dfrac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$

$\Rightarrow \quad x^2 + y^2 = 1$

Thus, required area is π sq. units.

2. (4)



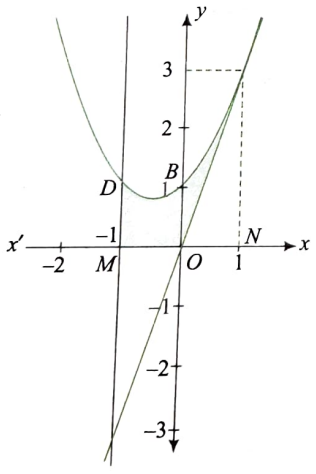
Required area = Shaded region

$$= \int_0^1 (x_2 - x_1)dy \text{ (integrating along } y\text{-axis)}$$

$$= \int_0^1 [(1-y^2)-(y-1)]dy$$

$$= \frac{7}{6} \text{ sq. unit}$$

3. (3) Given $y = x^2 + x + 1 = \left(x+\dfrac{1}{2}\right)^2 + \dfrac{3}{4} \Rightarrow y - \dfrac{3}{4} = \left(x+\dfrac{1}{2}\right)^2$.



This is a parabola with vertex at $\left(-\dfrac{1}{2}, \dfrac{3}{4}\right)$ and the curve is concave upwards.

$$y = x^2 + x + 1 \text{ or } \frac{dy}{dx} = 2x + 1 \text{ or } \left(\frac{dy}{dx}\right)_{(1,3)} = 3$$

Equation of the tangent at $A(1, 3)$ is $y = 3x$

Required (shaded) area = Area $ABDMN$ – Area ONA

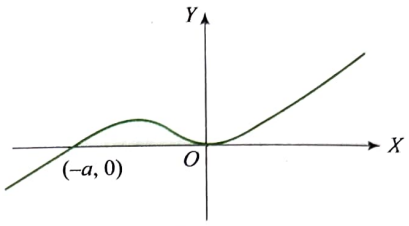
$$\text{Now, area } ABDMN = \int_{-1}^{1}(x^2+x+1)dx = 2\int_0^1 (x^2+1) = \frac{8}{3}$$

$$\text{Area of } ONA = \frac{1}{2}\times 1\times 3 = \frac{3}{2}.$$

$$\therefore \quad \text{Required area} = \frac{8}{3} - \frac{3}{2} = \frac{16-9}{6} = \frac{7}{6} \text{ sq. units.}$$

4. (4) The curve is $y = \dfrac{x^2(x+a)}{a^2}$, which is a cubic polynomial.

Since $\dfrac{x^2(x+a)}{a^2} = 0$ has a repeated root $x = 0$, it touches x-axis at $(0, 0)$ and intersects at $(-a, 0)$.



Required area $= \int_{-a}^{0} y\,dx$

$$= \int_{-a}^{0}\left[\frac{x^2(x+a)}{a^2}\right]dx$$

$$= a^2/12 \text{ sq. units.}$$

5. (2) The curve is $y = 2x^4 - x^2 = x^2(2x^2 - 1)$

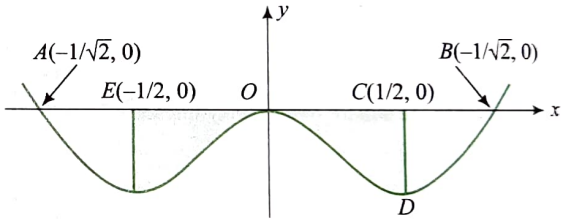
The curve is symmetrical about the axis of y.

Also, it is a polynomial of 4 degree having roots $0, 0, \pm\dfrac{1}{\sqrt{2}}$.

$x = 0$ is a repeated root. Hence, graph touches at $(0, 0)$.

The curve intersects the axes at $O(0, 0)$, $A(-1/\sqrt{2}, 0)$ and $B(1/\sqrt{2}, 0)$.

Thus, the graph of the curve is as shown in the figure.



Here, $y \le 0$, as x varies from $x = -1/2$ to $x = 1/2$

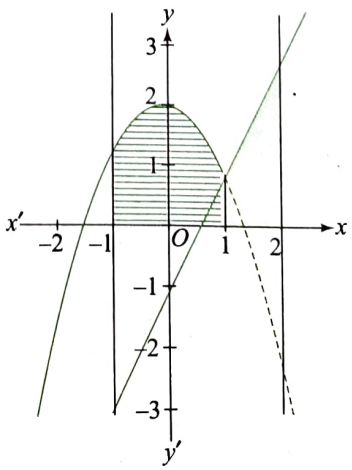
$\therefore$ required area = 2 Area $OCDO$

$$= 2\left|\int_0^{1/2} y\,dx\right|$$

$$= 2\left|\int_0^{1/2}(2x^4 - x^2)dx\right|$$

$$= 7/120 \text{ sq. units}$$

6. (1)



$$A = \int_{-1}^{1}(-x^2+2)dx + \int_{1}^{2}(2x-1)dx$$

$$= \left(-\frac{x^3}{3}+2x\right)_{-1}^{1} + (x^2-x)_1^2$$

$$= \frac{16}{3} \text{ sq. units}$$

7. (3) $a^2x^2 + ax + 1$ is clearly positive for all real values of x. Area under consideration

$$A = \int_0^1 \left(a^2x^2+ax+1\right)dx$$

$$= \frac{a^2}{3}+\frac{a}{2}+1$$

$$= \frac{1}{6}\left(2a^2+3a+6\right)$$

$$= \frac{1}{6}\left(2\left(a^2+\frac{3}{2}a+\frac{9}{16}\right)+6-\frac{18}{16}\right)$$

$$= \frac{1}{6}\left(2\left(a+\frac{3}{4}\right)^2+\frac{39}{8}\right),$$

which is clearly minimum for $a = -\frac{3}{4}$.

8. (3) $$A = \int_{\frac{2\pi}{3k}}^{\frac{\pi}{k}}(\sin kx)dx - \int_{\frac{\pi}{k}}^{\frac{5\pi}{3k}}(\sin kx)dx$$

$$= -\left(\frac{\cos kx}{k}\right)_{2\pi/3k}^{\pi/k} + \left(\frac{\cos kx}{k}\right)_{\pi/k}^{5\pi/3k}$$

$$= -\frac{1}{k}\left[-1-\cos\frac{2\pi}{3}\right]+\frac{1}{k}\left[\cos\frac{5\pi}{3}-\cos\pi\right]$$

$$= -\frac{1}{k}\left[-1+\frac{1}{2}\right]+\frac{1}{k}\left[\frac{1}{2}+1\right]$$

$$= \frac{1}{2k}+\frac{3}{2k}$$

$$= \frac{2}{k} < 2$$

9. (1) $y = x(1 - \log_e x)$

It meets x-axis, if $x(1 - \log_e x) = 0$

$\therefore$ $x = 0$ or $x = e$

$\therefore$ Required area $= \int_0^e x(1-\log_e x)dx$

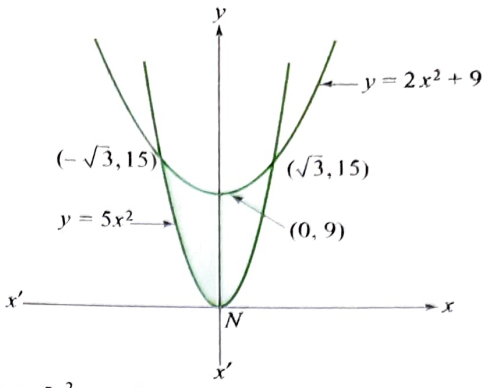
$$= \int_0^e x\,dx - \int_0^e x\log x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^e - \left[\frac{x^2}{2}\log_e x\right]_0^e + \int_0^e \frac{x}{2}dx$$

$$= \frac{e^2}{2} - \left[\frac{e^2}{2} - \lim_{x\to 0}\frac{x^2}{2}\log_e x\right] + \frac{e^2}{4}$$

$$= \frac{e^2}{4}$$

10. (1)



Given $5x^2 - y = 0$ ...(1)

and $2x^2 - y + 9 = 0$ ...(2)

Eliminating y, we get

$5x^2 - (2x^2 + 9) = 0$

or $3x^2 = 9$ or $x = -\sqrt{3}, \sqrt{3}$

$\therefore$ Required area $= 2\int_0^{\sqrt{3}}\left((2x^2+9)-5x^2\right)dx$

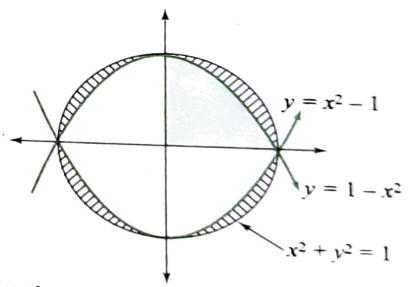
$$= 2\int_0^{\sqrt{3}}\left(9-3x^2\right)dx$$

$$= 2\left[9x-x^3\right]_0^{\sqrt{3}}$$

$$= 2[9\sqrt{3}-3\sqrt{3}]$$

$$= 12\sqrt{3} \text{ sq. units}$$

11. (1)



The dotted area is

$$A = \int_0^1(1-x^2)dx = \left(x-\frac{x^3}{3}\right)_0^1 = 1-\frac{1}{3} = \frac{2}{3}.$$

Hence, area bounded by circle $x^2 + y^2 = 1$ and

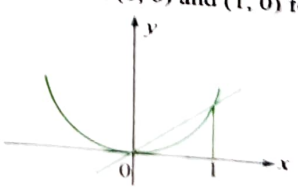
$|y| = 1 - x^2$

= Lined area

= Area of circle − Area bounded by $|y| = 1 - x^2$

$$= \pi - 4\times\left(\frac{2}{3}\right) = \frac{3\pi-8}{3} \text{ sq. units.}$$

12. (4) $y = x$ intersect $y = x^n$ at (0, 0) and (1, 0) for all $n \in N$



Area $A_n = \int_0^1(x-x^n)dx$

$$= \left[\frac{x^2}{2} - \frac{x^{n+1}}{n+1}\right]_0^1$$

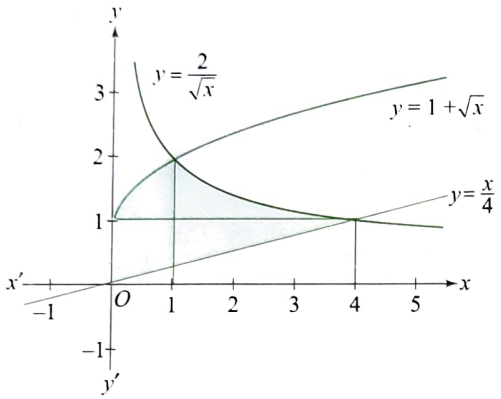
$$= \frac{1}{2} - \frac{1}{n+1}$$

$$= \frac{n-1}{2(n+1)}$$

Thus, $A_2 \cdot A_3 \cdot A_4 \ldots A_n = \frac{1}{2^{n-1}}\left(\frac{1}{3}\cdot\frac{2}{4}\cdot\frac{3}{5}\ldots\frac{n-1}{n+1}\right)$

$$= \frac{1}{2^{n-2}\cdot n(n+1)}$$

13. (3)



$$A_1 = \int_0^1 \left(1 + \sqrt{x} - \frac{x}{4}\right)dx$$

$$= \left[x + \frac{2x^{3/2}}{3} - \frac{x^2}{8}\right]_0^1 = 1 + \frac{2}{3} - \frac{1}{8} = \frac{37}{24}.$$

$$A_2 = \int_1^4 \left(\frac{2}{\sqrt{x}} - \frac{x}{4}\right)dx$$

$$= \left[4\sqrt{x} - \frac{x^2}{8}\right]_1^4$$

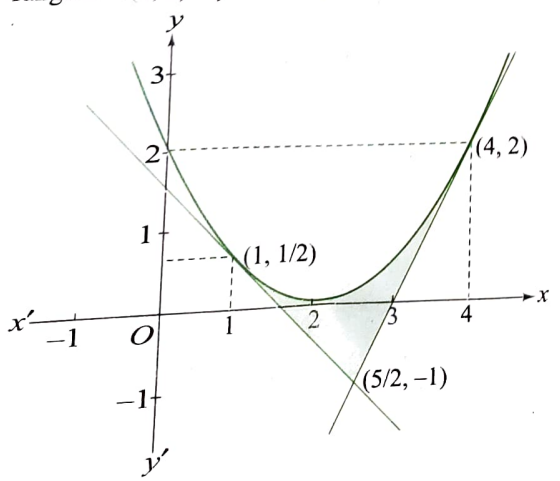
$$= \left[8 - 2 - 4 + \frac{1}{8}\right] = \frac{17}{8}$$

or $A = A_1 + A_2 = \frac{88}{24} = \frac{11}{3}$ sq. units

14. (1) $y = \frac{x^2}{2} - 2x + 2 = \frac{(x-2)^2}{2}$,

$$\frac{dy}{dx} = x - 2, \quad \left(\frac{dy}{dx}\right)_{x=1} = -1, \quad \left(\frac{dy}{dx}\right)_{x=4} = 2$$

Thus, tangent at (1, 1/2) is $y - 1/2 = -1(x - 1)$ or $2x + 2y - 3 = 0$
Tangent at (4, 2) is $y - 2 = 2(x - 4)$ or $2x - y - 6 = 0$



Hence, $A = \int_1^{5/2}\left(\frac{x^2}{2} - 2x + 2 - \frac{3-2x}{2}\right)dx$

$$+ \int_{5/2}^4 \left(\frac{x^2}{2} - 2x + 2 - (2x-6)\right)dx$$

$$= \int_1^4 \left(\frac{x^2}{2} - 2x + 2\right)dx - \int_1^{5/2}\left(\frac{3-2x}{2}\right)dx - \int_{5/2}^4 (2x-6)dx$$

$$= \left(\frac{x^3}{6} - x^2 + 2x\right)_1^4 - \frac{1}{2}\left(3x - x^2\right)_1^{5/2} - \left(x^2 - 6x\right)_{5/2}^4$$

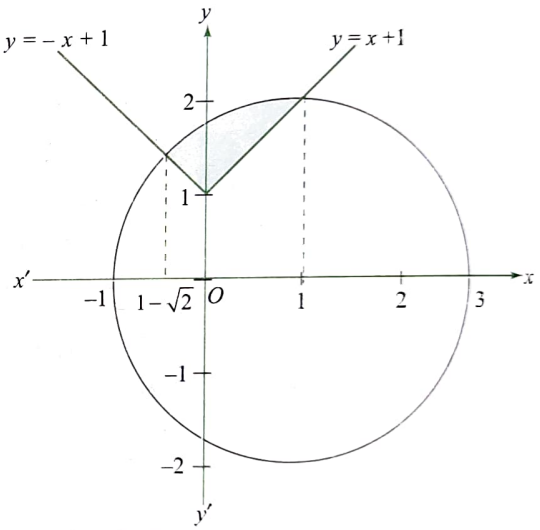
$$= \left(\frac{63}{6} - 15 + 6\right) - \frac{1}{2}\left(3\times\frac{3}{2} - \left(\frac{25}{4} - 1\right)\right)$$

$$-\left(\left(16 - \frac{25}{4}\right) - 6\left(4 - \frac{5}{2}\right)\right)$$

$$= \frac{3}{2} - \frac{1}{2}\left(\frac{9}{2} - \frac{21}{4}\right) - \left(\frac{39}{4} - 6\left(\frac{3}{2}\right)\right)$$

$$= \frac{9}{8} \text{ sq. units}$$

15. (1)



$x^2 + y^2 - 2x - 3 = 0$

or $(x-1)^2 + y^2 = 4$

$$A = \int_{1-\sqrt{2}}^0 \left(\sqrt{4-(x-1)^2} - (-x+1)\right)dx$$

$$+ \int_0^1 \left(\sqrt{4-(x-1)^2} - (x+1)\right)dx$$

$$= \frac{x-1}{2}\sqrt{4-(x-1)^2} + \frac{4}{2}\sin^{-1}\frac{x-1}{2} + \frac{x^2}{2} - x\Big|_{1-\sqrt{2}}^0$$

$$+ \frac{x-1}{2}\sqrt{4-(x-1)^2} + \frac{4}{2}\sin^{-1}\frac{x-1}{2} - \frac{x^2}{2} - x\Big|_0^1$$

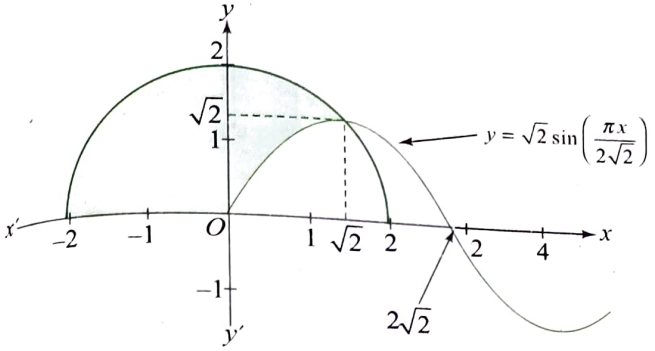
$$= \left(-\frac{\sqrt{3}}{2} - \frac{\pi}{3}\right) - \left(\frac{-\sqrt{2}}{2}\sqrt{2} - \frac{\pi}{2} + \frac{3-2\sqrt{2}}{2} - 1 + \sqrt{2}\right)$$

$$+ \left(-\frac{1}{2} - 1\right) - \left(-\frac{\sqrt{3}}{2} - \frac{\pi}{3}\right)$$

$$= -\left(-1 - \frac{\pi}{2} + \frac{3}{2} - \sqrt{2} - 1 + \sqrt{2}\right) - \frac{3}{2}$$

$$= \frac{\pi}{2} - 1 \text{ sq. units.}$$

6. (4) $y = \sqrt{4-x^2}$, $y = \sqrt{2}\sin\left(\frac{x\pi}{2\sqrt{2}}\right)$ intersect at $x = \sqrt{2}$



Area to the left of y-axis is π

Area to the right of y-axis

$$= \int_0^{\sqrt{2}} \left(\sqrt{4-x^2} - \sqrt{2}\sin\frac{x\pi}{2\sqrt{2}}\right)dx$$

$$= \left(\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right)_0^{\sqrt{2}} + \left(\frac{4}{\pi}\cos\frac{x\pi}{2\sqrt{2}}\right)_0^{\sqrt{2}}$$

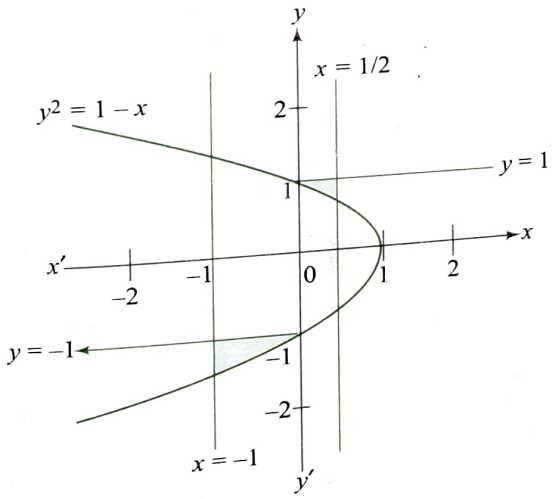
$$= \left(1 + 2\times\frac{\pi}{4}\right) + \frac{4}{\pi}(0-1)$$

$$= 1 + \frac{\pi}{2} - \frac{4}{\pi}$$

$$= \frac{2\pi + \pi^2 - 8}{2\pi} \text{ sq. units.}$$

$\therefore$ Ratio $= \dfrac{2\pi^2}{2\pi + \pi^2 - 8}$.

7. (1)



From the figure,

$$A = \int_{-1}^{0}\left(-1 - \left(-\sqrt{1-x}\right)\right)dx + \int_0^{1/2}(1-\sqrt{1-x})dx$$

$$= \left[-x - \frac{(1-x)^{3/2}}{3/2}\right]_{-1}^{0} + \left[x + \frac{(1-x)^{3/2}}{3/2}\right]_0^{1/2}$$

$$= \left[-\frac{2}{3} - \left(1 - \frac{2\times 2^{3/2}}{3}\right)\right] + \left[\frac{1}{2} + \frac{2}{3\times 2^{3/2}} - \frac{2}{3}\right]$$

$$= \frac{2}{3\times 2^{3/2}} + \frac{2\times 2^{3/2}}{3} - \frac{4}{3} - \frac{1}{2}$$

$$= \frac{3}{\sqrt{2}} - \frac{4}{3} - \frac{1}{2}$$

$$= \frac{3}{\sqrt{2}} - \frac{11}{6} \text{ sq. units}$$

18. (2) Given curves are $y = \log_e x$ and $y = (\log_e x)^2$

Solving $\log_e x = (\log_e x)^2 \Rightarrow \log_e x = 0, 1 \Rightarrow x = 1$ and $x = e$

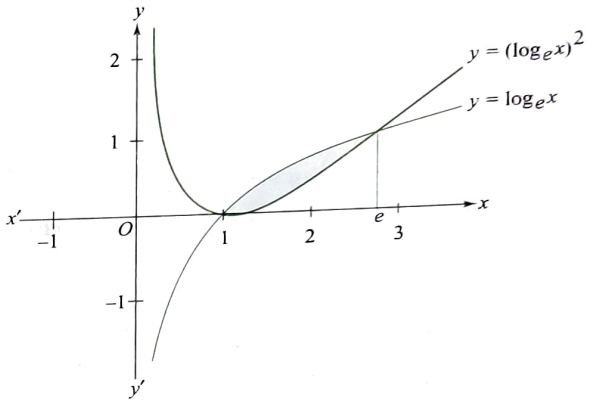
Also, for $1 < x < e$, $0 < \log_e x < 1 \Rightarrow \log_e x > (\log_e x)^2$

For $x > e$, $\log_e x < (\log_e x)^2$

$y = (\log_e x)^2 > 0$ for all $x > 0$

and when $x \to 0$, $(\log_e x)^2 \to \infty$.

From these information, we can plot the graph of the functions.



$\therefore$ Required area $= \int_1^e (\log x - (\log_e x)^2)dx$

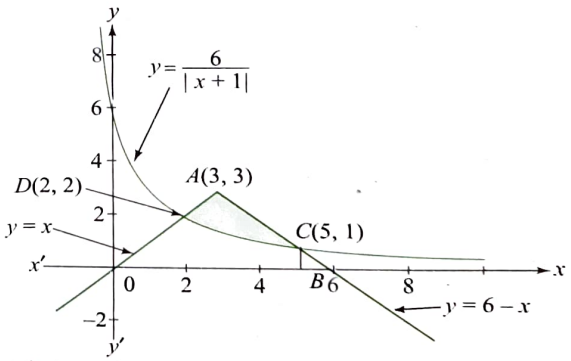
$$= \int_1^e \log x\,dx - \int_1^e (\log_e x)^2\,dx$$

$$= \left[x\log_e x - x\right]_1^e - \left[x(\log_e x)^2\right]_1^e + \int_1^e \frac{2\log_e x}{x}x\,dx$$

$$= 1 - e + 2\left[x\log_e x - x\right]_1^e$$

$$= 3 - e \text{ sq. units.}$$

19. (3)



First consider $y = 3 - |3 - x|$

For $x < 3$; $y = 3 - (3 - x) = x$

For $x \geq 3$; $y = 3 - (x - 3) = 6 - x$

Consider $y = \dfrac{6}{|x+1|}$

For $x < -1$; $y = \dfrac{6}{-1-x}$

$\Rightarrow \quad (1+x)y = -6$

For $x > -1$; $\quad y = \dfrac{6}{x+1}$

$$\text{Required area} = \left[\int_2^3 \left(x - \frac{6}{x+1}\right)dx + \int_3^5 \left((6-x) - \frac{6}{x+1}\right)dx\right]$$

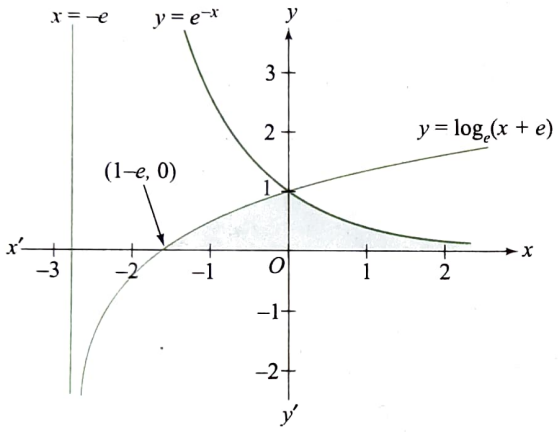
$$= \left[\left(\frac{x^2}{2}\right)_2^3 + \left(6x - \frac{x^2}{2}\right)_3^5 - \left(6\log(x+1)\right)_2^5\right]$$

$$= \left[\frac{5}{2} + 4 - 6\log 2\right]$$

$$= \frac{13}{2} - 6 \ln 2 \text{ sq. units.}$$

20. (1) $y = \log_e(x+e)$, $x = \log_e\left(\dfrac{1}{y}\right)$ or $y = e^{-x}$.

For $y = \log_e(x+e)$, shift the graph of $y = \log_e x$, e units to the left hand side.



$$\text{Required area} = \int_{1-e}^{0} \log_e(x+e)dx + \int_0^\infty e^{-x}\,dx$$

$$= |x \log_e(x+e)|_{1-e}^{0} - \int_{1-e}^{0} \frac{x}{x+e}dx - |e^{-x}|_0^\infty$$

$$= \int_0^{1-e}\left(1 - \frac{e}{x+e}\right)dx - e^{-\infty} + e^0$$

$$= |x - e\log(x+e)|_0^{1-e} - 0 + 1$$

$$= 1 - e + e\log e + 1 = 2 \text{ sq. units.}$$

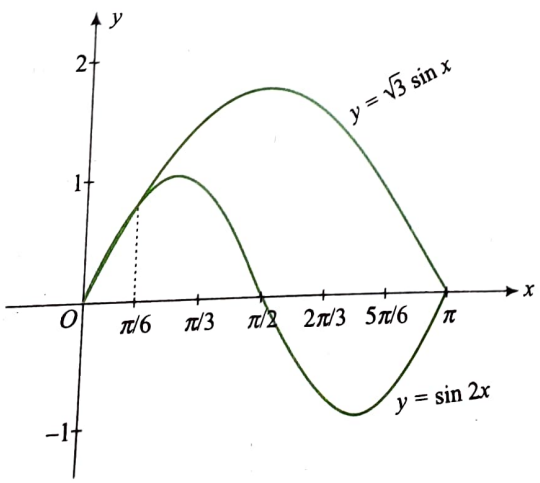
21. (1) We have $y = \sin 2x$, $y = \sqrt{3}\sin x$

Solving, we get $\sin 2x = \sqrt{3}\sin x$

$\therefore \quad 2\sin x \cos x = \sqrt{3}\sin x$

$\therefore \quad \sin x = 0$ or $\cos x = \sqrt{3}/2$

$\therefore \quad x = 0, \pi$ or $x = \pi/6$



$\therefore \quad$ Required area

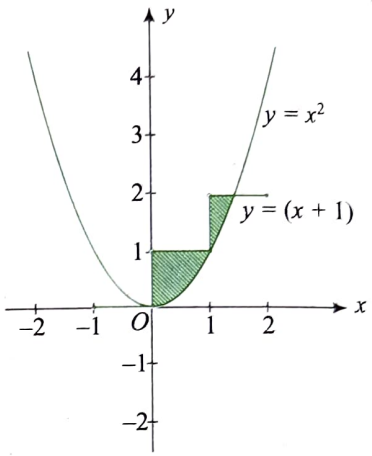
$$= \int_0^{\pi/6} (\sin 2x - \sqrt{3}\sin x)dx + \int_{\pi/6}^{\pi} (\sqrt{3}\sin x - \sin 2x)dx$$

$$= \left[-\frac{\cos 2x}{2} + \sqrt{3}\cos x\right]_0^{\pi/6} + \left[-\sqrt{3}\cos x + \frac{\cos 2x}{2} + \right]_{\pi/6}^{\pi}$$

$$= \left[\frac{7}{4} - \sqrt{3}\right] + \left[\sqrt{3} + \frac{7}{4}\right]$$

$$= \frac{7}{2}$$

22. (2)



$$\text{Area} \int_0^1 (1-x^2)\,dx + \int_1^{\sqrt{2}} (1-x^2)\,dx = \left[x - \frac{x^3}{3}\right]_0^1 + \left[x - \frac{x^3}{3}\right]_1^{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{3}$$

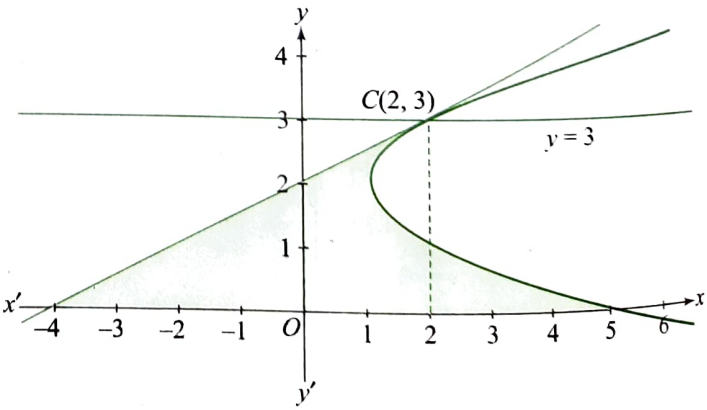
23. (3) Given parabola is $(y-2)^2 = x - 1$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2(y-2)}$$

When $y = 3$, $x = 2$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2(3-2)} = \frac{1}{2}.$$

Tangent at (2, 3) is $y - 3 = \dfrac{1}{2}(x-2)$ or $x - 2y + 4 = 0$



$\therefore \quad$ Required area $= \int_0^3 \left((y-2)^2 + 1\right)dy - \int_0^3 (2y-4)\,dy$

$$= \left|\frac{(y-2)^3}{3} + y\right|_0^3 - |y^2 - 4y|_0^3$$

$$= \frac{1}{3} + 3 + \frac{8}{3} - (9 - 12) = 9 \text{ sq. units.}$$

24. (1) Curve tracing : $y = x\,e^x$

Let $\frac{dy}{dx} = 0 \Rightarrow e^x + xe^x = 0$ or $x = -1$.

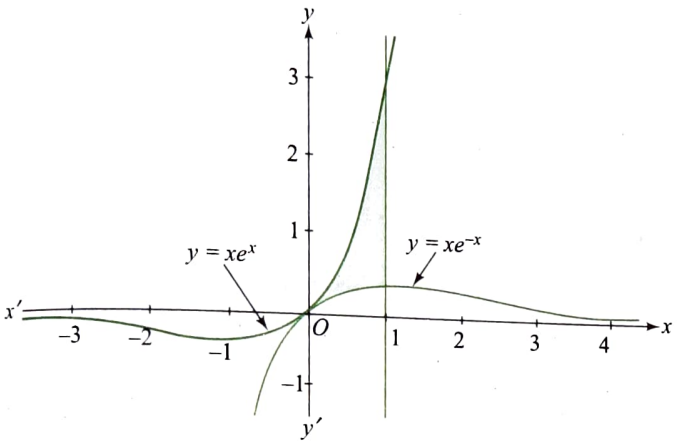
Also, at $x = -1$, $\frac{dy}{dx}$ changes sign from –ve to +ve,

Hence, $x = -1$ is a point of minima.

When $x \to \infty$, $y \to \infty$

Also $\lim\limits_{x\to-\infty} xe^x = \lim\limits_{x\to-\infty} \frac{x}{e^{-x}} = \lim\limits_{x\to-\infty} \frac{1}{-e^{-x}} = 0$

With similar types of arguments, we can draw the graph of $y = x\,e^{-x}$.



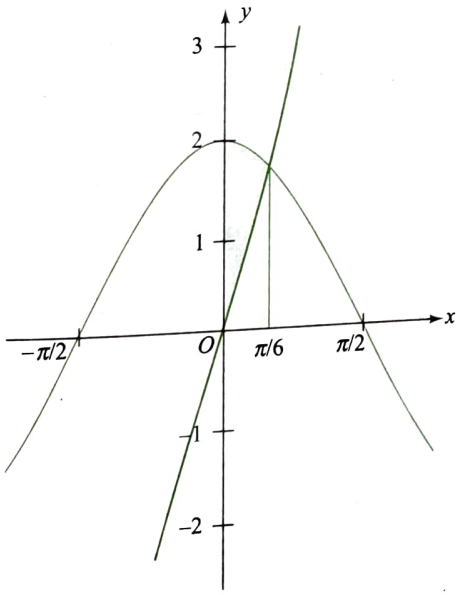
Required area $= \int_0^1 xe^x dx - \int_0^1 xe^{-x} dx$

$= \left[xe^x\right]_0^1 - \int_0^1 e^x dx - \left(\left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x} dx\right)$

$= e - (e-1) - \left(-e^{-1} - (e^{-1} - 1)\right) = \frac{2}{e}$ sq. units.

25. (2) Solving $2\cos x = 3\tan x$, we get

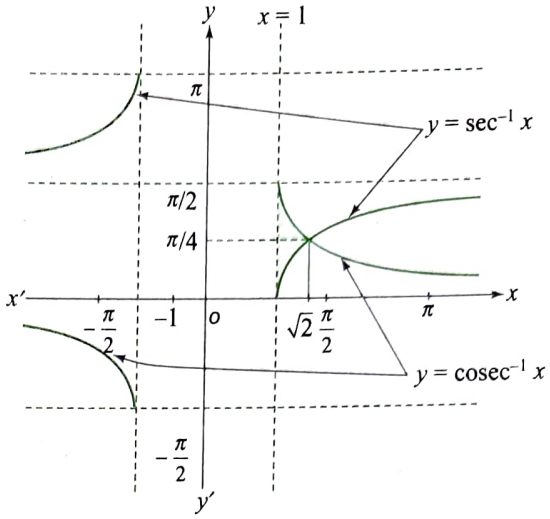
$2 - 2\sin^2 x = 3\sin x$ or $\sin x = \frac{1}{2}$ or $x = \frac{\pi}{6}$.



Required area $= \int_0^{\pi/6} (2\cos x - 3\tan x)dx$

$= 2\sin x - 3\log\sec x\Big|_0^{\pi/6} = 1 - 3\ln 2 + \frac{3}{2}\ln 3$ sq. units.

26. (1)



Integrating along x-axis, we get

$$A = \int_1^{\sqrt{2}} \left(\text{cosec}^{-1} x - \sec^{-1} x\right) dx$$

Integrating along y-axis, we get

$$A = 2\int_0^{\pi/4} (\sec y - 1)dy$$

$$= 2\left[\log|\sec y + \tan y| - y\right]_0^{\pi/4}$$

$$= 2\left[\log\left|\sqrt{2}+1\right| - \frac{\pi}{4}\right]$$

$$= \log(3 + 2\sqrt{2}) - \frac{\pi}{2} \text{ sq. units.}$$

27. (4) $y^2(2-x) = x^3$

$\therefore \quad y^2 = \frac{x^3}{2-x}$

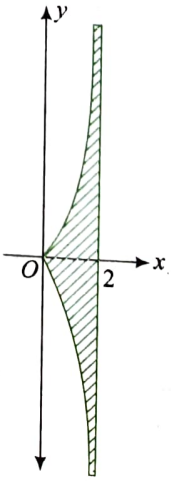
$\therefore \quad y = \pm\frac{x^{3/2}}{\sqrt{2-x}}$, clearly $x \in [0, 2)$

Consider $y = \frac{x^{3/2}}{\sqrt{2-x}}$

When $x = 0$, $y = 0$

Also, when x increases from '0' to '2', y increases from '0' to '∞'

Hence graph of given relation is as shown in the following figure:



Required Area is $A = 2\int_0^2 \frac{x^{3/2}}{\sqrt{2-x}}\,dx$

Put $x = 2\sin^2\theta$

$$\therefore \quad A = 2\int_0^{\pi/2} \frac{2\sqrt{2}\sin^3\theta}{\sqrt{2-2\sin^2\theta}} 4\sin\theta\cos\theta\,d\theta$$

$$= 16\int_0^{\pi/2} \sin^4\theta\,d\theta$$

$$= 16\int_0^{\pi/2} \frac{(1-\cos 2\theta)^2}{4}\,d\theta$$

$$= 4\int_0^{\pi/2} (1-2\cos 2\theta+\cos^2 2\theta)\,d\theta$$

$$= 4\int_0^{\pi/2} \left(1-2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)d\theta$$

$$= 4\left(\theta+\sin 2\theta+\frac{\theta+\frac{\sin 4\theta}{4}}{2}\right)_0^{\pi/2}$$

$$= 4\left(\frac{\pi}{2}+0+\frac{\pi}{4}\right)$$

$$= 3\pi$$

28. (4) $f(x) = x^2 + \cos x$

$\Rightarrow \quad f'(x) = 2x - \sin x$

$\Rightarrow \quad f'\left(\frac{\pi}{2}\right) = \pi - 1$

$\Rightarrow$ Equation of normal at $x = \frac{\pi}{2}$ is

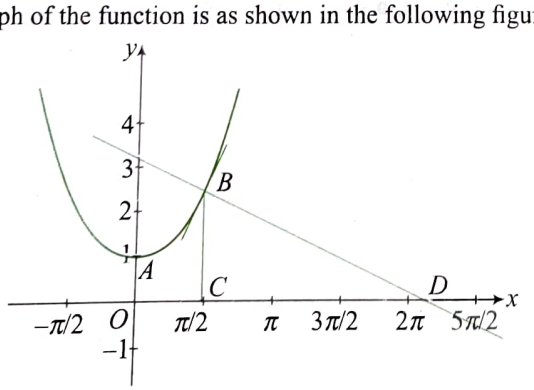
$$\left(y-\frac{\pi^2}{4}\right) = \frac{1}{1-\pi}\left(x-\frac{\pi}{2}\right)$$

It meets x-axis at $x = \frac{(\pi-1)\pi^2}{4} + \frac{\pi}{2}$

Also $f'(x) = 2x - \sin x > 0$ for $x > 0$ and < 0 for $x < 0$

So $f(x)$ increases for $x \in (0, \infty)$ and decreases for $x \in (-\infty, 0)$

Graph of the function is as shown in the following figure.



Required area

= Area $OABCO$ + Area of ΔBCD

$$= \int_0^{\pi/2} (x^2+\cos x)\,dx + \frac{1}{2}\left[\frac{(\pi-1)\pi^2}{4}+\frac{\pi}{2}-\frac{\pi}{2}\right]\times\frac{\pi^2}{4}$$

$$= \left[\frac{x^3}{3}+\sin x\right]_0^{\pi/2} + \frac{(\pi-1)\pi^2}{8}$$

$$= \frac{\pi^3}{24}+1+\frac{\pi^4}{32}(\pi-1)$$

$$= \frac{\pi^5}{32}-\frac{\pi^4}{32}+\frac{\pi^3}{24}+1$$

29. (2) $\int_0^x e^t(\log_e \sec t - \sec^2 t)\,dt$

$$= \int_0^x e^t[(\log_e \sec t - \tan t)+(\tan t - \sec^2 t)]\,dt$$

$$= \left[e^t(\log_e \sec t - \tan t)\right]_0^x$$

$$= e^x(\log_e \sec x - \tan x)$$

$\therefore$ Required area,

$$A = \int_0^{\pi/3} [e^x(\log_e \sec x - \tan x) - (-2e^x \tan x)]\,dx$$

$$= \int_0^{\pi/3} e^x(\log_e \sec x + \tan x)]\,dx$$

$$= \left[e^x \log_e \sec x\right]_0^{\pi/3}$$

$$= e^{\pi/3}\log_e \sec\frac{\pi}{3}$$

$$= e^{\pi/3}\log_e 2$$

30. (2) $ay^2 = x^2(a-x)$ or $y = \pm x\sqrt{\frac{a-x}{a}}$

Curve tracing : $y = x\sqrt{\frac{a-x}{a}}$

We must have $x \le a$

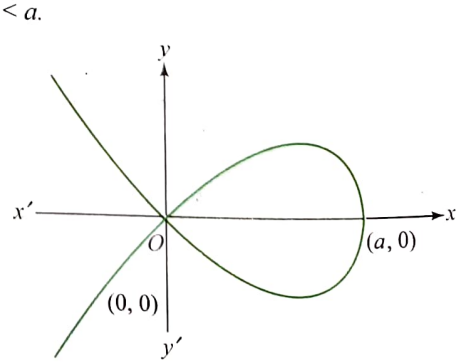
For $0 < x \le a$, $y > 0$ and for $x < 0$, $y < 0$

Also $y = 0 \Rightarrow x = 0, a$

Curve is symmetrical about x-axis.

When $x \to -\infty$, $y \to -\infty$

Also, it can be verified that y has only one point of maxima for $0 < x < a$.



Area $= 2\int_0^a x\sqrt{\frac{a-x}{a}}\,dx$

$\sqrt{\frac{a-x}{a}} = t$

$\Rightarrow \quad 1-\frac{x}{a}=t^2$ or $x=a(1-t^2)$

$\Rightarrow \quad A=2\int_1^0 a(1-t^2)t(-2at)dt$

$=4a^2\int_0^1(t^2-t^4)dt$

$=4a^2\left[\frac{t^3}{3}-\frac{t^5}{5}\right]_0^1$

$=4a^2\left[\frac{1}{3}-\frac{1}{5}\right]$

$=\frac{8a^2}{15}$ sq. units.

1. **(4)** $x=y^2-1$ is a parabola having vertex at $(-1, 0)$ and concave right hand side.

$x=|y|\sqrt{1-y^2}$

Here $x \geq 0$ as R.H.S. is positive

When $y>0$,

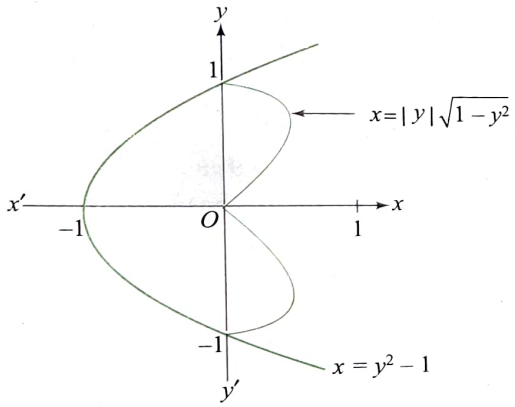
$x=y\sqrt{1-y^2}$ (1)

(1) meets y-axis at $(0, 0)$, $(0, 1)$; thus,

$\frac{dx}{dy}=\sqrt{1-y^2}-\frac{y^2}{\sqrt{1-y^2}}=\frac{1-2y^2}{\sqrt{1-y^2}}$

For $\frac{dx}{dy}=0$, $y=\frac{1}{\sqrt{2}}$

Hence, graph of (1) is as shown in the figure, with $x=-y\sqrt{1-y^2}$ $(y<0)$ as its mirror image.



$A=2\int_0^1\left[y\sqrt{1-y^2}-(y^2-1)\right]dy$

$=2$ sq. units (1)

2. **(4)** $4y^2=x^2(4-x^2)$

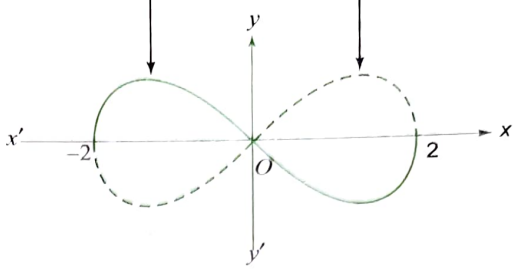
or $y=\pm\frac{1}{2}\sqrt{x^2(4-x^2)}$

$=\pm\frac{x}{2}\sqrt{(4-x^2)}$

$\Rightarrow \quad y=-\frac{x}{2}\sqrt{(4-x^2)}$;

$y=\frac{x}{2}\sqrt{(4-x^2)}$

$y=-\frac{x}{2}\sqrt{(4-x^2)}$: $y=\frac{x}{2}\sqrt{(4-x^2)}$



$\therefore$ Area $(A)=4\times\int_0^2\frac{x}{2}\sqrt{(4-x^2)}\,dx$

Let $4-x^2=t \Rightarrow -2x\,dx=dt$

$\Rightarrow \quad A=\int_0^4\sqrt{t}\,dt=\left[\frac{t^{3/2}}{3/2}\right]_0^4=\frac{2}{3}\times[\sqrt{64}-0]$

$=\frac{16}{3}$ sq.units

33. (1) The two curves are

$xy^2=a^2(a-x)$ or $x=\frac{a^3}{a^2+y^2}$ (1)

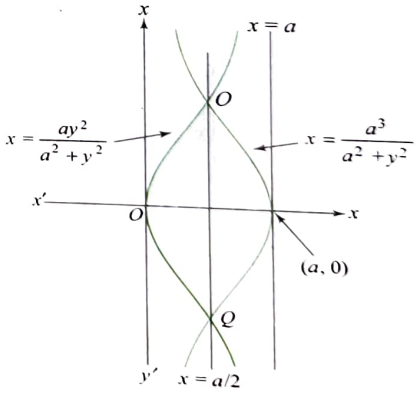
and $(a-x)y^2=a^2x$

$\Rightarrow \quad x=\frac{ay^2}{a^2+y^2}=\frac{ay^2+a^3-a^3}{a^2+y^2}=a-\frac{a^3}{a^2+y^2}$ (2)

Curve (1) is symmetrical about x-axis and have y-axis as the asymptote.

Curve (2) is symmetrical about x-axis, tangent at origin as y-axis, and the asymptote $x=a$.

The two curves intersect at the point $P(a/2, a)$ and $Q(a/2, -a)$.



Required area $=2\int_0^a\left[-a+\frac{a^3}{a^2+y^2}+\frac{a^3}{a^2+y^2}\right]dy$

(integrating along y-axis)

$=2\left[-ay+2a^2\tan^{-1}\frac{y}{a}\right]_0^a$

$=2\left[-a^2+2a^2\frac{\pi}{4}\right]$

$=(\pi-2)a^2$ sq. units.

34. (3) $(y-x)^2=x^3$, where $x \geq 0 \quad \Rightarrow \quad y-x=\pm x^{3/2}$

or $y=x+x^{3/2}$ (1)

$y=x-x^{3/2}$ (2)

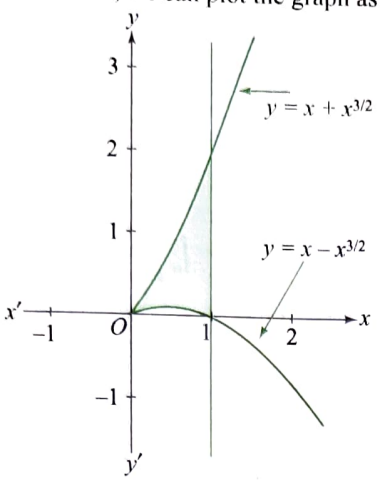
Function (1) is an increasing function.

Function (2) meets x-axis, when $x - x^{3/2} = 0$ or $x = 0, 1$.

Also, for $0 < x < 1$, $x - x^{3/2} > 0$ and for $x > 1$, $x - x^{3/2} < 0$.

When $x \to \infty$, $x - x^{3/2} \to -\infty$.

From these information, we can plot the graph as shown.



$$\text{Required area} = \int_0^1 \left[\left(x + x^{3/2}\right) - \left(x - x^{3/2}\right)\right] dx$$

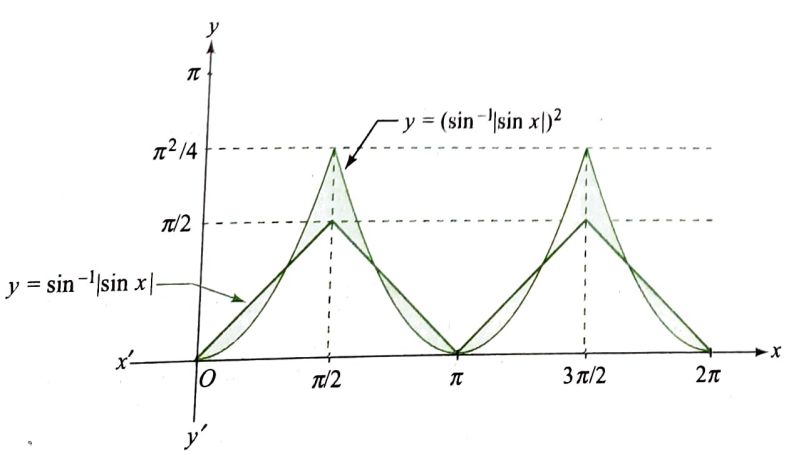
$$= 2\int_0^1 x^{3/2}\,dx$$

$$= 2\left[\frac{x^{5/2}}{5/2}\right]_0^1 = \frac{4}{5} \text{ sq. units.}$$

35. (4) $y = \sin^{-1}|\sin x| = \begin{cases} x, & 0 \le x < \dfrac{\pi}{2} \\ \pi - x, & \dfrac{\pi}{2} \le x < \pi \\ x - \pi, & \pi \le x < \dfrac{3\pi}{2} \\ 2\pi - x, & \dfrac{3\pi}{2} \le x < 2\pi \end{cases}$

$$y = (\sin^{-1}|\sin x|)^2 = \begin{cases} x^2, & 0 \le x < \dfrac{\pi}{2} \\ (\pi - x)^2, & \dfrac{\pi}{2} \le x < \pi \\ (x - \pi)^2, & \pi \le x < \dfrac{3\pi}{2} \\ (2\pi - x)^2, & \dfrac{3\pi}{2} \le x < 2\pi. \end{cases}$$

The required area A is shown as shaded region in the figure.



$$\Rightarrow \quad 4\int_0^1 (x - x^2)\,dx + 4\int_1^{\pi/2} (x^2 - x)\,dx$$

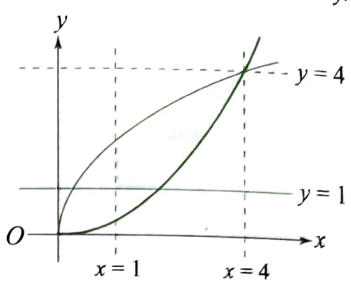
$$= \frac{4}{3} + \pi^2\left[\frac{\pi - 3}{6}\right] \text{ sq. units.}$$

36. (3) $y^2 = 4[\sqrt{y}]x$

For $y \in [1, 4)$, $[\sqrt{y}] = 1$ or $y^2 = 4x$.

Similarly, for $x \in [1, 4)$, $[\sqrt{x}] = 1$ and

$x^2 = 4\,[\sqrt{x}]\,y$ would transform into $x^2 = 4y$.

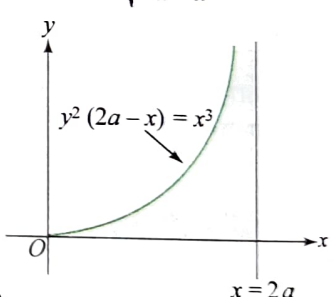


The required area is the shaded region.

$$A = \int_1^2 (2\sqrt{x} - 1)\,dx + \int_2^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx$$

$$= \left(\frac{4}{3}x^{3/2} - x\right)_1^2 + \left(\frac{4}{3}x^{3/2} - \frac{x^3}{12}\right)_2^4 = \frac{11}{3} \text{ sq. units.}$$

37. (2) The required area $A = \int_0^{2a} \sqrt{\dfrac{x^3}{2a - x}}\,dx$



Put $x = 2a\sin^2\theta$

$\Rightarrow \quad dx = 2a\,2\sin\theta\cos\theta\,d\theta$

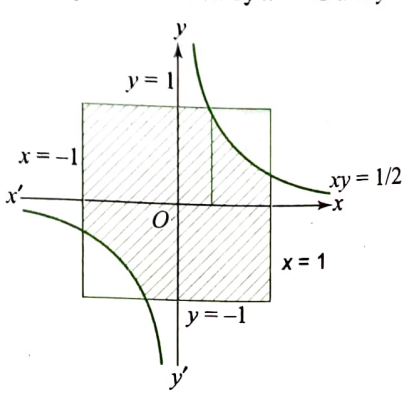
$$\Rightarrow \quad A = 8a^2 \int_0^{\pi} \left(\frac{1 - \cos 2\theta}{2}\right)^2 d\theta$$

$$= 2a^2 \int_0^{\pi} (1 - 2\cos 2\theta + \cos^2 2\theta)\,d\theta$$

$$= 2a^2 \int_0^{\pi} \left(1 - 2\cos 2\theta + \frac{1 + \cos 4\theta}{2}\right) d\theta = \frac{3\pi a^2}{2}$$

38. (2) $\max(|x|, |y|) \le 1 \Rightarrow |x| \le 1$ and $|y| \le 1$

which represent square bounded by $x = \pm 1$ and $y = \pm 1$



The required area is the lined area.

Now, shaded area is given by

$$2\int_{1/2}^{1}\left(1-\frac{1}{2x}\right)dx = 2\left(x-\frac{1}{2}\ln x\right)_{1/2}^{1}$$

$$= 2\left[(1-0)-\left(\frac{1}{2}-\frac{1}{2}\ln\frac{1}{2}\right)\right]$$

$$= 1 - \ln 2 \text{ sq. units.}$$

or Horizontal lined area $= 4 - (1 - \ln 2) = 3 + \ln 2$ sq. units.

9. (1) The points in the required region satisfy

$$4 \le x^2 + y^2 \le 2(|x| + |y|) \quad \ldots(1)$$

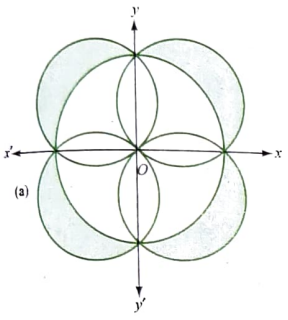
Since the curve (1) is symmetrical about both the axes, the required area is 4 times the area of the region in the first quadrant. Therefore, it is sufficient to sketch the region and to find the area in the first quadrant.

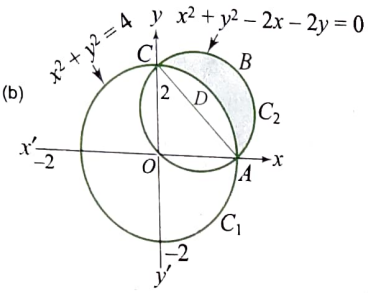
In the first quadrant, the curve (1) consist of two curves

$x^2 + y^2 \ge 4$

and $x^2 + y^2 - 2x - 2y \ge 0$ (C_1)

(C_2)





$\therefore$ Required area = 4 (area $ABCDA$)

= 4(area of semi-circle $ABCA$) – (area of sector $ADCA$)

= 4(area of semi-circle $ABCA$) – (area of sector $OADCO$ – area of triangle OAC)

$= 4\{\pi - (\pi - 2\} = 8$ sq. units.

10. (3) $\int_{\pi/4}^{\beta} f(x)dx = \beta\sin\beta + \frac{\pi}{4}\cos\beta + \sqrt{2}\beta$

Differentiating both sides w.r.t. β, we get

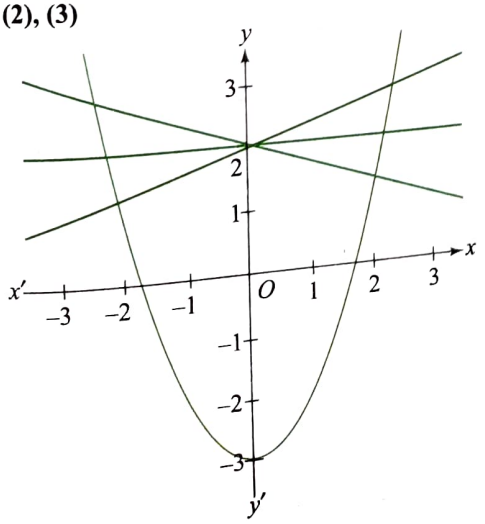
$\therefore \quad f(\beta) = \beta\cos\beta + \sin\beta - \angle\frac{\pi}{4}\sin\beta + \sqrt{2}$

$\Rightarrow \quad f'(\beta) = -\beta\sin\beta + \cos\beta + \cos\beta - \frac{\pi}{4}\cos\beta$

$\Rightarrow \quad f'\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$

Multiple Correct Answers Type

1. (2), (3)



Line $y = kx + 2$ passes through fixed point (0, 2) for different value of k.

Also, it is obvious that minimum $A(k)$ occurs when $k = 0$, as when line is rotated from this position about point (0, 2), the increased part of area is more than the decreased part of area. Therefore,

$$\text{Minimum area} = 2\int_0^{\sqrt{5}}(2-(x^2-3))dx$$

$$= 2\int_0^{\sqrt{5}}(5-x^2)dx$$

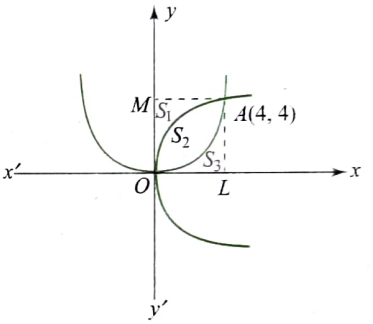
$$= 2\left[5x - \frac{x^3}{3}\right]_0^{\sqrt{5}}$$

$$= 2\left[5\sqrt{5} - \frac{5\sqrt{5}}{3}\right]$$

$$= \frac{20\sqrt{5}}{3} \text{ sq. units}$$

2. (1), (3), (4)

$y^2 = 4x$ and $x^2 = 4y$ meet at $O(0, 0)$ and $A(4, 4)$.



Now $S_3 = \int_0^4 \frac{x^2}{4}dx = \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4 = \frac{1}{12}[64-0] = \frac{16}{3}$.

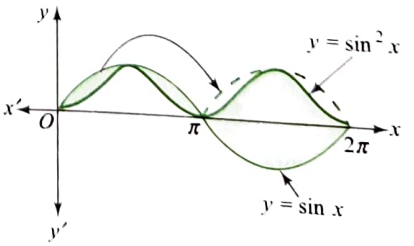
$$S_2 = \int_0^4 2\sqrt{x}\,dx - S_3 = 2\left[\frac{x^{3/2}}{3/2}\right]_0^4 - \frac{16}{3}$$

$$= \frac{4}{3}[8-0] - \frac{16}{3} = \frac{16}{3}.$$

And $S_1 = 4 \times 4 - (S_2 + S_3) = 16 - \left(\frac{16}{3} + \frac{16}{3}\right) = \frac{16}{3}$.

Hence, $S_1:S_2:S_3 = 1:1:1$

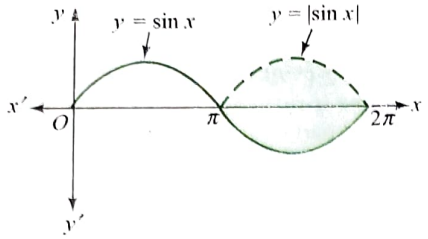
3. (1), (3), (4)



We know that area bounded by $y = \sin x$ and x-axis for $x \in [0, \pi]$ is 2 sq. units.

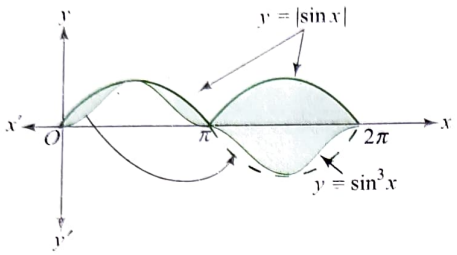
Then area bounded by $y = \sin x$ and $y = \sin^2 x$ is 4 sq. units for $x \in [0, 2\pi]$.

Then for $x \in [0, 10\pi]$, the area bounded is 20 sq. units.



The area bounded by $y = \sin x$ and $y = |\sin x|$ for $x \in [0, 2\pi]$ is 4 sq. units.

Then for $x \in [0, 20\pi]$, the area bounded is 40 sq. units.



The area bounded by $y = \sin x$ and $y = \sin^3 x$ for $x \in [0, 2\pi]$ is 4 sq. units.

Then for $x \in [0, 10\pi]$, the area bounded is 20 sq. units.

Similarly, the area bounded by $y = \sin x$ and $y = \sin^4 x$ for $x \in [0, 10\pi]$ is 20 sq. units.

4. (3), (4)

Since the curve $y = ax^{1/2} + bx$ passes through the point (1, 2)

$\therefore \quad 2 = a + b \qquad (1)$

By observation the curve also passes through (0, 0). Therefore, the area enclosed by the curve, x-axis and $x = 4$ is given by

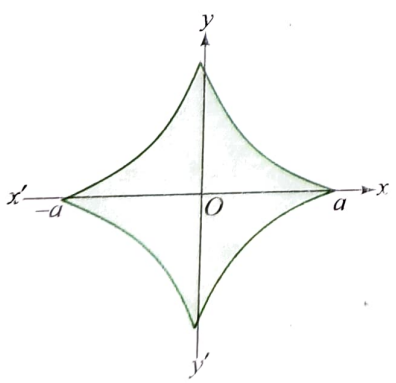
$$A = \int_0^4 (ax^{1/2} + bx)dx = 8 \text{ or } \frac{2a}{3} \times 8 + \frac{b}{2} \times 16 = 8$$

or $\quad \dfrac{2a}{3} + b = 1. \qquad (2)$

Solving (1) and (2), we get $a = 3$, $b = -1$.

5. (1), (3), (4)

Eliminating t, we have $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \Rightarrow y = \left(a^{2/3} - x^{2/3}\right)^{3/2}$.



From the figure,

$$A = 2\int_{-a}^{a} \left(a^{2/3} - x^{2/3}\right)^{3/2} dx$$

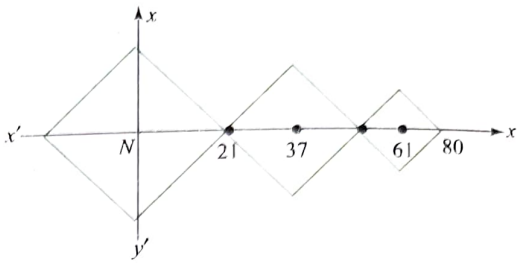
$$= 4\int_0^a \left(a^{2/3} - x^{2/3}\right)^{3/2} dx = 4\int_0^a y\,dx$$

$$= 4a^2 \int_0^{\pi/2} 3\cos^3 t \sin^2 t \cos t\, dt.$$

6. (1), (3)

$$a_1 = 0,\ b_1 = 32,\ a_2 = a_1 + \frac{3}{2} b_1 = 48,\ b_2 = \frac{b_1}{2} = 16$$

$$a_3 = 48 + \frac{3}{2} \times 16 = 72,\ b_3 = \frac{16}{2} = 8$$



So the three loops from $i = 1$ to $i = 3$ are alike.

Now area of ith loop (square) $= \dfrac{1}{2}$ (diagonal)^2

$$A_i = \frac{1}{2}(2b_i)^2 = 2(b_i)^2$$

So, $\dfrac{A_{i+1}}{A_i} = \dfrac{2(b_{i+1})^2}{2(b_i)^2} = \dfrac{1}{4}$.

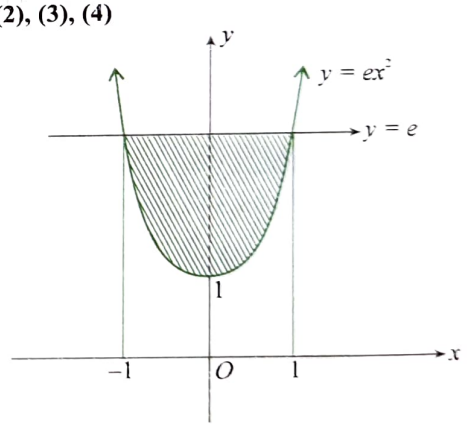
So the areas form a G.P. series.

So, the sum of the G.P. up to infinite terms is

$$A_1 \frac{1}{1-r} = 2(32)^2 \times \frac{1}{1-\frac{1}{4}}$$

$$= 2 \times (32)^2 \times \frac{4}{3} = \frac{8}{3}(32)^2 \text{ sq. units.}$$

7. (1), (2), (3), (4)



Area, $A = \int_{-1}^{1} (e - e^{x^2})dx = 2e - \int_{-1}^{1} e^{x^2} dx$

$y = e^{x^2}$

$\therefore \quad \log_e y = x^2$

$\therefore \quad x = \pm\sqrt{\log_e y}$

$\therefore \quad$ Area, $A = 2\int_1^e \sqrt{\log_e y}\, dy$

Now put $\log_e y = t$.

$\therefore \quad y = e^t$

$\therefore \quad dy = e^t\, dt$

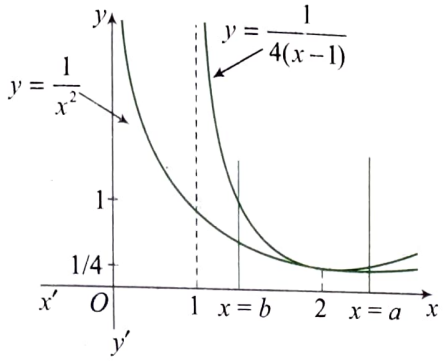
So, area $A = 2\int_0^1 \sqrt{t}\, e^t dt$

8. (1), (4)

Solving given curves $y = \frac{1}{x^2}; y = \frac{1}{4(x-1)}$, we get

$x^2 = 4(x-1) \Rightarrow (x-2)^2 = 0$

So, curves touch other at $x = 2$.



$\therefore \quad A = \int_2^a \left(\frac{1}{4(x-1)} - \frac{1}{x^2}\right)dx = \frac{1}{a}$

$\Rightarrow \quad \left[\frac{1}{4}\log(x-1) + \frac{1}{x}\right]_2^a = \frac{1}{a}$

$\Rightarrow \quad \frac{1}{4}\log(a-1) + \frac{1}{a} - \frac{1}{2} = \frac{1}{a}$

$\Rightarrow \quad \log(a-1) = 2$

$\Rightarrow \quad a = e^2 + 1$

Also $1 - \frac{1}{b} = \int_b^2 \left(\frac{1}{4(x-1)} - \frac{1}{x^2}\right)dx$

$\Rightarrow \quad b = 1 + e^{-2}$

9. (1), (2), (4)

$\sqrt{|x|} + \sqrt{|y|} = \sqrt{a}$

Consider $x, y > 0$

$\therefore \quad \sqrt{x} + \sqrt{y} = \sqrt{a}$

$\therefore \quad y = (\sqrt{a} - \sqrt{x})^2$

$y' = 2(\sqrt{a} - \sqrt{x})\left(-\frac{1}{\sqrt{x}}\right) = 2\left(1 - \frac{\sqrt{a}}{\sqrt{x}}\right)$

$\therefore \quad y'' = \frac{\sqrt{a}}{x^{3/2}} > 0$

So, graph is concave upward.

Area bounded by S_1 and coordinate axis in first quadrant

$= \int_0^a (\sqrt{a} - \sqrt{x})^2 dx$

$= \int_0^a [a + x - 2\sqrt{a}\sqrt{x}]dx$

$= \left[ax + \frac{x^2}{2} - 4\sqrt{a}\frac{x^{3/2}}{3}\right]_0^a$

$= a^2 + \frac{a^2}{2} - \frac{4a^2}{3}$

$= \frac{a^2}{6}$

Area bounded by S_2 and coordinate axis in first quadrant

$= \frac{\pi a^2}{4}$

Area bounded by S_3 and coordinate axis in first quadrant

$= \frac{a^2}{2}$

$\therefore$ Area bounded by S_1 and S_2, $\alpha = 4\left(\frac{\pi a^2}{4} - \frac{a^2}{6}\right)$

$= a^2\left(\pi - \frac{2}{3}\right)$

Area bounded by S_1 and S_3, $\beta = 4\left(\frac{a^2}{2} - \frac{a^2}{6}\right) = \frac{4a^2}{3}$

Area bounded by S_2 and S_3, $\gamma = 4\left(\frac{\pi a^2}{4} - \frac{a^2}{2}\right) = a^2(\pi - 2)$

Also, $\frac{\beta}{\gamma} = \frac{4}{3(\pi - 2)}$

10. (1), (3)

x_1 and x_2 are the roots of the equation

$x^2 + 2x - 3 = kx + 1$

or $x^2 + (2-k)x - 4 = 0$

$\left.\begin{aligned} x_1 + x_2 &= k-2 \\ x_1 x_2 &= -4 \end{aligned}\right\}$

$A = \int_{x_1}^{x_2} [(kx+1) - (x^2 + 2x - 3)]dx$

$= \left[(k-2)\frac{x^2}{2} - \frac{x^3}{3} + 4x\right]_{x_1}^{x_2}$

$= \left[(k-2)\frac{x_2^2 - x_1^2}{2} - \frac{1}{3}(x_2^3 - x_1^3) + 4(x_2 - x_1)\right]$

$= (x_2 - x_1)\left[\frac{(k-2)^2}{2} - \frac{1}{3}\left((x_2 - x_1)^2 - x_1 x_2\right) + 4\right]$

$= \sqrt{(x_2 + x_1)^2 - 4x_1 x_2}\left[\frac{(k-2)^2}{2} - \frac{1}{3}\left((k-2)^2 + 4\right) + 4\right]$

$= \frac{\sqrt{(k-2)^2 + 16}}{6}\left[\frac{1}{6}(k-2)^2 + \frac{8}{3}\right]$

$= \frac{\left[(k-2)^2 + 16\right]^{3/2}}{6}$

which is least when $k = 2$ and $A_{least} = 32/3$ sq. units

Linked Comprehension Type

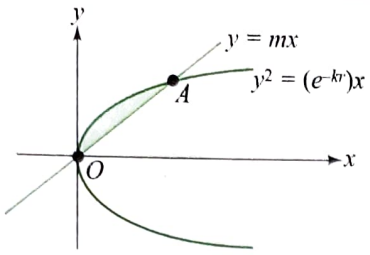
For Problems 1 and 2

1 (2), 2. (3)

Solving the two equations, we get

$m^2 x^2 = (e^{-kx})x$

$x_1 = 0, x_2 = \frac{e^{-kx}}{m^2}$.



So, $A_r = \int_0^{x_2} \left(e^{-\frac{kr}{2}} \sqrt{x} - mx \right) dx$

$= \frac{2}{3} e^{-kr/2} x_2^{3/2} - m\frac{x_2^2}{2}$

$= \frac{2}{3} e^{-kr/2} \frac{e^{-3kr/2}}{m^3} - \frac{m}{2}\frac{e^{-2kr}}{m^4} = \frac{e^{-2kr}}{6m^3}$.

Now, $\frac{A_{r+1}}{A_r} = \frac{e^{-2k(r+1)}}{e^{-2kr}} = e^{-2k}$ = constant.

So, the sequence $A_1, A_2, A_3, \ldots$ is in G.P.

Sum of n terms $= \frac{e^{-2k}}{6m^3}\frac{e^{-2nk}-1}{e^{-2k}-1} = \frac{1}{6m^3}\frac{e^{-2nk}-1}{1-e^{2k}}$

Sum to infinite terms $= A_1 \frac{1}{1-e^{-2k}}$

$= \frac{e^{-2k}}{6m^3} \times \frac{e^{2k}}{e^{2k}-1} = \frac{1}{6m^3(e^{2k}-1)}$.

For Problems 3–5

3. (4) $f(x) = \frac{x^3}{3} - x^2 + a$

$f'(x) = x^2 - 2x = x(x-2) < 0$ (note that $f(x)$ is monotonic in (0, 2))

Hence, for the minimum and $f(x)$ must cross the x-axis at $\frac{0+2}{2} = 1$.

Hence, $f(1) = \frac{1}{3} - 1 + a = 0$

$\Rightarrow \quad a = \frac{2}{3}$.

4. (3) $f(x) = x^3 + 3x^2 + x + a$

$f'(x) = 3x^2 + 6x + 1 = 0$

$\Rightarrow \quad x = -1 \pm \frac{\sqrt{6}}{3}$.

Hence, $f(x)$ cuts the x-axis at

$\frac{1}{2}\left[\left(-1+\frac{\sqrt{6}}{3}\right)+\left(-1-\frac{\sqrt{6}}{3}\right)\right] = -1$.

$f(-1) = -1 + 3 - 1 + a = 0$

$a = -1$.

5. (1) $f(x) = \sin x + \cos x$

or $\frac{df(x)}{dx} = \cos x - \sin x$

If $\frac{df(x)}{dx} = 0$, then $\cos x = \sin x \Rightarrow x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$

(considering any two of consecutive points of extremum).

For minimum area bounded by $y = f(x)$ and $y = a$, between $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$, graphs of $g(x)$ must cut $y = a$ at

$c = \frac{\frac{\pi}{4}+\frac{5\pi}{4}}{2} = \frac{3\pi}{4}$.

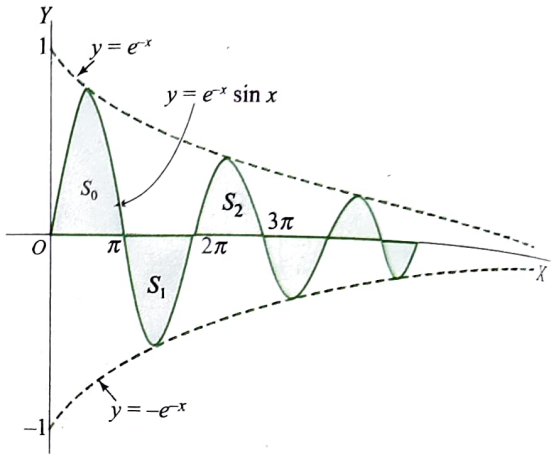
$a = f\left(\frac{3\pi}{4}\right)$

$\Rightarrow \quad a = \sin\left(\frac{3\pi}{4}\right) + \cos\left(\frac{3\pi}{4}\right) = 0.$

For Problems 6–8

6. (1), 7. (3), 8. (2)

Since $-1 \le \sin x \le 1$, the curve $y = e^{-x} \sin x$ is bounded by the curves $y = e^{-x}$ and $y = e^{-x}$.



Also, the curve $y = e^{-x} \sin x$ intersects the positive semi-axis OX at the points where $\sin x = 0$, where $x_n = n\pi$, $n \in Z$.

Also $|y_n|$ = |y coordinate in the half-wave S_n|

$= (-1)^n e^{-x} \sin x$, and in S_n, $n\pi \le x \le (n+1)\pi$

$\therefore \quad S_n = (-1)^n \int_{n\pi}^{(n+1)\pi} e^{-x} \sin x dx$

$= \frac{(-1)^{n+1}}{2}\left[e^{-x}(-\sin x + \cos x)\right]_{n\pi}^{(n+1)\pi}$

$= \frac{(-1)^{n+1}}{2}\left[e^{-(n+1)\pi}(-1)^{n+1} - e^{n\pi}(-1)^n\right]$

$= \frac{e^{-n\pi}}{2}\left(1+e^{\pi}\right)$

$\Rightarrow \quad \frac{S_{n+1}}{S_n} = e^{-\pi}$

and $S_0 = \frac{1}{2}\left(1+e^{\pi}\right)$.

Therefore, the sequence $S_0, S_1, S_2, \ldots$ forms an infinite G.P. with common ratio $e^{-\pi}$. we have

$$\sum_{n=0}^{\infty} S_n = \frac{\frac{1}{2}(1+e^{\pi})}{1-e^{-\pi}}$$

For Problems 9–11

9. (2) Given

$(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2-y^2)$

$= (x^2-y^2)[(x+y)^2 - (x-y)^2]$

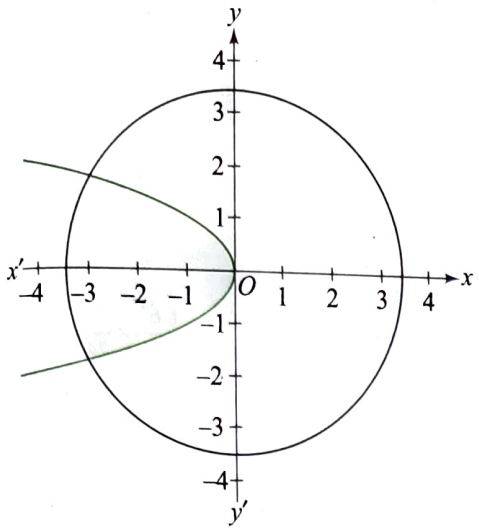
$= (x-y)(x+y)^3 - (x+y)(x-y)^3$

$\Rightarrow \quad f(x+y) = (x+y)^3 \Rightarrow f(x) = x^3, f(y) = y^3$

Now equations of given curves are

$y^2 + x = 0$ ...(1)

$x^2 + y^2 = 12$ ...(2)



Solving equations (1) and (2), we get $x = -3, y = \pm\sqrt{3}$

The area bounded by curves

$$A = 2\left[\left|\int_{-2\sqrt{3}}^{-3}\sqrt{12-x^2}\,dx\right| + \left|\int_{-3}^{0}\sqrt{-x}\,dx\right|\right]$$

$$I_1 = 2\int_{-2\sqrt{3}}^{-3}\sqrt{12-x^2}\,dx = 2\int_{-\pi/2}^{-\pi/3} 12\cos^2\theta\,d\theta$$

$$= 12\left[\int_{-\pi/2}^{-\pi/3}(1+\cos 2\theta)\,d\theta\right]$$

$$= 12\left[\theta + \frac{\sin\theta}{2}\right]_{-\pi/2}^{-\pi/3} = 12\left[-\frac{\pi}{3} - \frac{\sqrt{3}}{4} + \frac{\pi}{2}\right]$$

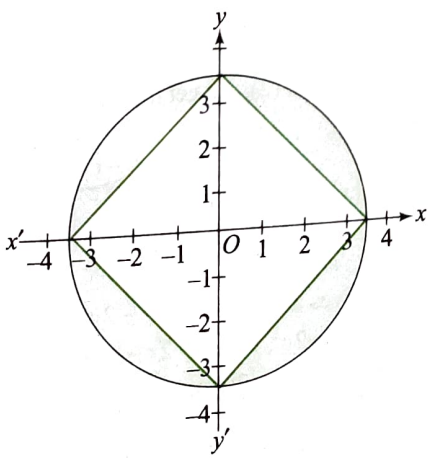
$$= 12\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right] = 2\pi - 3\sqrt{3}.$$

$$I_2 = 2\int_{-3}^{0}\sqrt{-x}\,dx = \frac{2[(-x)^{3/2}]_{-3}^{0}}{-3/2} = -\frac{4}{3}[0 - 3^{3/2}] = 4\sqrt{3}.$$

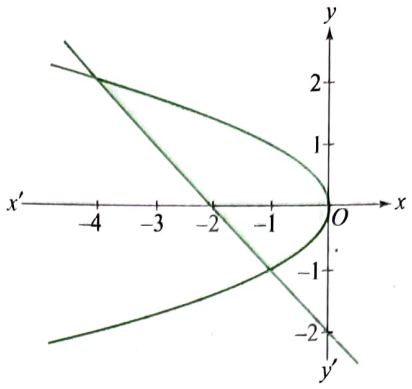
$A = 2\pi - 3\sqrt{3} + 4\sqrt{3} = 2\pi + \sqrt{3}$ sq. units.

(l) Required area = Area of circle − Area of square

$= 12\pi - 24$ sq. units.



11. (3)



Required area $= \int_{-1}^{2}(-y^2 - (-y-2))\,dy$

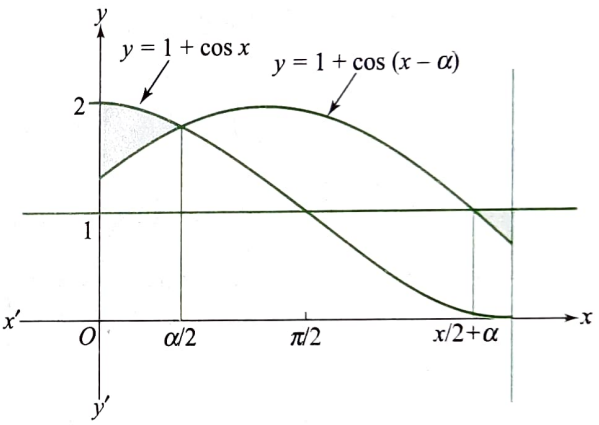
$$= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{2}$$

$$= \left[\frac{4}{2} + 4 - \frac{8}{3} - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)\right]$$

$= 9/2$ sq. units.

For Problems 12 and 13

12. (3)



$1 + \cos x = 1 + \cos(x - \alpha)$

or $\quad x = \alpha - x$ or $x = \frac{\alpha}{2}$

Now $\int_0^{\alpha/2}\left((1+\cos x) - (1+\cos(x-\alpha))\right)dx$

$$= -\int_{\frac{\pi}{2}+\alpha}^{\pi}\left(1 - (1+\cos(x-\alpha))\right)dx$$

or $\quad [\sin x - \sin(x-a)]_0^{\alpha/2} = \left[\sin(x-\alpha)\right]_{\pi}^{\frac{\pi}{2}+\alpha}$

or $\quad \left[\sin\frac{\alpha}{2} - \sin\left(-\frac{\alpha}{2}\right)\right] - [0 - \sin(-\alpha)]$

$$= \sin\left(\frac{\pi}{2}\right) - \sin(\pi - \alpha)$$

or $\quad 2\sin\frac{\alpha}{2} - \sin\alpha = 1 - \sin\alpha$

Hence, $2\sin\frac{\alpha}{2} = 1$ or $\alpha = \frac{\pi}{3}$.

13. (2) $\int_0^{\pi/6}\left((1+\cos x)-\left(1+\cos\left(x-\frac{\pi}{3}\right)\right)\right)dx$

$$+\int_{\pi/6}^{\pi}\left(\left(1+\cos\left(x-\frac{\pi}{3}\right)\right)-(1+\cos x)\right)dx$$

$$=\left[\sin x-\sin\left(x-\frac{\pi}{3}\right)\right]_0^{\pi/6}+\left[\sin\left(x-\frac{\pi}{3}\right)-\sin x\right]_{\pi/6}^{\pi}$$

$$=\left[\left(\frac{1}{2}+\frac{1}{2}\right)-\frac{\sqrt{3}}{2}\right]+\left[\frac{\sqrt{3}}{2}-\left(-\frac{1}{2}-\frac{1}{2}\right)\right]$$

$= 2$ sq. units.

For Problems 14–16

14. (1) For $-1 \le x < 0$

$$(y-e^{\sin^{-1}x})^2 = 2-x^2$$

$$y = e^{\sin^{-1}x} \pm \sqrt{2-x^2}$$

$$A=\int_{-1}^{0}\left(e^{\sin^{-1}x}+\sqrt{2-x^2}\right)-\left(e^{\sin^{-1}x}-\sqrt{2-x^2}\right)dx$$

$$=2\int_{-1}^{0}\sqrt{2-x^2}\,dx$$

$$=2\left(\frac{1}{2}x\sqrt{2-x^2}\Big|_{-1}^{0}+\frac{2}{2}\sin^{-1}\frac{x}{\sqrt{2}}\Big|_{-1}^{0}\right)$$

$$=\left[1+2\left(0-\left(-\frac{\pi}{4}\right)\right)\right]$$

$$=\frac{\pi}{2}+1 \text{ sq. units.}$$

For $0 \le x < 1$, $y = \sin^{-1}x \pm \sqrt{1-x^2}$

$$A=2\int_0^1\sqrt{1-x^2}\,dx$$

$$=2\left[\frac{x}{2}\sqrt{1-x^2}\Big|_0^1+\frac{1}{2}\sin^{-1}\frac{x}{1}\Big|_0^1\right]$$

$$=0+\sin^{-1}(1)=\frac{\pi}{2} \text{ sq. units.}$$

Total area $=\left(\frac{\pi}{2}+1\right)+\frac{\pi}{2}=\pi+1.$

15. (4) Ratio $=\dfrac{\frac{\pi}{2}+1}{\frac{\pi}{2}}=\dfrac{\pi+2}{\pi}.$

16. (1) $A=2\int_0^{1/2}\sqrt{1-x^2}\,dx$

$$=2\left[\frac{x}{2}\sqrt{1-x^2}\Big|_0^{\frac{1}{2}}+\frac{1}{2}\sin^{-1}x\Big|_0^{\frac{1}{2}}\right]$$

$$=\frac{\sqrt{3}}{4}+\frac{\pi}{6} \text{ sq. unit.}$$

Problems 17–19

(2) $S=\left|-\int_0^{2\pi}a(1-\cos t)a(1-\cos t)dt\right|$

$$=\left|-a^2\int_0^{2\pi}(1-2\cos t+\cos^2 t)dt\right|$$

$$=\left|-a^2\int_0^{2\pi}\left(1-2\cos t+\left(\frac{1+\cos 2t}{2}\right)\right)dt\right|$$

$$=\left|-\frac{a^2}{2}\int_0^{2\pi}(3-4\cos t+\cos 2t)dt\right|$$

$$=\left|-\frac{a^2}{2}[3t-4\cos t+\cos 2t]_0^{2\pi}\right|$$

$$=|-3\pi a^2|=3\pi a^2 \text{ sq. units.}$$

18. (1) $\int_0^6\left(\frac{3}{2}t^2-\frac{1}{2}t^3+\frac{1}{24}t^4\right)dt=\frac{3}{2}\times\frac{6^3}{3}\times\frac{1}{2}\frac{6^4}{4}\times\frac{1}{24}\frac{6^5}{5}$

$$=\frac{6^3}{2}-\frac{6^4}{8}+\frac{6^4}{20}$$

$$=6^4\left(\frac{1}{12}-\frac{1}{8}+\frac{1}{20}\right)$$

$$=\frac{54}{5}.$$

$\therefore \quad \frac{1}{2}\int_0^6(xy'-yx')dx=\frac{1}{2}\times\frac{54}{5}=\frac{27}{5}$ sq. units.

19. (3) $\frac{dx}{dt}=1-3t^2$ and $\frac{dy}{dt}=1-4t^3$.

So, $x\frac{dy}{dt}-y\frac{dx}{dt}=(t-t^3)(1-4t^3)-(1-t^4)(1-3t^2)$

$$=t^6-3t^4-t^3+3t^2+t-1$$

$\therefore$ Required area $=\frac{1}{2}\int_{-1}^{1}(t^6-3t^4-t^3+3t^2+t-1)dt$

$=\frac{16}{35}$ sq. units (taking absolute value)

For Problems 20 and 21

20. (3), 21. (2)

$$f(x)=\begin{cases}x^2+ax+b; & x<-1\\ 2x; & -1\le x\le 1\\ x^2+ax+b; & x>1\end{cases}$$

$\because$ $f(x)$ is continuous at $x=-1$ and $x=1$

$\therefore$ $(-1)^2+a(-1)+b=-2 \Rightarrow b-a=-3$ ..

and $2=(1)^2+a.1+b \Rightarrow a+b=1$ ..

Solving (1) and (2), we get $a=2$, $b=-1$

$$\therefore \quad f(x)=\begin{cases}x^2+2x-1; & x<-1\\ 2x; & -1\le x\le 1\\ x^2+2x-1; & x>1\end{cases}$$

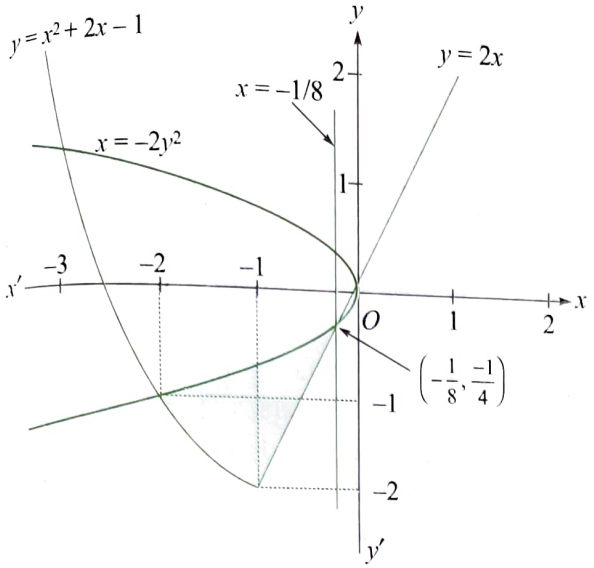
Given curves are $y=f(x)$, $x=-2y^2$ and $8x+1=0$

Solving $x=-2y^2$, $y=x^2+2x-1$ $(x<-1)$ we get $x=-2$

Also, $y=2x$, $x=-2y^2$ meet at $(0, 0)$

$y=2x$ and $x=-1/8$ meet at $\left(-\frac{1}{8},\frac{-1}{4}\right)$

The required area is the shaded region in the figure



∴ Required area

$$= \int_{-2}^{-1}\left[-\sqrt{\frac{-x}{2}} - \left(x^2+2x-1\right)dx\right] + \int_{-1}^{-1/8}\left[-\sqrt{\frac{-x}{2}} - 2x\right]dx$$

$$= \left[\frac{1}{\sqrt{2}}\,\frac{2(-x)^{3/2}}{3} - \frac{x^3}{3} - x^2 + x\right]_{-2}^{-1} + \left[\frac{1}{\sqrt{2}}\,\frac{2(-x)^{3/2}}{3} - x^2\right]_{-1}^{-1/8}$$

$$= \left(\frac{\sqrt{2}}{3}+\frac{1}{3}-1-1\right) - \left(\frac{4}{3}+\frac{8}{3}-4-2\right) + \left(\frac{\sqrt{2}}{3}\cdot\frac{1}{16\sqrt{2}} - \frac{1}{64}\right) - \left(\frac{\sqrt{2}}{3}-1\right)$$

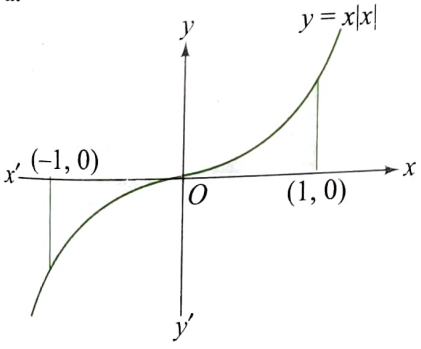
$$= \left(\frac{\sqrt{2}-5}{3}\right) - \left(\frac{4+8-18}{3}\right) + \left(\frac{4-3}{192}\right) - \left(\frac{\sqrt{2}-3}{3}\right)$$

$$= \frac{257}{192} \text{ sq. units}$$

rix Match Type

1. **a → r; b → p; c → s; d → q**

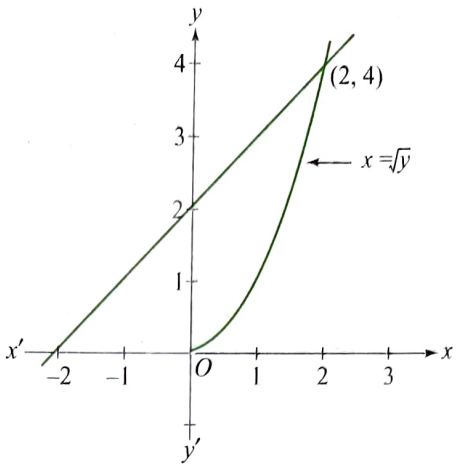
a.



Required area $= 2\int_0^1 x|x|\,dx$

$$= 2\left(\frac{x^3}{3}\right)_0^1 = \frac{2}{3}$$

b.

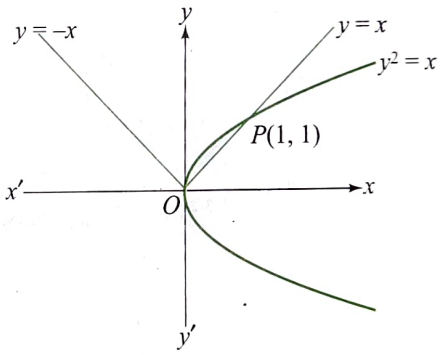


Required area $= \int_0^2[(x+2)-(x^2)]dx = \left[\frac{x^2}{2}+2x-\frac{x^3}{3}\right]_0^2$

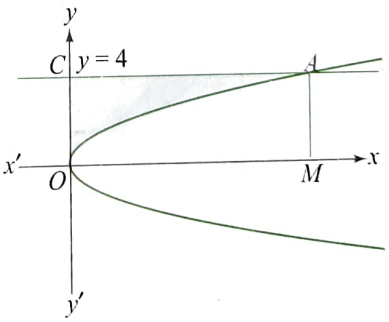
$$= 2+4-\frac{8}{3} = \frac{10}{3} \text{ sq. units.}$$

c. Reqd. area $= \int_0^1\left(\sqrt{x}-x\right)dx = \left[\frac{x^{3/2}}{3/2}-\frac{x^2}{2}\right]_0^1$

$$= \left(\frac{1}{3/2}-\frac{1}{2}\right) = \frac{2}{3}-\frac{1}{2} = \frac{1}{6} \text{ sq. units.}$$



d. $y = 4$ meets the parabola $y^2 = x$ at A is $(16, 4)$



Required area = Area of rectangle $OMAC$ − Area OMA

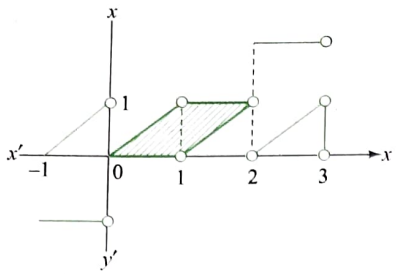
$$= 4\times 16 - \int_0^{16}\sqrt{x}\,dx = 64 - \left|\frac{x^{3/2}}{3/2}\right|_0^{16}$$

$$= 64 - \frac{2}{3}(4)^3$$

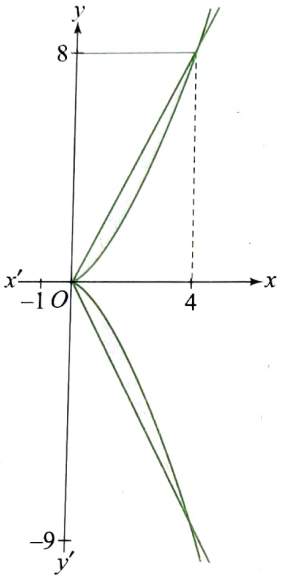
$$= 64 - \frac{128}{3} = \frac{64}{3} \text{ sq. units.}$$

2. **a → q; b → p; c → s; d → r**

a. Area $= 2\left(\frac{1}{2}1\times 1\right) = 1$ sq. units.



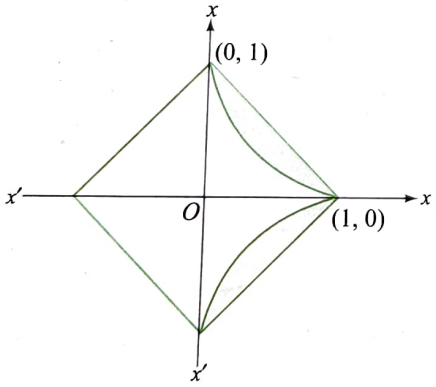
b. $y^2 = x^3$ and $|y| = 2x$, both the curve are symmetric about y-axis



$4x^2 = x^3$ or $x = 0, 4$.

Required area $= 2\int_0^4 \left(2x - x^{3/2}\right)dx = \frac{32}{5}$ sq. units.

c. $\sqrt{x} + \sqrt{|y|} = 1$



The curve is symmetrical about x-axis

$\sqrt{|y|} = 1 - \sqrt{x}$ and $\sqrt{x} = 1 - \sqrt{|y|}$

$\Rightarrow$ for $x > 0, y > 0$ $\sqrt{y} = 1 - \sqrt{x}$

$$\frac{1}{2\sqrt{y}}\frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = -\sqrt{\frac{y}{x}}$$

$\frac{dy}{dx} < 0$, function is decreasing

Required area $= 2\int_0^1 \left((1-x) - (1 - 2\sqrt{x} + x)\right)dx$

$$= 4\int_0^1 \left(\sqrt{x} - x\right)dx$$

$$= 4\left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2}\right]_0^1$$

$$= 4\left[\frac{2}{3} - \frac{1}{2}\right]$$

$$= \frac{2}{3} \text{ sq. units.}$$

d. If $-8 < x < 8$, then $y = 2$.

If $x \in (-8\sqrt{2}, -8] \cup [8, 8\sqrt{2})$, then $y = 3$, and so on

Intersection of $y = x - 1$ and $y = 2$. We get $x = 3 \in (-8, 8)$.

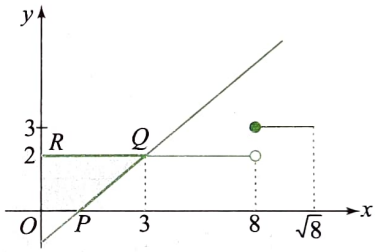
Intersection of $y = x - 1$ and $y = 3$.

We get $x = 4 \notin (-8\sqrt{2}, -8] \cup [8, 8\sqrt{2})$.

Similarly, $y = x - 1$ will not intersect $y = \left[\frac{x^2}{64} + 2\right]$ at any other integral, except in the interval $x \in (-8, 8)$.

Required area (shaded region) $= 2 \times 3 - \frac{1}{2} \times 2 \times 2$

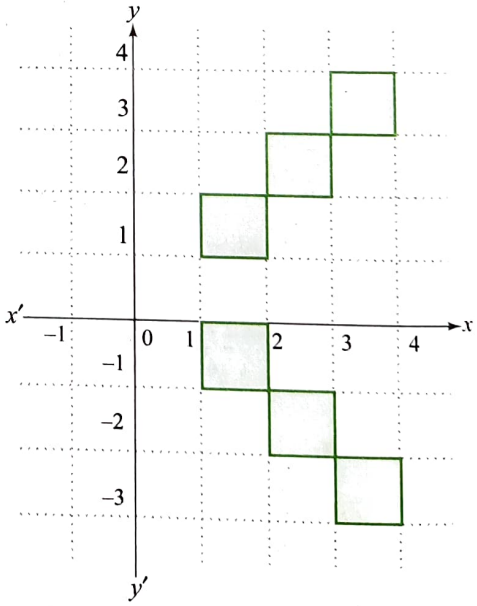
$= 4$ sq. units.



3. **a → q; b → s; c → p; d → p**

a. $[x]^2 = [y]^2$, where $1 \le x \le 4$

$\Rightarrow$ $[x] = \pm [y]$

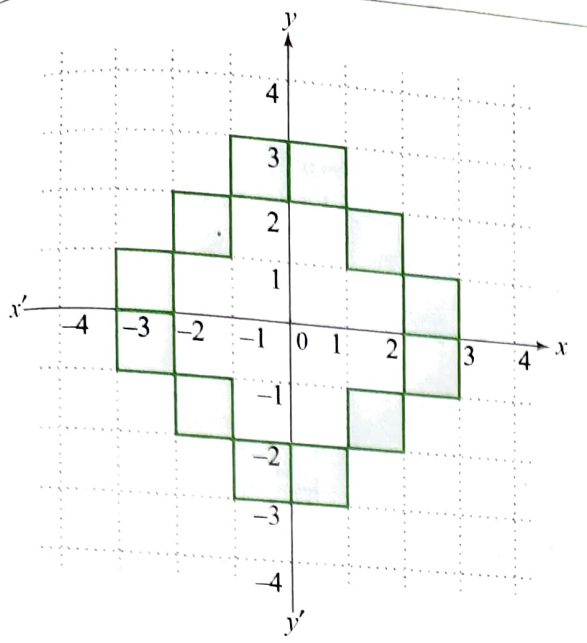


b. $[|x|] + [|y|] = 2$

The graph is symmetrical about both x-axis and y-axis.

For $x, y > 0$; $[x] + [y] = 2$.

$\Rightarrow$ $[x] = 0$ and $[y] = 2$, $[x] = 1$ and $[y] = 1$ or $[x] = 2$ and $[y] = 0$.

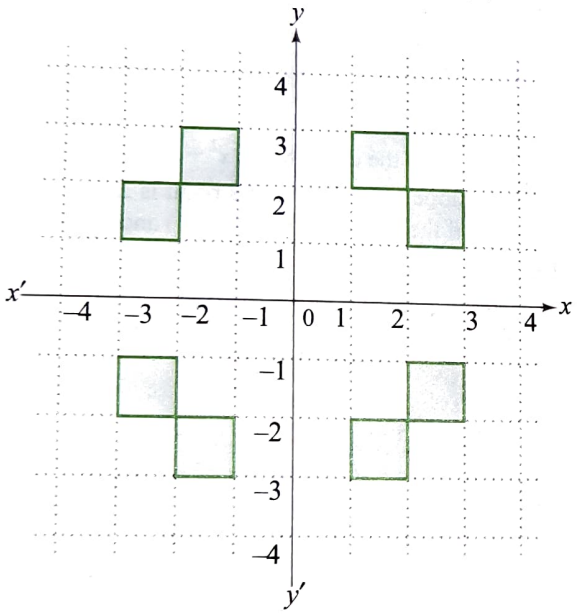


c. $[|x|]\,[|y|] = 2$

The graph is symmetrical about both x-axis and y-axis.

For $x, y > 0$; $[x][y] = 2$

$\Rightarrow$ $[x] = 1$ and $[y] = 2$ or $[x] = 2$ and $[y] = 1$.

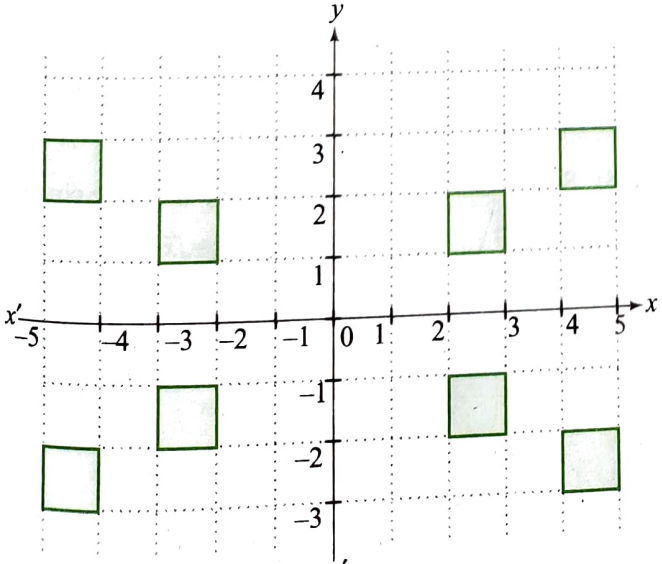


d. $\dfrac{[|x|]}{[|y|]} = 2$, where $-5 \le x \le 5$.

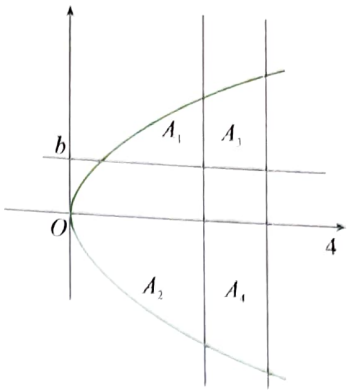
The graph is symmetrical about both the axes.

For $x, y > 0$, $[x] = 2[y]$, $[y] \neq 0$.

$\Rightarrow$ $[x] = 2$ and $[y] = 1$ or $[x] = 4$ and $[y] = 2$.



4. (3) a.



$$\text{Total area} = 2\int_0^4 \sqrt{x}\,dx = \frac{32}{3}$$

Area of each part = 8/3

$$A_3 = A_4 \Rightarrow \int_a^4 (\sqrt{x} - b)dx = \int_a^4 (b + \sqrt{x})dx = \frac{8}{3}$$

$\therefore \quad b = 0$

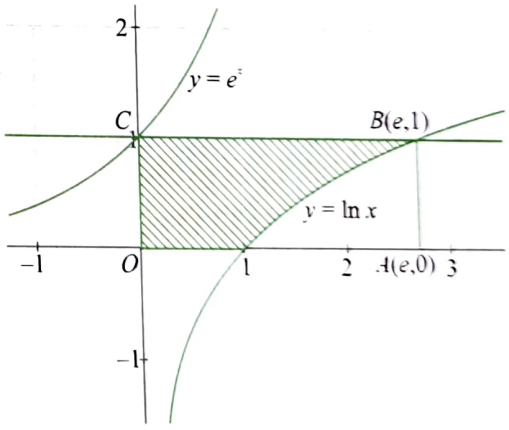
$$\therefore \quad \int_a^4 \sqrt{x}\,dx = \frac{8}{3}$$

$\therefore \quad a^3 = 16$

b. $f(x) = \begin{cases} x^{\frac{1}{\log_e x}}, & x \neq 1 \\ e, & x = 1 \end{cases} = \begin{cases} x^{\log_x e}, & x \neq 1 \\ e, & x = 1 \end{cases} = e$

Hence required area bound by $y = f(x)$ and $y = |x - e|$ is e^2

c.



$$\text{Area} = \text{Area of rectangle } OABC - \int_1^e \ln x\,dx$$

$$= e - 1 \text{ sq. units.}$$

d. Solving $2\cos x = 3\tan x$ we get, $2 - 2\sin^2 x = 3\sin x$

$$\Rightarrow \quad \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$$

$$\text{Required area} = \int_0^{\pi/6} (2\cos x - 3\tan x)dx$$

$$= (2\sin x - 3\ln\sec x)_0^{\pi/6}$$

$$= 1 + \frac{3}{2}\log_e 3 - 3\log_e 2$$

Numerival Value Type

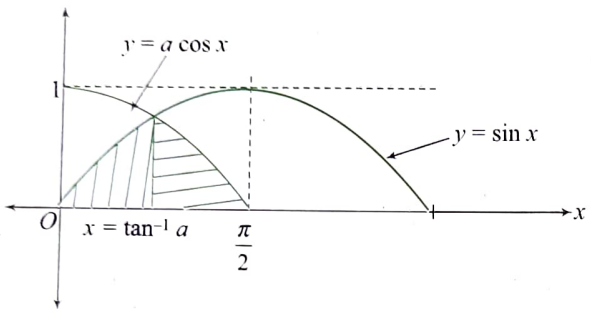
1. (9) Required area

$$A = \int_0^3 x\sqrt{9-x^2}\,dx;\ \text{Put } 9-x^2 = t^2 \Rightarrow -2x\,dx = 2t\,dt$$

$$\therefore \quad A = \int_0^3 t^2 dt = 9$$

2. (4) We have $S = \int_0^{\pi} \sin x\,dx = 2$, so $T = \frac{2}{3}$, where $a > 0$.

$$\text{Now } T = \int_0^{\tan^{-1}a} \sin x\,dx + \int_{\tan^{-1}a}^{\pi/2} a\cos x\,dx = \frac{2}{3}$$



$$\text{i.e.,} \quad -\cos(\tan^{-1}a) + 1 + a\{[1-\sin(\tan^{-1}a)]\} = \frac{2}{3},$$

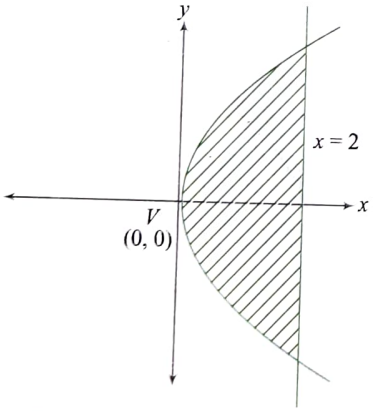
$$\text{i.e.,} \quad -\frac{1}{\sqrt{1+a^2}} + 1 + a - \frac{a^2}{\sqrt{1+a^2}} = \frac{2}{3}$$

$$\text{or} \quad (a+1) - \sqrt{a^2+1} = \frac{2}{3} \ \text{ or } \ a + \frac{1}{3} = \sqrt{a^2+1}$$

$$\text{or} \quad a = \frac{4}{3}$$

Hence, $3a = 4$.

3. (8)



Let $P(x, y)$ be any point on the curve C.

$$\text{Now, } \frac{dy}{dx} = \frac{1}{y}$$

$$\text{or} \quad y\,dy = dx \ \Rightarrow \ \frac{y^2}{2} = x + k$$

Since the curve passes through $M(2,2)$, so $k = 0$

$$\Rightarrow \quad y^2 = 2x$$

$$\therefore \quad \text{Required area} = 2\int_0^2 \sqrt{2x}\,dx = 2\sqrt{2} \times \frac{2}{3}\left(x^{3/2}\right)_0^2$$

$$= \frac{4}{3}\sqrt{2} \times 2\sqrt{2} = \frac{16}{3} \text{ sq. unit}$$

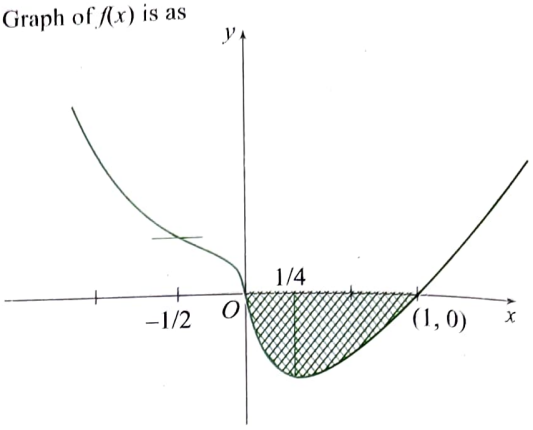
4. (8) $\int_0^3 (-x^2 + ax + 12)\,dx = 45$ gives $a = 4$

Hence, $f(x) = 12 + 4x - x^2 = (2 + x)(6 - x)$

Hence, $m = -2$ and $n = 6$

$$m + n + a = 6 - 1 + 4 = 8$$

5. (9) Graph of $f(x)$ is as



$$A = \int_0^1 (x^{4/3} - x^{1/3})dx = \frac{3}{7}x^{3/7} - \frac{3}{4}x^{4/3}\Big]_0^1$$

$$= \left|\frac{3}{7} - \frac{3}{4}\right| = 3\left|\frac{4-7}{28}\right| = \frac{9}{28}$$

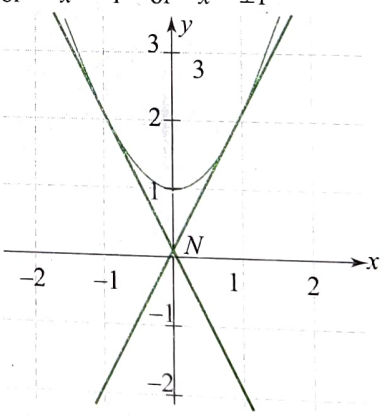
or $\quad 28A = 9$

6. (2) Let the point of the curve be $(x, x^2 + 1)$.

Now, the slope of tangent at this point is $2x$, which is equal to the slope of the line joining $(x, x^2 + 1)$ and $(0, 0)$.

Hence, $2x = (x^2 + 1)/x$ or $2x^2 = x^2 + 1$

or $\quad x^2 = 1$ or $x = \pm 1$

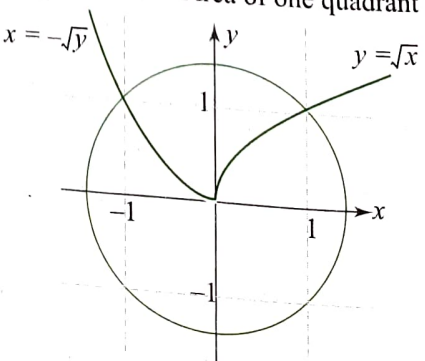


Hence, equation of tangent is $y = \pm 2x$

$$\text{Now, Area } 2\int_0^1 \left(x^2 + 1 - 2x\right)dx = 2\int_0^1 (x-1)^2\,dx$$

$$= 2\left[\frac{(x-1)^3}{3}\right]_0^1 = 2/3$$

7. (8) Required area = Area of one quadrant of the circle = $\pi/2$



8. (1) $f(a) = \int_a^{2a}\left(\frac{x}{6}+\frac{1}{x^2}\right)dx = \left(\frac{x^2}{12} - \frac{1}{x}\right)_a^{2a}$

$= \left(\frac{4a^2}{12} - \frac{1}{2a} - \frac{a^2}{12} + \frac{1}{a}\right) = \frac{a^2}{4} + \frac{1}{2a}$

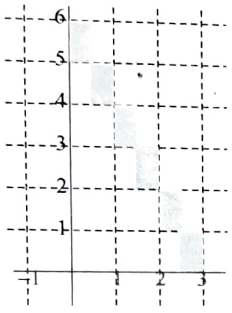
Let $f'(a) = \frac{2a}{4} - \frac{1}{2a^2} = 0$

$\Rightarrow \quad a = 1$ which is point of minima.

9. (3) $[2x] = 0 \quad \Rightarrow \quad 2x \in [0, 1)$

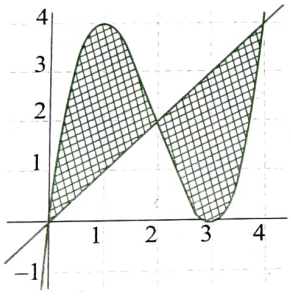
$\Rightarrow \quad x \in [0, 1/2) \quad \Rightarrow \quad [y] = 5 \quad \Rightarrow \quad y \in [5, 6)$

Similarly we can consider $[2x] = 1, 2, 3, 4$ and 5



From the graph, area is 3 sq. units

(8) Required area $= 2\int_0^2 \left(x(x-3)^2 - x\right)dx = 8$ sq. units

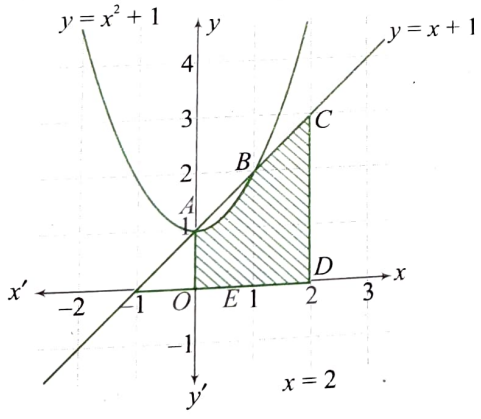


(6) Draw the given region point of intersection of $y = x^2 + 1$

$y = x + 1$

$x + 1 = x^2 + 1$

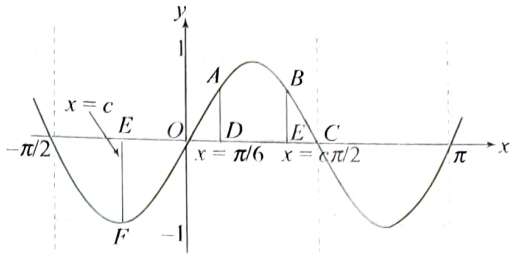
$x = 0, 1$



Required area $OABCDE = \int_0^1 (x^2+1)dx + \int_1^2 (x+1)dx$

$= \left(\frac{x^3}{3} + x\right)_0^1 + \left(\frac{x^2}{2} + x\right)_1^2 = \frac{23}{6}$ sq. units

12. (6)



Area $OABC = \int_0^{\pi/2} \sin 2x\, dx = 1$

Area $OAD = \int_0^{\pi/6} \sin 2x\, dx = \frac{1}{4}$

Since $\sin 2x$ is symmetric about origin,

so $c = -\frac{\pi}{6}$, because area OAD = area OEF

$\int_{\frac{\pi}{6}}^{c} \sin 2x\, dx = \frac{1}{2}$

$\cos 2c = -\frac{1}{2} \quad \cos 2c = \frac{3}{2}$ (not posible)

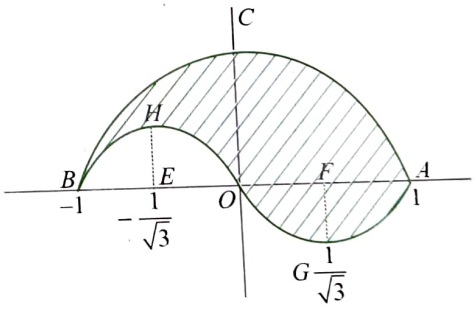
$c = \frac{\pi}{3}$

so $\quad c = -\frac{\pi}{6}, \frac{\pi}{3}$

13. (2) $y = \sqrt{1-x^2}$ (1)

$y = x^3 - x$ (2)

$y = 0$ in (2) $x = 0, 1, -1$



Required area = Area of region $BCAGOHB$

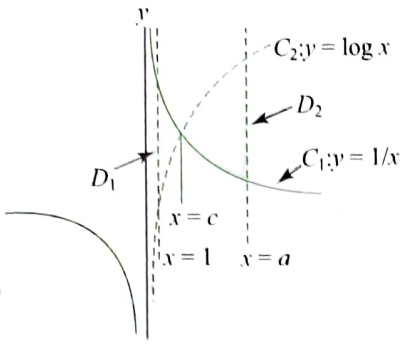
= Area of semi-circle $BCAOB$

$= \frac{\pi}{2}$ ($\because$ area of $BHOEB$ = area of $OFAGO$)

14. (1) Given that $D_1 = D_2$

$\int_1^c \left(\frac{1}{x} - \log x\right)dx = \int_c^a \left(\log x - \frac{1}{x}\right)dx$

$\left(\frac{-1}{x^2} - x(\log x - 1)\right)_1^c = \left(x(\log x - 1) + \frac{1}{x^2}\right)_c^a$



$\therefore \quad 0 = a(\log a - 1) + \dfrac{1}{a^2}$

$\therefore \quad a = 1$

15. (3) $y = \dfrac{a^2 - ax}{1 + a^4}$ (1)

$= \dfrac{x^2 + 2ax + 3a^2}{1 + a^4}$ c (2)

Point of intersection of (1) and (2)

$$\frac{a^2 - ax}{1 + a^4} = \frac{x^2 + 2ax + 3a^2}{1 + a^4}$$

$(x + a)(x + 2a) = 0$

$x = -a, -2a$

$$\text{Req. area} = \int_{-2a}^{-a} \left[\left(\frac{a^2 - ax}{1 + a^4} \right) - \left(\frac{x^2 + 2ax + 3a^2}{1 + a^4} \right) \right]$$

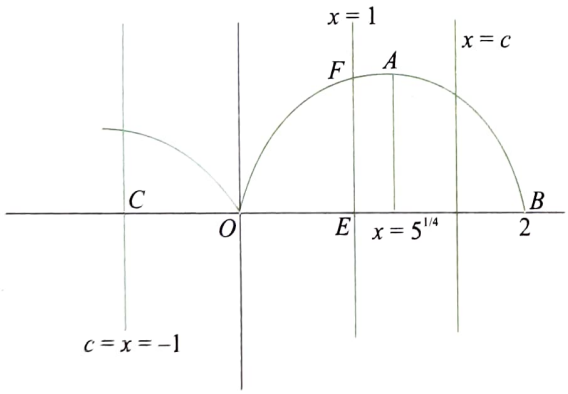
$\therefore \quad f(a) = \dfrac{a^3}{6(1 + a^4)}$

If $f(a)$ is max, then $f'(a) = 0$

$\Rightarrow \quad 3 + 3a^4 - 4a^4 = 0$

or $\quad a^4 = 3$

16. (2)



Given that $\int_1^c y\,dx = \dfrac{16}{3}$

$$\Rightarrow \quad \int_1^c (8x^2 - x^5)\,dx = \frac{16}{3} \quad c = (8 - \sqrt{17})^{1/3} \qquad (c > 0)$$

$$\text{Area } OFE = \int_0^c (8x^2 - x^5)\,dx = \frac{8}{3} \qquad (c > 0)$$

so $c = -1$

Hence, $c = -1$ and $(8 - \sqrt{17})^{1/3}$

Archives

JEE Main

Single Correct Answer Type

1. (3) The given parabola is $(y-2)^2 = x - 1$. So

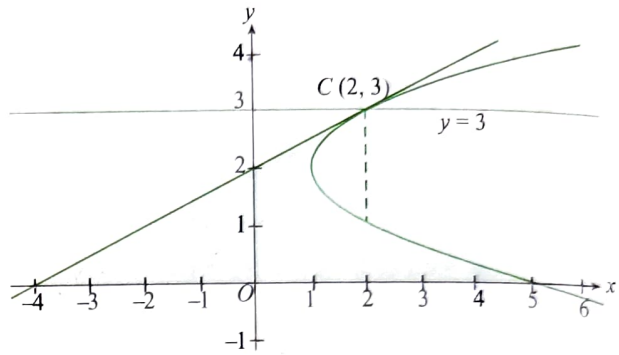
$$\frac{dy}{dx} = \frac{1}{2(y-2)}$$

At (2, 3),

$$\frac{dy}{dx} = \frac{1}{2(3-2)} = \frac{1}{2}$$

Tangent at (2, 3) is

$$y - 3 = \frac{1}{2}(x - 2) \quad \Rightarrow \quad x - 2y + 4 = 0$$



Therefore, the required area

$$= \int_0^3 [(y-2)^2 + 1]\,dy - \int_0^3 (2y - 4)\,dy$$

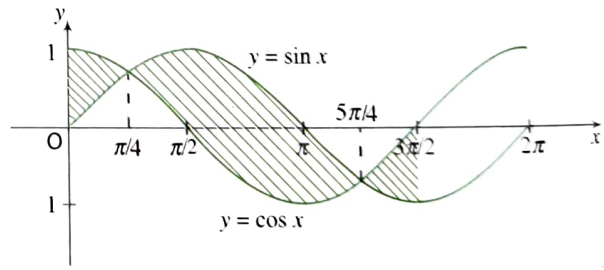
$$= \left| \frac{(y-2)^3}{3} + y \right|_0^3 - |y^2 - 4y|_0^3$$

$$= \frac{1}{3} + 3 + \frac{8}{3} - (9 - 12)$$

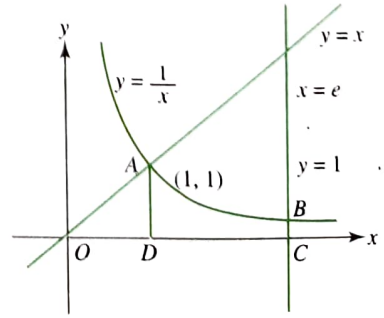
$= 9$ sq. units

2. (4) $\int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\,dx$

$$+ \int_{5\pi/4}^{3\pi/2} (\cos x - \sin x) = 4\sqrt{2} - 2$$



3. (4)



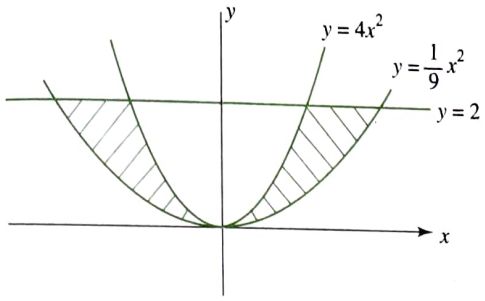
Required area = Area of ΔAOD + Area of region $ADCB$

$$= \frac{1}{2} \times 1 \times 1 + \int_1^e \frac{1}{x} dx$$

$$= \frac{1}{2} + [\ln x]_1^e$$

$$= \frac{1}{2} + 1 - 0 = \frac{3}{2} \text{ sq. units}$$

4. (3)

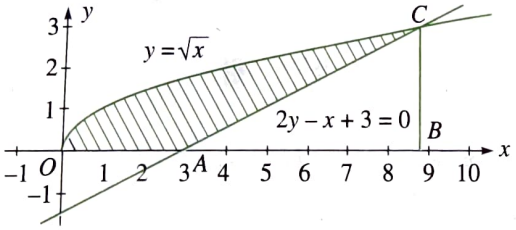


Intergrating along y-axis,

$$\text{Area} = 2\int_0^2 \left(3\sqrt{y} - \frac{\sqrt{y}}{2}\right) dy = 2\left[\frac{5}{2}\frac{y\sqrt{y}}{3/2}\right]_0^2$$

$$= 2 \cdot \frac{5}{3} 2\sqrt{2} = \frac{20\sqrt{2}}{3}$$

. **(1)** $y = \sqrt{x}$ and $2y - x + 3 = 0$



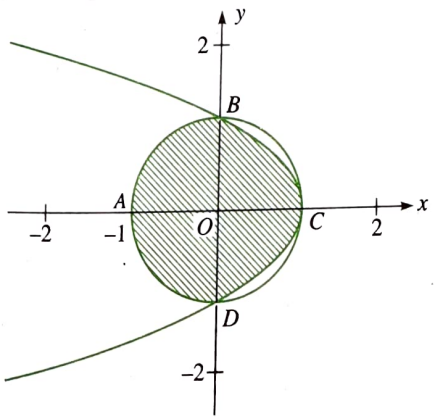
∴ Required area = Area of region OABCO − Area of ΔABC

$$= \int_0^9 \sqrt{x}\, dx - \frac{1}{2} \times BC \times AB$$

$$= \left(\frac{x^{3/2}}{3/2}\right)_0^9 - \frac{1}{2} \times 3 \times 6$$

$$= 9 \text{ sq. units}$$

(4) $x^2 + y^2 \le 1$, represents the region inside the circle $x^2 + y^2 = 1$.



$y^2 \le 1 - x$, represents the region inside parabola $y^2 = 1 - x$ towards focus.

The common region is as shown in the figure.

The area of shaded region

= Area of semicircle $ADOBA$ + Area of region $ODCBO$

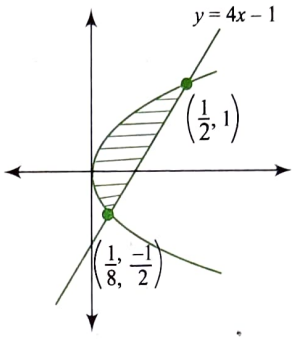
$$= \frac{\pi}{2} + 2\int_0^1 (1 - y^2) dy$$

$$= \frac{\pi}{2} + 2\left[y - \frac{y^3}{3}\right]_0^1$$

$$= \frac{\pi}{2} + 2\left[1 - \frac{1}{3}\right]$$

$$= \frac{\pi}{2} + \frac{2}{3}$$

7. (4) $y^2 \le 2x$ represents the interior of parabola $y^2 = 2x$ (towards focus) and $y \ge 4x - 1$ represents the half region of plane to the left of the line $y = 4x - 1$.

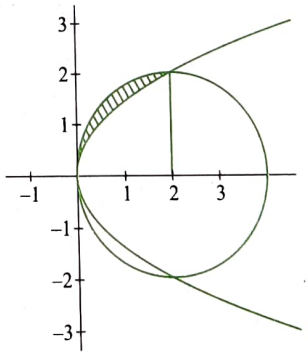


$$\int_{-1/2}^{1} \left(\frac{y+1}{4} - \frac{y^2}{2}\right) dy = \frac{1}{4}\left\{\frac{y^2}{2} + y\right\}_{-1/2}^{1} - \frac{1}{6}\{y^3\}_{-1/2}^{1}$$

$$= \frac{1}{4}\left\{\left(\frac{1}{2} + 1\right) - \left(\frac{1}{8} - \frac{1}{2}\right)\right\} - \frac{1}{6}\left\{1 + \frac{1}{8}\right\}$$

$$= \frac{1}{4}\left\{\frac{3}{2} + \frac{3}{8}\right\} - \frac{1}{6}\left\{\frac{9}{8}\right\} = \frac{15}{32} - \frac{6}{32} = \frac{9}{32}$$

8. (1) $y^2 = 2x$ and $x^2 + y^2 = 4x$ meet at $O(0, 0)$ and $B(2, 2)$ {$(2, -2)$ is not considered as $x, y \ge 0$}



Now required area $= \frac{1}{4}\pi \times 4 - \int_0^2 \sqrt{2x}\, dx$

$$= \pi - \sqrt{2}\frac{2}{3}\left[x^{3/2}\right]_0^2$$

$$= \pi - \frac{8}{3}$$

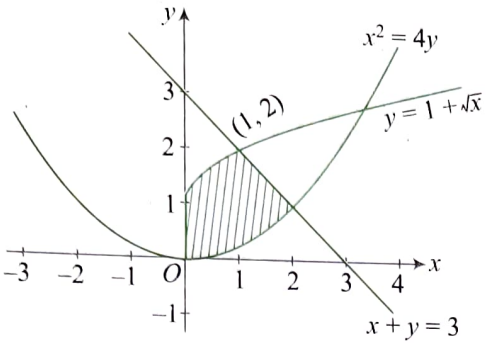
9. (1) $x \ge 0$ is the region of first quadrant and fourth quadrant.

$x + y \le 3$ is half the region of the plane divided by the line $x + y = 3$ where origin lies.

$x^2 \le 4y$ is the region inside the parabola $x^2 = 4y$.

$y \geq 1+\sqrt{x}$ is the region inside the parabola $y = 1+\sqrt{x}$, which has vertex at (0, 1).

So, the required region is as shown in the following figure.



From the figure,

$$\text{area} = \int_0^1 (1+\sqrt{x})dx + \int_1^2 (3-x)dx - \int_0^2 \frac{x^2}{4}dx = \frac{5}{2}$$

10. (2) $18x^2 - 9\pi x + \pi^2 = 0$

$\Rightarrow \quad (6x-\pi)(3x-\pi) = 0$

$\therefore \quad \alpha = \frac{\pi}{6}, \beta = \frac{\pi}{3}$

$$y = g(f(x)) = g(\sqrt{x}) = \cos x$$

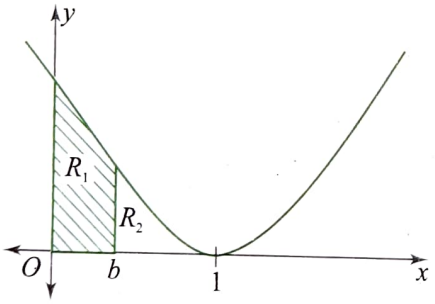
Area bounded by curves $y = \cos x$, $x = \frac{\pi}{6}$, $x = \frac{\pi}{3}$ and $y = 0$ is

$$\int_{\pi/6}^{\pi/3} \cos x\, dx = [\sin x]_{\pi/6}^{\pi/3} = \sin\frac{\pi}{3} - \sin\frac{\pi}{6} = \frac{1}{2}(\sqrt{3}-1)$$

JEE ADVANCED

Single Correct Answer Type

1. (2) $\because \quad \int_0^b (1-x)^2\,dx - \int_b^1 (1-x)^2\,dx = \frac{1}{4}$



$$\Rightarrow \quad \frac{(x-1)^3}{3}\Big|_0^b - \frac{(x-1)^3}{3}\Big|_b^1 = \frac{1}{4}$$

$$\Rightarrow \quad \frac{(b-1)^3}{3} + \frac{1}{3} - \left(0 - \frac{(b-1)^3}{3}\right) = \frac{1}{4}$$

or $\quad \frac{2(b-1)^3}{3} = -\frac{1}{12}$

or $\quad (b-1)^3 = -\frac{1}{8} \quad$ or $\quad b = \frac{1}{2}$

2. (2) Since $\sin x$ and $\cos x > 0$ for $x \in [0, \pi/2]$, the graph of $y = \sin x + \cos x$ always lies above the graph of $y = |\cos x - \sin x|$

Also $\cos x > \sin x$ for $x \in [0, \pi/4]$ and

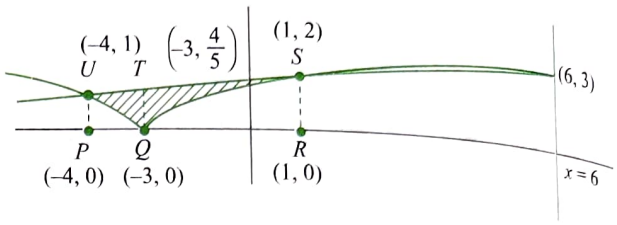
$\sin x > \cos x$ for $x \in [\pi/4, \pi/2]$

$$\Rightarrow \quad \text{Area} = \int_0^{\pi/4} ((\sin x + \cos x) - (\cos x - \sin x))\,dx + \int_{\pi/4}^{\pi/2} ((\sin x + \cos x) - (\sin x - \cos x))\,dx$$

$$= 4 - 2\sqrt{2}$$

3. (3) $y \geq \sqrt{|x+3|}$

$$y^2 \geq \begin{cases} x+3 & \text{if } x \geq -3 \\ -x-3 & \text{if } x < -3 \end{cases}$$



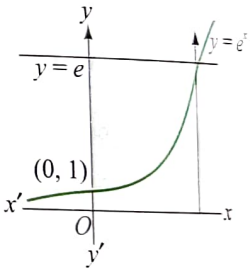
$$A = \left(A(\text{trapezium PQTU} - \int_{-4}^{-3} \sqrt{-x-3}\,dx\right) + \left(A(\text{trapezium QRST}) - \int_{-3}^{1} \sqrt{x+3}\,dx\right)$$

$$= \left(\frac{11}{10} - \frac{2}{3}\right) + \frac{16}{15} = \frac{3}{2}$$

Multiple Correct Answers Type

1. (2), (3), (4)

$$\text{Required area} = \int_1^e \ln y\,dy = (y\ln y - y)_1^e = (e-e) - \{-1\} = 1$$



Also,

$$\int_1^e \ln y\,dy \quad \int_1^e \ln(e+1-y)\,dy$$

Further Required area $= e \times 1 - \int_0^1 e^x\,dx$

2. (1), (4)

Area between $y = x^3$ and $y = x$ in $x \in (0, 1)$ is

$$A = \int_0^1 (x - x^3)dx = \frac{1}{4}$$

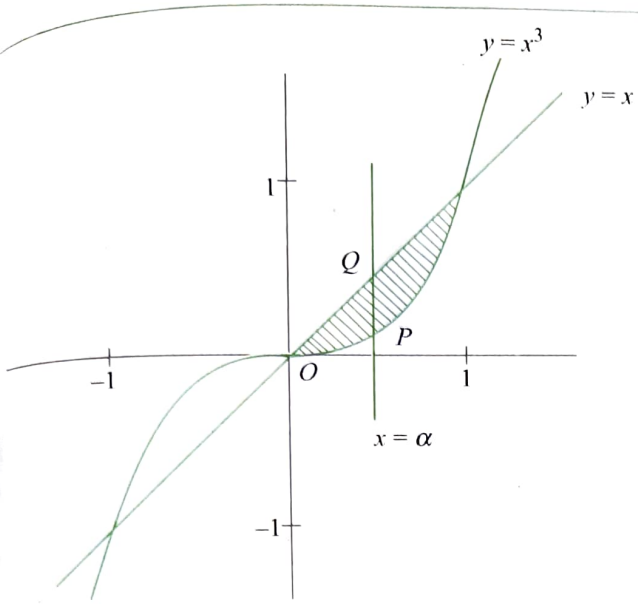
Area of curvilinear triangle $OPQ = \frac{A}{2} = \frac{1}{8}$

$$\Rightarrow \quad \int_0^\alpha (x-x^3)dx = \frac{1}{8}$$

$$\Rightarrow \quad 2\alpha^4 - 4\alpha^2 + 1 = 0$$

$$\Rightarrow \quad (\alpha^2 - 1)^2 = \frac{1}{2}$$

$$\Rightarrow \quad \alpha^2 = \frac{\sqrt{2}-1}{\sqrt{2}}$$



3. (2), (3)

$$f(x) = 1 - 2x + \int_0^x e^{x-t} f(t)dt$$

$$\Rightarrow \quad f(x) = 1 - 2x + e^x \int_0^x e^{-t} f(t)dt \qquad \ldots(i)$$

Differentiating w.r.t. x, we get

$$f'(x) = -2 + e^x e^{-x} f(x) + e^x \int_0^x e^{-t} f(t)dt \qquad \ldots(ii)$$

Subtracting (i) from (ii), we get

$$f'(x) - f(x) = -2 + f(x) - 1 + 2x$$

$$\Rightarrow \quad f'(x) - 2f(x) = 2x - 3$$

This is linear differential equation.

$$\text{I.F.} = e^{\int -2dx} = e^{-2x}$$

Therefore, solution is

$$y \cdot e^{-2x} = \int (2x-3)e^{-2x}dx + c$$

$$\Rightarrow \quad y \cdot e^{-2x} = (2x-3)\cdot\frac{e^{-2x}}{-2} - \int 2 \times \frac{e^{-2x}}{-2}dx + c$$

$$\Rightarrow \quad y \cdot e^{-2x} = -\frac{(2x-3)\,e^{-2x}}{2} - \frac{e^{-2x}}{2} + c$$

$$\Rightarrow \quad y \cdot e^{-2x} = (1-x)e^{-2x} + c \qquad \ldots(iii)$$

$$\Rightarrow \quad y = (1-x) + c.e^{2x}$$

Putting $x = 0$ in (i), we get $f(0) = 1$.

So, from above equation, we have

$$1 = 1 + c \quad \Rightarrow \quad c = 0$$

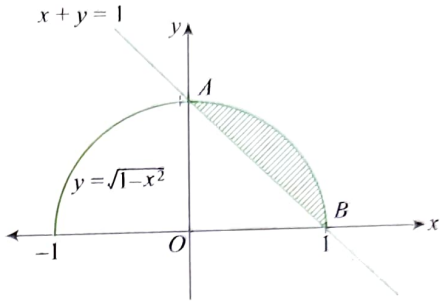
Now, (iii) reduces to $y = 1 - x$, which passes through point $(2, -1)$.

Now, $f(x) \le y \le \sqrt{1-x^2}$

$$\Rightarrow \quad 1 - x \le y \le \sqrt{1-x^2}$$

$1 - x \le y$ represents the region above the line $1 - x = y$ and $y \le \sqrt{1-x^2}$ represents the region inside the semicircle (lying above x-axis) $x^2 + y^2 = 1$.

So, common region is as the shaded one in the following figure:



Area of the shaded region

= Area of quarter of the circle − Area of triangle OAB

$$= \frac{1}{4} \times \pi(1)^2 - \frac{1}{2} \times 1 \times 1 = \frac{\pi - 2}{4}$$

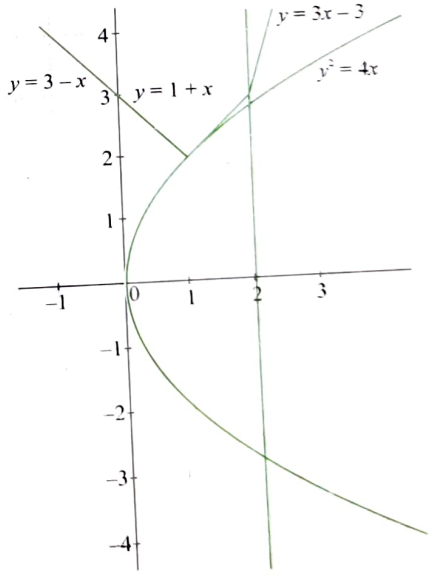
Matrix Match Type

1. d → s, t

For $\alpha = 1$

$$y = |x-1| + |x-2| + x = \begin{cases} 3 - x; & x < 1 \\ 1 + x; & 1 \le x < 2 \\ 3x - 3; & x \ge 2 \end{cases}$$

For $\alpha = 0, y = 3$

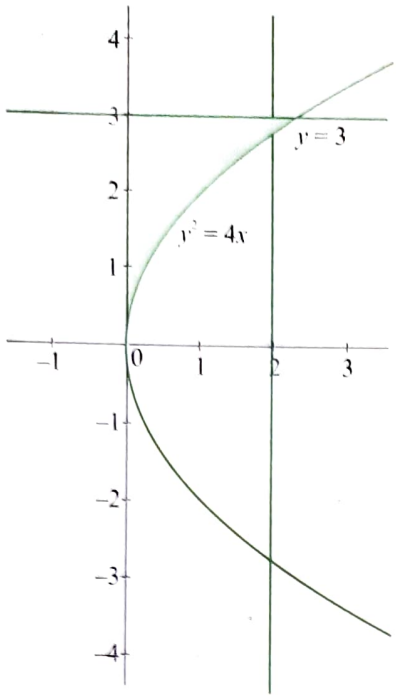


$$A = \frac{1}{2}(2+3) \times 1 + \frac{1}{2}(2+3) \times 1 - \int_0^2 2\sqrt{x}dx$$

$$\Rightarrow \quad A = 5 - \frac{8}{3}\sqrt{2}$$

$$\therefore \quad F(1) + \frac{8}{3}\sqrt{2} = 5$$

For $\alpha = 0, y = |-1| + |-2| = 3$



$$A = 6 - \int_0^2 2\sqrt{x}\,dx$$

$$\Rightarrow \quad A = 6 - \frac{8}{3}\sqrt{2}$$

$$\therefore \quad F(0) + \frac{8}{3}\sqrt{2} = 6$$

Note: Solutions of the remaining parts are given in their respective chapters.

Numerical Value Type

1. (6) For $P(x, y)$, we have

$$2 \le d_1(P) + d_2(P) \le 4$$

$$\Rightarrow \quad 2 \le \frac{|x-y|}{\sqrt{2}} + \frac{|x+y|}{\sqrt{2}} \le 4$$

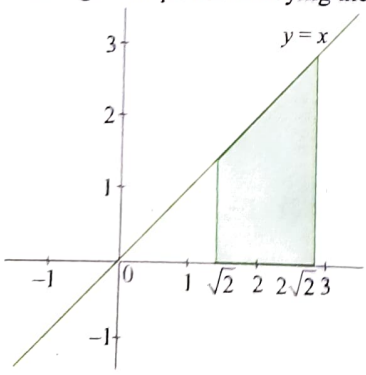
$$\Rightarrow \quad 2\sqrt{2} \le |x-y| + |x+y| \le 4\sqrt{2}$$

In first quadrant if $x > y$, we have

$$2\sqrt{2} \le x - y + x + y \le 4\sqrt{2}$$

or $\quad \sqrt{2} \le x \le 2\sqrt{2}$

The region of points satisfying these inequalities is

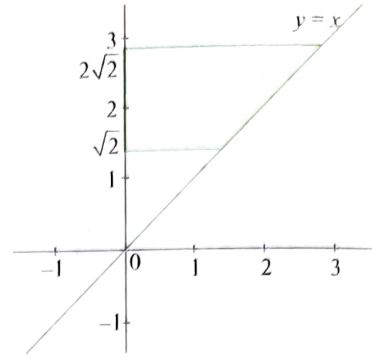


In first quadrant if $x < y$, we have

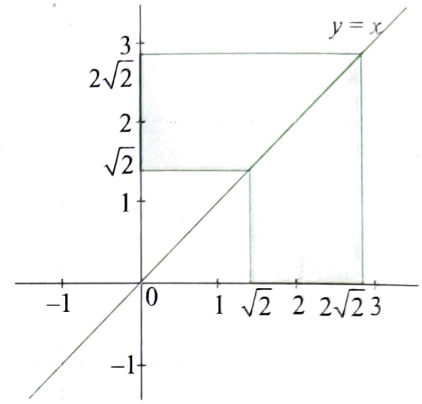
$$2\sqrt{2} \le y - x + x + y \le 4\sqrt{2}$$

or $\quad \sqrt{2} \le y \le 2\sqrt{2}$

The region of points satisfying these inequalities is



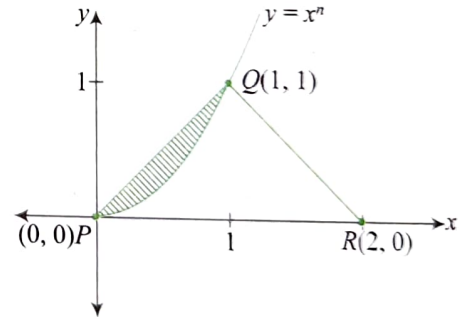
Combining above two regions we have



Area of the shaded region = $((2\sqrt{2})^2 - (\sqrt{2})^2)$

$= 8 - 2 = 6$ sq. units

2. (4)



According to the question,

$$\int_0^1 (x - x^n)\,dx = \frac{3}{10}\left(\frac{1}{2} \times 2 \times 1\right)$$

$$\Rightarrow \quad \left|\frac{x^2}{2} - \frac{x^{n+1}}{n+1}\right|_0^1 = \frac{3}{10}$$

$$\Rightarrow \quad \frac{1}{2} - \frac{1}{n+1} = \frac{3}{10}$$

$$\Rightarrow \quad \frac{1}{n+1} = \frac{1}{2} - \frac{3}{10} = \frac{1}{5}$$

$$\Rightarrow \quad n = 4$$

Chapter 10

Concept Application Exercises

Exercise 10.1

1. $\frac{d^2y}{dx^2} = \left\{1+\left(\frac{dy}{dx}\right)^4\right\}^{5/3}$

or $\left(\frac{d^2y}{dx^2}\right)^3 = \left\{1+\left(\frac{dy}{dx}\right)^4\right\}^5$

Hence, the order is 2 and the degree is 3.

2. $\frac{d^3y}{dx^3} = x\ln\left(\frac{dy}{dx}\right)$

Clearly, the order is 3 and the degree is not defined due to $\ln\left(\frac{dy}{dx}\right)$ term.

3. $\left(\frac{d^4y}{dx^4}\right)^3 + 3\left(\frac{d^2y}{dx^2}\right)^6 + \sin x = 2\cos x$

Clearly, order is 4 and degree is 3.

4. We have $\left(\frac{d^3y}{dx^3}\right)^{2/3} + 4 - 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 0$

or $\left(\frac{d^3y}{dx^3}\right)^2 = \left(3\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 4\right)^3$

Clearly, it is a differential equation of degree 2 and order 3.

5. The given equation when expressed as a polynomial in derivative as

$$a^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3$$

Clearly, it is a second-order differential equation of degree 2.

6. $\frac{d^4y}{dx^4} - \sin\left(\frac{d^3y}{dx^3}\right) = 0$

The highest-order derivative present in the differential equation is $\frac{d^4y}{dx^4}$. Thus, its order is four.

However, the given differential equation is not a polynomial equation. Hence, its degree is not defined.

Exercise 10.2

1. All such lines are given by $y = mx + c$.

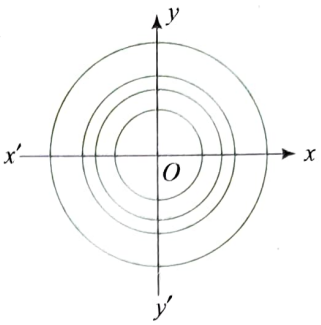
Here, we have two effective constants m and c. So, it is required to differentiate twice.

$y = mx + c$

$\therefore \quad \frac{dy}{dx} = m$

or $\frac{d^2y}{dx^2} = 0$

2.



Equation of family of such concentric circles is given by $x^2 + y^2 = r^2$.

Differentiating w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0$$

or $x + y\frac{dy}{dx} = 0$, which is required differential equation.

3. Equation of parabola whose axes is parallel to x-axis and having latus rectum 'a' is $(y-\beta)^2 = a(x-\alpha)$

Here we have two effective constants α and β.

So it is required to differentiate twice.

Differentiating both sides,

we get $2(y-\beta)\frac{dy}{dx} = a$ ...(2)

Differentiating (2), w.r.t. x

$$\Rightarrow \quad 2(y-\beta)\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 0 \qquad \text{...(3)}$$

Eliminating β from (2) and (3),

$$\Rightarrow \quad a.\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^3 = 0,$$

which is required differential equation.

4. $y = Ae^{2x} + Be^{-2x}$

$\therefore \quad \frac{dy}{dx} = 2Ae^{2x} - 2Be^{-2x}$

or $\frac{d^2y}{dx^2} = 4Ae^{2x} + 4Be^{-2x}$

$= 4(Ae^{2x} + Be^{-2x})$

$= 4y$, which is the required differential equation.

5. Putting $x = \tan A$ and $y = \tan B$ in the given relation, we get

$\cos A + \cos B = \lambda(\sin A - \sin B)$

or $\tan\left(\frac{A-B}{2}\right) = \frac{1}{\lambda}$

or $\tan^{-1}x - \tan^{-1}y = 2\tan^{-1}\left(\frac{1}{\lambda}\right)$

Differentiating w.r.t. to x, we get

$$\frac{1}{1+x^2} - \frac{1}{1+y^2}\frac{dy}{dx} = 0$$

or $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

6. $y = (c_1 + c_2)\cos(x + c_3) - c_4 e^{x+c_5}$

or $y = A\cos(x + B) - Ce^x$, where

$A = c_1 + c_2$, $B = c_3$ and $C = c_4 e^{c_5}$

$\Rightarrow \quad \frac{dy}{dx} = -A\sin(x + B) - Ce^x$

$\Rightarrow \quad \frac{d^2y}{dx^2} = -A\cos(x + B) - Ce^x$

$\Rightarrow \quad \frac{d^2y}{dx^2} + y = -2Ce^x$

$\Rightarrow \quad \frac{d^3y}{dx^3} + \frac{dy}{dx} = -2Ce^x = \frac{d^2y}{dx^2} + y$

$\Rightarrow \quad \frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0.$

Exercise 10.3

1. $e^{\frac{dy}{dx}} = x + 1$

or $\frac{dy}{dx} = \log(x + 1)$

or $\int dy = \int \log(x+1)dx$

or $y = (x + 1)\log(x + 1) - x + c$

when $x = 0$, $y = 3$ gives $c = 3$

Hence, the solution is $y = (x + 1)\log(x + 1) - x + 3$.

2. We have $x(1 - y^2)dx = y(1 - x^2)dy$

$\therefore \quad \frac{2x}{x^2 - 1}dx = \frac{2y}{y^2 - 1}dy$

Integrating both sides, we get

$\log_e(x^2 - 1) = \log_e(y^2 - 1) + \log_e C$

$\therefore \quad (x^2 - 1) = C(y^2 - 1)$

3. The given equation can be rewritten as

$$\int \frac{\sec^2 x}{\tan x}dx + \int \frac{\sec^2 y}{\tan y}dy = \log c$$

or $\log \tan x + \log \tan y = \log c$, where c is an arbitrary positive constant

or $\tan x \tan y = c$.

4. $e^x \tan y\, dx + (1 - e^x)\sec^2 y\, dy = 0$

or $(1 - e^x)\sec^2 y\, dy = -e^x \tan y\, dx$

or $\int \frac{\sec^2 y}{\tan y}dy = \int \frac{-e^x}{1 - e^x}dx$

or $\int \frac{(\tan y)'}{\tan y}dy = \int \frac{(1 - e^x)'}{1 - e^x}dx$

or $\log(\tan y) = \log(1 - e^x) + \log C$

or $\log(\tan y) = \log[C(1 - e^x)]$

or $\tan y = C(1 - e^x)$

5. $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

or $\int \frac{dx}{a + x} = \int \frac{dy}{y - ay^2} = \int\left(\frac{1}{y} + \frac{a}{1 - ay}\right)dy$

[By partial fractions]

Integrating, we get

$\log(a + x) + \log c = \log y - \log(1 - ay)$

where c is an arbitrary positive constant.

Thus, the solution can be written as $\frac{y}{1 - ay} = c(a + x)$

6. $(x - y)(dx + dy) = dx - dy$

or $(x - y + 1)dy = (1 - x + y)dx$

or $\frac{dy}{dx} = \frac{1 - x + y}{x - y + 1}$

or $\frac{dy}{dx} = \frac{1 - (x - y)}{1 + (x - y)}$ (1)

Let $x - y = t$.

$\therefore \quad \frac{d}{dx}(x - y) = \frac{dt}{dx}$

or $1 - \frac{dt}{dx} = \frac{dy}{dx}$

Substituting the values of $x - y$ and $\frac{dy}{dx}$ in equation (1), we get

$1 - \frac{dt}{dx} = \frac{1 - t}{1 + t}$

or $\frac{dt}{dx} = 1 - \left(\frac{1 - t}{1 + t}\right) = \frac{2t}{1 + t}$

or $\left(1 + \frac{1}{t}\right)dt = 2\,dx$

Integrating both sides, we get

$t + \log|t| = 2x + C$

or $(x - y) + \log|x - y| = 2x + C$

or $\log|x - y| = x + y + C$ (2)

Now, $y = -1$ at $x = 0$.

Therefore, equation (2) becomes

$\log 1 = 0 - 1 + C$

$\therefore \quad C = 1$

Substituting $C = 1$ in equation (2) we get, $\log|x - y| = x + y + 1$.

This is the required particular solution of the given differential equation.

7. $\frac{dy}{dx} + yf'(x) = f(x)\,f'(x)$

or $\frac{dy}{dx} = [f(x) - y]\,f'(x)$

Put $f(x) - y = t$

$\therefore \quad f'(x) - \frac{dy}{dx} = \frac{dt}{dx}$

Then the given equation transforms to

$f'(x) - \frac{dt}{dx} = tf'(x)$

or $(1 - t)f'(x) = \frac{dt}{dx}$

or $\int \frac{dt}{1 - t} = \int f'(x)dx$

or $-\log(1 - t) = f(x) + c$

or $\log[1 + y - f(x)] + f(x) + c = 0$

8. $\frac{dy}{dx} = \cos(x + y) - \sin(x + y)$

Putting $x + y = t$, we get $\frac{dy}{dx} = \frac{dt}{dx} - 1$

Therefore, $\frac{dt}{dx} - 1 = \cos t - \sin t$

or $\dfrac{dt}{1+\cos t-\sin t}=dx$ or $\dfrac{\sec^2\frac{t}{2}\,dt}{2\left(1-\tan\frac{t}{2}\right)}=dx$

or $-\ln\left|1-\tan\dfrac{x+y}{2}\right|=x+c.$

9. We have $f(x)f''(x)-f(x)f'(x)-(f'(x))^2=0$

Dividing by $f(x)f'(x)$, we get

$$\frac{f''(x)}{f'(x)}-1=\frac{f'(x)}{f(x)}$$

Integrating both sides, we get

$$\log_e f'(x)-x=\log_e f(x)+C$$

Since, $f(0)=1=f'(0)$, $C=0$.

$\therefore$ $\log_e f'(x)-\log_e f(x)=x$

$\Rightarrow$ $\log_e\dfrac{f'(x)}{f(x)}=x$

$\Rightarrow$ $\dfrac{f'(x)}{f(x)}=e^x$

Integrating both sides, we get

$$\log_e f(x)=e^x+C$$

Using $f(0)=1$, we have $C=-1$

$\therefore$ $f(x)=e^{e^x-1}$

ercise 10.4

1. Putting $y=vx$ and $\dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$, we get

$$xv+x^2\frac{dv}{dx}=vx+2x\sqrt{v^2-1}$$

or $\displaystyle\int\frac{dv}{2\sqrt{v^2-1}}=\int\frac{dx}{x}$,

Integrating, we get

$$\frac{1}{2}\ln\left(v+\sqrt{v^2-1}\right)=\ln cx$$

or $\dfrac{1}{2}\ln\left(\dfrac{y+\sqrt{y^2-x^2}}{x}\right)=\ln cx$

2. We have $\dfrac{dy}{dx}=\dfrac{\frac{y}{x}}{1-2\sqrt{\frac{y}{x}}}$, which is homogeneous.

Put $y=vx$ so that $\dfrac{dy}{dx}=x\dfrac{dv}{dx}+v$

$\therefore$ $x\dfrac{dv}{dx}=\dfrac{v}{1-2\sqrt{v}}-v=\dfrac{2v^{3/2}}{1-2\sqrt{v}}$

or $\dfrac{dx}{x}=\dfrac{1-2\sqrt{v}}{2v^{3/2}}\,dv=\left(\dfrac{1}{2v^{3/2}}-\dfrac{1}{v}\right)dv$

Integrating, we get

$$-C+\log x=-v^{-1/2}-\log v=-\sqrt{\frac{x}{y}}-\log y+\log x$$

or $\log y+\sqrt{\dfrac{x}{y}}=C$

3. $x(dy/dx)=y(\log y-\log x+1)$

or $\dfrac{dy}{dx}=\dfrac{y}{x}\left[\log\dfrac{y}{x}+1\right]$

Putting $y=vx$, we get $\dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$

and the given equation transforms to

$$v+x\frac{dv}{dx}=v[\log v+1]$$

or $x\dfrac{dv}{dx}=v\log v$

or $\displaystyle\int\frac{dv}{v\log v}=\int\frac{dx}{x}$

or $\log\log v=\log x+\log c,\ c>0$

or $cx=\log(y/x)$

or $y=xe^{cx},\ c>0$

4. $\left[x\sin^2\left(\dfrac{y}{x}\right)-y\right]dx+xdy=0$

or $\dfrac{dy}{dx}=-\sin^2\left(\dfrac{y}{x}\right)+\dfrac{y}{x}$ (1)

The given differential equation is a homogeneous equation.

Put $y=vx$

$\therefore$ $\dfrac{dy}{dx}=v+\dfrac{xdv}{dx}$

Therefore, given equation reduces to

$$v+x\frac{dv}{dx}=v-\sin^2 v$$

or $x\dfrac{dv}{dx}=-\sin^2 v$

or $\operatorname{cosec}^2 v\,dv=-\dfrac{dx}{x}$

Integrating both sides, we get

$$-\cot v=-\log|x|-\log C$$

or $\cot\left(\dfrac{y}{x}\right)=\log|Cx|$ (2)

Now, $y=\dfrac{\pi}{4}$ at $x=1$.

$\therefore$ $\cot\left(\dfrac{\pi}{4}\right)=\log|C|$

or $1=\log C$

or $C=e$

Substituting $C=e$ in equation (2), we get

$$\cot\left(\frac{y}{x}\right)=\log|ex|$$

This is the required solution of the given differential equation

5. $y^3dy+(x+y^2)dx=0$

or $y\dfrac{dy}{dx}=\dfrac{x+y^2}{y^2}$ (1)

Let $y^2=t$ or $2y\dfrac{dy}{dx}=\dfrac{dt}{dx}$

Equation (1) transforms to $\dfrac{dt}{dx}=2\dfrac{x+t}{t}$

or $\dfrac{dt}{dx}=2\left(\dfrac{x}{t}+1\right)$, which is homogeneous.

6. Here, $\frac{dy}{dx}=\frac{2x-y+1}{x+2y-3}$

Cross multiplying, we get

$xdy + (2y - 3)dy = (2x + 1)dx - ydx$

or $(xdy + ydx) + (2y - 3)dy = (2x + 1)dx$

or $d(xy) + (2y - 3)dy = (2x + 1)dx$

Integrating, we get

$xy + y^2 - 3y = x^2 + x + c$

Exercise 10.5

1. $y\,dx - x\,dy + 3x^2 y^2 e^{x^3} dx = 0$

or $\frac{y\,dx - x\,dy}{y^2} + 3x^2 e^{x^3} dx = 0$

or $\int d\left(\frac{x}{y}\right) + \int d(e^{x^3}) = c$

or $\frac{x}{y} + e^{x^3} = c$

2. $\frac{dy}{dx}=\frac{2xy}{x^2-1-2y}$

or $x^2\,dy - (1 + 2y)\,dy = 2xy\,dx$

or $2xy\,dx - x^2\,dy = -(1 + 2y)\,dy$

or $\frac{y\,d(x^2) - x^2\,dy}{y^2} = -\left(\frac{1}{y^2} + \frac{2}{y}\right)dy$

or $d\left(\frac{x^2}{y}\right) = -\left(\frac{1}{y^2} + \frac{2}{y}\right)dy$

Integrating, we get $\frac{x^2}{y} = \frac{1}{y} - 2\log y + c$

3. $y\,dx + (x + x^2 y)\,dy = 0$

or $(x\,dy + y\,dx) + x^2 y\,dy = 0$

or $d(xy) + x^2 y\,dy = 0$

or $\frac{d(xy)}{(xy)^2} + \frac{1}{y}dy = 0$

Integrating, we get $-\frac{1}{xy} + \log y = c$.

4. The given equation is $xy^4\,dx + y\,dx - x\,dy = 0$

Dividing by y^4, we get

$$x\,dx + \frac{y\,dx - x\,dy}{y^4} = 0 \qquad (1)$$

or $$x^3\,dx + \left(\frac{x}{y}\right)^2 d(x/y) = 0 \qquad (2)$$

Integrating equation (2), we get $\frac{x^4}{4} + \frac{1}{3}\left(\frac{x}{y}\right)^3 = c$

or $3x^4 y^3 + 4x^3 = cy^3$, which is the required solution.

5. $y(x^2 y + e^x)dx - e^x\,dy = 0$

or $y\,e^x dx - e^x\,dy + x^2 y^2 dx = 0$

or $\frac{ye^x dx - e^x dy}{y^2} + x^2 dx = 0$

or $d\left(\frac{e^x}{y}\right) + x^2 dx = 0$

Integrating, we get

$\frac{e^x}{y} + \frac{x^3}{3} = k$

or $x^3 y + 3e^x = 3ky$

or $x^3 y + 3e^x = Cy$

6. We have $-xdy = ydx + \sin x\,dx$

$\therefore$ $-\sin x\,dx = xdy + ydx$

$\Rightarrow$ $-\sin x\,dx = d(xy)$

$\Rightarrow$ $\int -\sin x = \int d(xy)$

$\Rightarrow$ $\cos x = xy + c$

Given that $y(0) = 1$

$\therefore$ $c = 1$

So, required particular solution is $\cos x = xy + 1$

7. $y(xy + 1)dx + x(1 + xy + x^2y^2)dy = 0$

$\therefore$ $(xy^2 dx + x^2 y dy) + (ydx + xdy) + x^3 y^2 dy = 0$

$\Rightarrow$ $xyd(xy) + d(xy) + x^3 y^2 dy = 0$

$\Rightarrow$ $\frac{d(xy)}{x^3 y^3} + \frac{d(xy)}{x^2 y^2} + \frac{dy}{y} = 0$

$\Rightarrow$ $-\frac{1}{2x^2 y^2} - \frac{1}{xy} + \log_e y = c$

Exercise 10.6

1. The given differential equation is $(1 - y^2)\frac{dx}{dy} + yx = ay$

or $\frac{dx}{dy} + \frac{yx}{1-y^2} = \frac{ay}{1-y^2}$

This is a linear differential equation of the form

$\frac{dx}{dy} + Py = Q$, where $P = \frac{y}{1-y^2}$ and $Q = \frac{ay}{1-y^2}$

The integrating factor (I.F.) is given by the relation

$$\text{I.F.} = e^{\int p\,dy} = e^{\int \frac{y}{1-y^2}dy}$$

$$= e^{-\frac{1}{2}\log(1-x^2)} = e^{\log\left[\frac{1}{\sqrt{1-y^2}}\right]}$$

$$= \frac{1}{\sqrt{1-y^2}}$$

2. Given equation is linear and

$P = \cot x$, $Q = \sin x$

$\therefore$ $\text{I.F.} = e^{\int \cot x\,dx} = e^{\log \sin x} = \sin x$

Hence, the solution is

$$y \sin x = \int \sin x \sin x\,dx + c$$

$$= \frac{1}{2}\int (1 - \cos 2x)dx + c$$

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + c$$

$\therefore$ $y \sin x = \frac{1}{4}[2x - \sin 2x] + c$

3. The given equation can be rewritten as

$\frac{dx}{dy} - 1x = (y + 1)$ [linear, y as independent variable]

Here, $P = -1$, $Q = (y + 1)$

$\text{I.F.} = e^{\int -dy} = e^{-y}$

Therefore, the solution is

$xe^{-y} = \int (y + 1)e^{-y}\,dy + c$

$= -(y+1)e^{-y} - e^{-y} + c$

or $x = ce^{y} - y - 2$

4. We have

$$\frac{dy}{dx} + \frac{2x}{1-x^2}y = \frac{x}{\sqrt{1-x^2}}$$

Here, $P = \frac{2x}{1-x^2}$ and $Q = \frac{x}{\sqrt{1-x^2}}$

$$\text{I.F.} = e^{\int \frac{2x}{1-x^2}dx} = e^{-\log(1-x^2)} = \frac{1}{1-x^2}$$

Therefore, the solution is

$$\frac{y}{1-x^2} = \int \frac{x}{\sqrt{(1-x^2)}} \times \frac{1}{(1-x^2)}\,dx + c$$

$$= \frac{1}{\sqrt{1-x^2}} + c$$

or $y = \sqrt{1-x^2} + c(1-x^2)$

5. Given equation is $\frac{dx}{dy} = \frac{2y\ln y + y - x}{y}$

or $\frac{dx}{dy} + \frac{1}{y}x = (2\ln y + 1)$

I.F. $= y$ and solution is $xy = \int (2\ln y + 1)\,y\,dy + c$

or $xy = y^2 \ln y + c$

6. $y\,dx + (x - y^2)\,dy = 0$

or $y\,dx = (y^2 - x)\,dy$

or $\frac{dx}{dy} = \frac{y^2 - x}{y} = y - \frac{x}{y}$

or $\frac{dx}{dy} + \frac{x}{y} = y$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Px = Q \text{ where } P = \frac{1}{y} \text{ and } Q = y$$

Now, I.F. $= e^{\int P dy} = e^{\int \frac{1}{y}dy} = e^{\log y} = y.$

The general solution of the given differential equation is given by the relation

$$x\,(\text{I.F.}) = \int (Q \times \text{I.F.})\,dy + C$$

or $xy = \int (y \cdot y)\,dy + C$

or $xy = \int y^2\,dy + C$

or $xy = \frac{y^3}{3} + C$

or $x = \frac{y^2}{3} + \frac{C}{y}$

7. We have $\frac{dy}{dx} = \frac{-(y + y^3)}{1 + x(1+y^2)}$

or $\frac{dx}{dy} = -\frac{1 + x(1+y^2)}{y + y^3}$

$$= -\left[\frac{1}{y(y^2+1)} + \frac{x(1+y^2)}{y(1+y^2)}\right]$$

$$\therefore \quad \frac{dx}{dy} + \frac{x}{y} = \frac{-1}{y(1+y^2)}.$$

$$\text{I.F.} = e^{\int \frac{dy}{y}} = e^{\log y} = y.$$

Hence, the solution is

$$x \cdot y = -\int \frac{1}{y(1+y^2)} \cdot y\,dy + C$$

$$= -\int \frac{dy}{1+y^2} + C = -\tan^{-1} y + C$$

or $xy + \tan^{-1} y = C.$

This passes through (0, 1). Therefore, $\tan^{-1} 1 = C$, i.e., $C = \frac{\pi}{4}$.

Thus, the equation of the curve is $xy + \tan^{-1} y = \frac{\pi}{4}$.

Exercise 10.7

1. Dividing by e^{y}, we get

$$e^{-y}\frac{dy}{dx} + e^{-y}\frac{1}{x} = \frac{1}{x^2}$$

Putting $e^{-y} = v$, we get

$$-\frac{dv}{dx} + \frac{v}{x} = \frac{1}{x^2}$$

or $\frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x^2}$ (linear)

$$\text{I.F.} = e^{-\int (1/x)dx} = e^{-\log x} = 1/x$$

Therefore, solution is

$$\frac{v}{x} = \int -\frac{1}{x^2}\frac{1}{x}dx + c = \frac{x^{-2}}{2} + c$$

$$\therefore \quad \frac{e^{-y}}{x} = \frac{x^{-2}}{2} + c$$

2. The given equation can be expressed as

$$\sec^2 y \frac{dy}{dx} + 2x\tan y = x^3$$

Put $\tan y = z$ so that $\sec^2 y \frac{dy}{dx} = \frac{dz}{dx}$

Given equation transforms to $\frac{dz}{dx} + 2xz = x^3$, which is linear in z.

$$\text{I.F.} = e^{2\int x\,dx} = e^{x^2}$$

Therefore, solution is given by

$$z\,e^{x^2} = \int x^3 e^{x^2}\,dx + c$$

or $\tan y\, e^{x^2} = \frac{1}{2}e^{x^2}(x^2 - 1) + c$

(substitute for $x^2 = t$ and then integrate by parts)

3. $\frac{dy}{dx} + \frac{xy}{(1-x^2)} = x\sqrt{y}$

Dividing by $\sqrt{y}$, we have

$$\frac{1}{\sqrt{y}}\frac{dy}{dx} + \frac{x}{(1-x^2)}\sqrt{y} = x \qquad (1)$$

Putting $\sqrt{y} = v$, we get $\frac{1}{2\sqrt{y}}\frac{dy}{dx} = \frac{dv}{dx}$

Then given equation transforms to

$$\frac{dv}{dx}+\frac{x}{2(1-x^2)}v=\frac{1}{2}x \qquad (2)$$

$$\text{I.F.} = e^{\frac{1}{2}\int[x/(1-x^2)]dx}$$

$$= e^{-\frac{1}{4}\log(1-x^2)}$$

$$= 1/(1-x^2)^{1/4}$$

Therefore, the solution is

$$v/(1-x^2)^{1/4} = \frac{1}{2}\int[x/(1-x^2)^{1/4}]dx+c$$

$$= -\frac{1}{4}\int[(-2x)/(1-x^2)^{1/4}]dx+c$$

$$= -\frac{1}{4}(4/3)(1-x^2)^{3/4}+c$$

Hence, the required solution is

$$\sqrt{y}/(1-x^2)^{1/4} = -\frac{1}{3}(1-x^2)^{3/4}+c.$$

4. $\frac{1}{1+y^2}\cdot\frac{dy}{dx}+2x(\tan^{-1}y)=x^3$ (1)

Put $\tan^{-1}y=z$

$\therefore \quad \frac{1}{1+y^2}\cdot\frac{dy}{dx}=\frac{dz}{dx}$

Thus, (1) reduces to $\frac{dz}{dx}+(2x)z=x^3$, which is liner differential equation

$$\text{I.F.} = e^{\int 2x dx}=e^{x^2}$$

Thus, solution is $z\cdot e^{x^2}=\frac{1}{2}\int 2e^{x^2}\cdot x^3\,dx+C$

or $2e^{x^2}(\tan^{-1}y)=x^2e^{x^2}-e^{x^2}+2C$

or $2\tan^{-1}y=x^2-1+2Ce^{-x^2}$

5. We have $\frac{dy}{dx}-\frac{\tan y}{1+x}=(1+x)e^x\sec y$

$\therefore \quad \cos y\,\frac{dy}{dx}+(-\sin y)\frac{1}{1+x}=e^x(1+x)$ ...(1)

Put $\sin y=v \Rightarrow \cos y\,\frac{dy}{dx}=\frac{dv}{dx}$

$\therefore$ (1) reduces to

$$\frac{dv}{dx}+\left(\frac{-1}{1+x}\right)v=e^x(1+x)$$

$$\text{I.F.} = e^{-\int\frac{1}{1+x}dx}=\frac{1}{1+x}$$

$\therefore$ Solution is

$$v\cdot\frac{1}{x+1}=\int\frac{1}{x+1}e^x(1+x)+c$$

$$\Rightarrow \quad \frac{\sin y}{x+1}=e^x+c$$

$$\Rightarrow \quad \sin y=(e^x+c)(x+1)$$

Exercise 10.8

1. Since subnormal is $y\frac{dy}{dx}$

we have $y\frac{dy}{dx}=ky^2$

or $\frac{dy}{y}=k\,dx$

Integrating, we get

$$\log y = kx+\log c \text{ or } y=ce^{kx}.$$

2. Length of normal $= y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

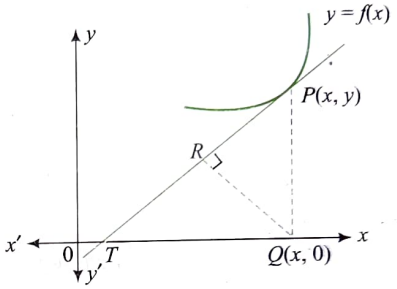
and radius vector $=\sqrt{x^2+y^2}$

$$\therefore \quad y^2\left[1+\left(\frac{dy}{dx}\right)^2\right]=x^2+y^2$$

or $y\frac{dy}{dx}=\pm x$

or $y\,dy\pm x\,dx=0$ or $y^2\pm x^2=c$

3.



Equation of the tangent at $P(x, y)$ is

$$Y-y=\frac{dy}{dx}(X-x)$$

or $\frac{dy}{dx}X-Y+\left(y-x\frac{dy}{dx}\right)=0$

Length of perpendicular QR upon the tangent from the foot of ordinate $Q(x, 0)$ is

$$\left|\frac{x\frac{dy}{dx}-0+y-x\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right|=k$$

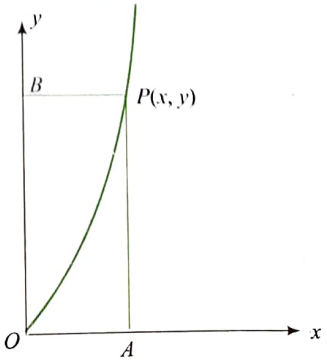
or $1+\left(\frac{dy}{dx}\right)^2=\frac{y^2}{k^2}$ or $\frac{dy}{dx}=\pm\sqrt{\frac{y^2-k^2}{k^2}}$

or $\int\frac{dy}{\sqrt{y^2-k^2}}=\pm\int\frac{1}{k}dx$

or $\log[y+\sqrt{y^2-k^2}]=\pm\frac{x}{k}+\log c$

or $y+\sqrt{y^2-k^2}=ce^{\pm x/k}$

4. Area of $OBPO$: area of $OPAP = m : n$



$$\therefore \quad \frac{xy - \int_0^x y\,dx}{\int_0^x y\,dx} = \frac{m}{n}$$

or $\quad nxy = (m+n)\int_0^x y\,dx$

Differentiating w.r.t .x, we get

$$n\left(x\frac{dy}{dx} + y\right) = (m+n)\,y$$

or $\quad nx\frac{dy}{dx} = my$ or $\frac{m}{n}\frac{dx}{x} = \frac{dy}{y}$

or $\quad y = cx^{m/n}$

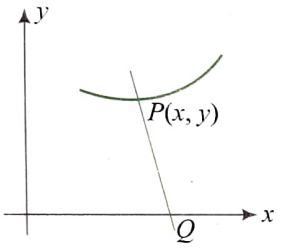
5. The equation of normal to required curve at $P(x, y)$ is given by

$$(X - x) + \frac{dy}{dx}(Y - y) = 0$$

For point Q, where this normal meets X-axis, put $Y = 0$. Then

$$X = x + y\frac{dy}{dx}$$

$$\therefore \quad Q \equiv \left(x + y\frac{dy}{dx}, 0\right)$$



According to question, length of $PQ = k$.

or $\quad \left(y\frac{dy}{dx}\right)^2 + y^2 = k^2$

or $\quad y\frac{dy}{dx} = \pm\sqrt{k^2 - y^2}$

which is the required differential equation of given curve.

Solving this, we get

$$\int \frac{y\,dy}{\sqrt{k^2 - y^2}} = \int \pm\,dx$$

or $\quad -\frac{1}{2}\,2\sqrt{k^2 - y^2} = \pm x + C$

or $\quad -\sqrt{k^2 - y^2} = \pm x + C$

As it passes through $(0, k)$, we get $C = 0$.

Thus, equation of curve is $-\sqrt{k^2 - y^2} = \pm x$

or $x^2 + y^2 = k^2$

6. $x^2 + y^2 = cx$ (1)

Differentiating w.r.t. x, we get $2x + 2y\frac{dy}{dx} = c$ (2)

Eliminating c between equations (1) and (2), we get

$$2x + 2y\frac{dy}{dx} = \frac{x^2 + y^2}{x} \quad \text{or} \quad \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we get $\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$.

This equation is homogeneous, and its solution gives the orthogonal trajectories as $x^2 + y^2 = ky$.

7. Equation of the tangent at point (x, y) on the curve is

$$Y - y = \frac{dy}{dx}(X - x).$$

This meet axis in $A\left(x - y\frac{dx}{dy}, 0\right)$ and $B\left(0, y - x\frac{dy}{dx}\right)$.

Midpoint of AB is $\left(\frac{1}{2}\left(x - y\frac{dx}{dy}\right), \frac{1}{2}\left(y - x\frac{dy}{dx}\right)\right)$.

We are given $\frac{1}{2}\left(x - y\frac{dx}{dy}\right) = x$ and $\frac{1}{2}\left(y - x\frac{dy}{dx}\right) = y$

or $\quad x\frac{dy}{dx} = -y$ or $\frac{dy}{y} = -\frac{dx}{x}$

Integrating both sides, we get $\int\frac{dy}{y} = -\int\frac{dy}{x}$

or $\quad \log y = -\log x + c$

Put $x = 1, y = 1$. Then $\log 1 - \log 1 = c$ or $c = 0$.

$\therefore \quad \log y + \log x = 0$ or $\log yx = 0$

or $\quad yx = e^0 = 1$ which is a rectangular hyperbola.

Exercise 10.9

1. Let $N(t)$ denote the balance in the account at any time t.

Initially, $N(0) = 500$.

For the first four years, $k = 0.085$. Therefore.

$$\frac{dN}{dt} - 0.085\,N = 0$$

Its solution is $N(t) = ce^{0.085t}\ (0 \le t \le 4)$ (1)

At $t = 0$, $N(0) = 500$. Then from (1), $5000 = ce^{0.085(0)} = c$

and equation (1) becomes $N(t) = 5000\,e^{0.085t}\ (0 \le t \le 4)$ (2)

Substituting $t = 4$ into equation (2), we find the balance after four years to be

$$N(4) = 5000\,e^{0.085(4)} = 5000\,(1.404948) = 7024.74.$$

This amount also represents the beginning balance for the last three-year period.

Over the last three years, the interest rate is 9.25%

$$\therefore \quad \frac{dN}{dt} - 0.0925\,N = 0 \qquad (4 \le t \le 7)$$

Its solution is $N(t) = ce^{0.0925t}\ (4 \le t \le 7)$ (3)

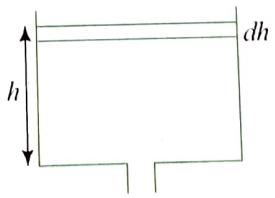
At $t = 4$, $N(4) = 7024.74$. Then from equation (3),

$$7024.74 = ce^{0.0925(4)} = c\,(1.447735) \text{ or } c = 4852.23$$

Then from equation (3),

$$N(7) = 4852.23\,e^{0.0925(7)} = 4852.23(1.910758)$$

2.



Let us allow the water to flow for time dt.

We suppose that in this time, the height of the water level reduces by dh. Therefore,

$$\pi(2.5)^2\,dh = -2.5\sqrt{h}\,\pi(0.025)^2\,dt$$

or $\quad \frac{dh}{dt} = -2.5 \times 10^{-4}\sqrt{h}$ (negative signs denotes that the rate of flow will decrease as t increases)

$$\int\frac{dh}{\sqrt{h}} = -2.5 \times 10^{-4}\int dt$$

or $2\sqrt{h} = -2.5 \times 10^{-4} t + c$

At $t = 0$, $h = 3$ or $c = 2\sqrt{3}$

Hence, for $h = 0$, $t = \dfrac{2\sqrt{3}}{2.5 \times 10^{-4}} = 8000\sqrt{3}$ s.

3. Let x denote the population at time t in years.

Then $\dfrac{dx}{dt} \propto x \Rightarrow \dfrac{dx}{dt} = kx$, where k is constant of proportionality.

Solving $\dfrac{dx}{dt} = kx$, we get $\int \dfrac{dx}{x} = \int k\, dt$

or $\log x = kt + c$ or $x = e^{kt+c}$ or $x = x_0 e^{kt}$,

where x_0 is the population at time $t = 0$.

Since it doubles in 50 years, at $t = 50$, we must have $x = 2x_0$

Hence, $2x_0 = x_0 e^{50k}$ or $50k = \log 2$

or $k = \dfrac{\log 2}{50}$ so that $x = x_0 e^{\frac{\log 2}{50} t}$

To find t, when it triples, i.e., $x = 3x_0$, we get

$$3x_0 = x_0 e^{\frac{\log 2}{50} t}$$

or $\log 3 = \dfrac{\log 2}{50} t$

or $t = \dfrac{50 \log 3}{\log 2} = 50 \log_2 3$

4. Let T be the temperature of the substance at time t.

$\therefore \quad -\dfrac{dT}{dt} = k(T - 290)$

or $\dfrac{dT}{dt} = -k(T - 290)$

$\left(\text{Negative sign because } \dfrac{dT}{dt} \text{ is rate of cooling}\right)$

or $\int \dfrac{dT}{T - 290} = -k \int dt$ (1)

Integrating the L.H.S. between the limits $T = 370$ to $T = 330$ and the R.H.S. between the limits $t = 0$ to $t = 10$, we get

$$\int_{370}^{330} \frac{dT}{T - 290} = -k \int_0^{10} dt$$

or $\log (T - 290)\big|_{370}^{330} = -kt\big|_0^{10}$

or $\log 40 - \log 80 = -k \times 10$

or $\log 2 = 10k$

or $k = \dfrac{\log 2}{10}$ (2)

Now, integrating equation (1) between $T = 370$ and $T = 295$ and $t = 0$ and $t = t$, we get

$$\int_{370}^{295} \frac{dT}{T - 290} = -k \int_0^{t} dt$$

or $\log (T - 290)\big|_{370}^{295} = -kt$

or $\log 5 - \log 80 = -kt$

or $-\log 16 = -kt$

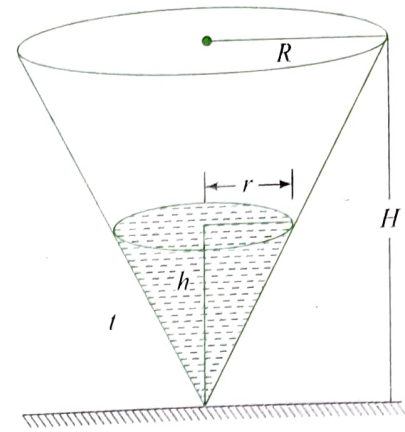
or $t = \dfrac{\log 16}{k}$

Hence, from equation (2), we get

$$t = \frac{\log 16}{\log 2} \times 10 = 40$$

i.e., after 40 min.

5. Let at time t, r and h be the radius and height of cone of water, respectively. Thus, at time t, surface area of liquid in contact with air $= \pi r^2$.



Now according to question, $-\dfrac{dV}{dt} \propto \pi r^2$

[$\because$ "–" sign shows that V decreases with time]

or $\dfrac{dV}{dt} = -k\pi r^2$ ... (1)

But from the figure, we get [Similar Δ's]

$$\frac{r}{h} = \frac{R}{H} \text{ or } h = \frac{rH}{R}$$

or $\dfrac{1}{3}\dfrac{d}{dt}\left[r^2 \dfrac{rH}{R}\right] = -kr^2$ (from (1))

or $\dfrac{r^2 H}{R}\dfrac{dr}{dt} = -kr^2$

or $\dfrac{dr}{dt} = -\dfrac{kR}{H}$

or $r = \dfrac{-kR}{H}t + C$ (integrating)

Now, at $t = 0$, $r = R$

$\therefore \quad R = 0 + C$ or $C = R$

$\therefore \quad r = \dfrac{-kRt}{H} + R$

Now, let the time at which cone is empty be T. Then at T, $r = 0$ (no liquid is left).

$\therefore \quad 0 = \dfrac{-kRT}{H} + R$ or $T = H/k$

Exercises

Single Correct Answer Type

1. (1) Putting $x = \sin A$ and $y = \sin B$ in the given relation, we get

$$\cos A + \cos B = a(\sin A - \sin B)$$

or $A - B = 2 \cot^{-1} a$

or $\sin^{-1} x - \sin^{-1} y = 2 \cot^{-1} a$

Differentiating w.r.t. x, we get

$$\frac{1}{\sqrt{1 - x^2}} - \frac{1}{\sqrt{1 - y^2}}\frac{dy}{dx} = 0$$

Clearly, it is a differential equation of degree one.

2. (3) $y = e^{mx}$ satisfies $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 12y = 0$

Then $e^{mx}(m^3 - 3m^2 - 4m + 12) = 0$

or $m = \pm 2, 3$

$m \in N$

Hence, $m \in \{2, 3\}$

3. (1) Equation of family of circles is $(x^2 + y^2 - 2x - 2y + 1) + \lambda(3x + 4y - 7) = 0$

Arbitrary constant is 1

So, order is 1.

4. (2) The general equation of all non-horizontal lines in xy-plane is $ax + by = 1$, where $a \neq 0$.

Now, $ax + by = 1$

or $a\dfrac{dx}{dy} + b = 0$ [Differentiating w.r.t. y]

or $a\dfrac{d^2x}{dy^2} = 0$ [Differentiating w.r.t. y]

or $\dfrac{d^2x}{dy^2} = 0$ [$\because a \neq 0$]

Hence, the required differential equation is $\dfrac{d^2x}{dy^2} = 0$

5. (1) $y = e^x(A\cos x + B\sin x)$

$$\frac{dy}{dx} = e^x[-A\sin x + B\cos x] + e^x[A\cos x + B\sin x]$$

$$\frac{dy}{dx} = e^x[-A\sin x + B\cos x] + y \qquad (1)$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = e^x[-A\sin x + B\cos x] + e^x[-A\cos x - B\sin x] + \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx} - y\right) - y + \frac{dy}{dx} \qquad \text{[using (1)]}$$

$$\frac{d^2y}{dx} - 2\frac{dy}{dx} + 2y = 0$$

6. (4) Equation of family of circles will be $x^2 + (y-2)^2 + \lambda(y-2) = 0$

Differentiating, we get $2x + 2(y-2)\dfrac{dy}{dx} + \lambda\dfrac{dy}{dx} = 0$

Thus, the equation is $x^2 + (y-2)^2 - (y-2)\left(2x\dfrac{dx}{dy} + 2y - 4\right) = 0$

7. (1) The parametric form of the given equation is $x = t, y = t^2$.

The equation of any tangent at t is $2xt = y + t^2$.

Differentiating, we get $2t = y_1$.

Putting this value in the equation of tangent, we get

$2x\,y_1/2 = y + (y_1/2)^2$

or $4xy_1 = 4y + y_1^2$

The order of this equation is one and degree is two.

8. (3) The given family of curves is $x^2 + y^2 - 2ay = 0$ (1)

Differentiating w.r.t. x, we get $2x + 2y\dfrac{dy}{dx} - 2a\dfrac{dy}{dx} = 0$

or $2x + 2y\dfrac{dy}{dx} - \dfrac{x^2 + y^2}{y}\dfrac{dy}{dx} = 0$ [Using equation (1)]

or $2xy + (2y^2 - x^2 - y^2)\,y' = 0$

or $(y^2 - x^2)\,y' + 2xy = 0$

or $(x^2 - y^2)\,y' = 2xy$

9. (2) $y = c_1\cos(x + c_2) - (c_3\,e^{-x+c_4}) + (c_5\sin x)$

$= c_1(\cos x\ \cos c_2 - \sin x\ \sin c_2) - (c_3\,e^{c_4}\,e^{-x}) + (c_5\sin x)$

$= (c_1\cos c_2)\cos x - (c_1\sin c_2 - c_5)\sin x - (c_3\,e^{c4})\,e^{-x}$

or $y = l\cos x + m\sin x - n\,e^{-x}$ (1)

where l, m, n are arbitrary constants

$\therefore \quad \dfrac{dy}{dx} = -l\sin x + m\cos x + n\,e^{-x}$ (2)

or $\dfrac{d^2y}{dx^2} = -l\cos x - m\sin x - n\,e^{-x}$ (3)

or $\dfrac{d^3y}{dx^3} = l\sin x - m\cos x + n\,e^{-x}$ (4)

From equations (1) + (3), $\dfrac{d^2y}{dx^2} + y = -2n\,e^{-x}$ (5)

From equations (2) + (4), $\dfrac{d^3y}{dx^3} + \dfrac{dy}{dx} = 2n\,e^{-x}$ (6)

From equations (5) + (6), $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0$

10. (4) $\log c + \log|x| = \dfrac{x}{y}$

Differentiating w.r.t. x, $\dfrac{1}{x} = \dfrac{y - x\dfrac{dy}{dx}}{y^2}$

or $\dfrac{y^2}{x} = y - x\dfrac{dy}{dx}$

or $\dfrac{dy}{dx} = \dfrac{y}{x} - \dfrac{y^2}{x^2}$

or $\phi\left(\dfrac{x}{y}\right) = -\dfrac{y^2}{x^2}$

11. (3) $\dfrac{x}{c-1} + \dfrac{y}{c+1} = 1$ (1)

or $\dfrac{1}{c-1} + \dfrac{y'}{c+1} = 0$ (2)

or $\dfrac{y'}{1} = \dfrac{c+1}{1-c}$

or $\dfrac{y'-1}{y'+1} = c$

Put value of c in equation (1). Thus,

$$\frac{x}{\dfrac{y'-1}{y'+1} - 1} + \frac{y}{\dfrac{y'-1}{y'+1} + 1} = 1$$

or $\dfrac{x(y'+1)}{-2} + \dfrac{y(y'+1)}{2y'} = 1$

or $\dfrac{(y'+1)}{2}\left(\dfrac{y}{y'} - x\right) = 1$

or $\left(1 + \dfrac{dy}{dx}\right)\left(y - x\dfrac{dy}{dx}\right) = 2\dfrac{dy}{dx}$

12. (1) $\dfrac{1}{y+1}\,dy = -\dfrac{\cos x}{2 + \sin x}\,dx$

Integrating, we get

$\log(y+1) + \log k + \log(2 + \sin x) = 0$

$\therefore \quad k(y+1)(2+\sin x) = 1$ when $x = 0, y = 1$ where k is constant.

$\therefore \quad 4k = 1$ or $k = 1/4$

$\therefore \quad (y+1)(2 + \sin x) = 4$

Now, put $x = \pi/2$

$\therefore \quad (y+1)\,3 = 4 \quad \therefore \quad y = 1/3$

13. (1) Slope $= \frac{dy}{dx} = \frac{y-1}{x^2+x}$

or $\frac{dy}{y-1} = \frac{dx}{x^2+x}$

or $\int \frac{1}{y-1}dy = \int\left(\frac{1}{x} - \frac{1}{x+1}\right)dx + C$

or $\frac{(y-1)(x+1)}{x} = k$

Putting $x = 1, y = 0$, we get $k = -2$.

The equation is $(y - 1)(x + 1) + 2x = 0$.

14. (4) $(y\cos y + \sin y)\,dy = (2x\log x + x)\,dx$

$$y\sin y - \int \sin y\,dy + \int \sin y\,dx$$

$$= x^2\log x - \int x^2\,\frac{1}{x}\,dx + \int x\,dx + c$$

$\therefore \quad y\sin y = x^2\log x + c$

15. (1) $x^2(y+1)\,dx + y^2(x-1)dy = 0$

or $\frac{x^2dx}{x-1} = -\frac{y^2dy}{y+1}$

or $\int\left[x+1+\frac{1}{x-1}\right]dx = -\int\left[y-1+\frac{1}{y+1}\right]dy$

or $\frac{x^2}{2} + x + \ln(x-1) = -\left[\frac{y^2}{2} - y + \ln(y+1)\right] + \ln c$

or $\frac{x^2+y^2}{2} + (x-y) + \ln\left(\frac{(x-1)(y+1)}{c}\right) = 0$

16. (1) $dy - \sin x\sin y\,dx = 0$

or $dy = \sin x\sin y\,dx$

or $\int \operatorname{cosec} y dy = \int \sin x dx$

or $\log\tan\frac{y}{2} = -\cos x + \log c$

or $\log\frac{\tan\frac{y}{2}}{c} = -\cos x$

or $\frac{\tan\frac{y}{2}}{c} = e^{-\cos x}$

or $e^{\cos x}\tan\frac{y}{2} = c$

17. (1) $\frac{dv}{dt} + \frac{k}{m}v = -g$

or $\frac{dv}{dt} = -\frac{k}{m}\left(v + \frac{mg}{k}\right)$

or $\frac{dv}{v + mg/k} = -\frac{k}{m}dt$

or $\log\left(v + \frac{mg}{k}\right) = -\frac{k}{m}t + \log c$

or $v + \frac{mg}{k} = ce^{-k/mt}$

or $v = ce^{-\frac{k}{m}t} - \frac{mg}{k}$

18. (2) We have $\frac{dy}{dx} = \sin\frac{x-y}{2} - \sin\frac{x+y}{2}$

$$= -2\cos\frac{x}{2}\sin\frac{y}{2}$$

or $\frac{1}{2}\operatorname{cosec}\frac{y}{2}dy = -\cos\frac{x}{2}dx$

or $\log\tan\frac{y}{4} = -\frac{\sin\frac{x}{2}}{\frac{1}{2}} + c$

or $\log\tan\left(\frac{y}{4}\right) = c - 2\sin\frac{x}{2}$

19. (1) Put $xy = v$, i.e., $y + x\frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \quad \frac{dv}{dx} = x\frac{\phi(v)}{\phi'(v)}$

$\therefore \quad \frac{\phi'(v)}{\phi(v)}dv = x\,dx.$

Integrating, we get

$$\log\phi(v) = \frac{x^2}{2} + \log k$$

or $\log\frac{\phi(v)}{k} = \frac{x^2}{2}$

or $\phi(v) = ke^{x^2/2}$

or $\phi(xy) = ke^{x^2/2}$

20. (1) $x^2 = e^{\left(\frac{x}{y}\right)^{-1}\left(\frac{dy}{dx}\right)} = e^{\left(\frac{y}{x}\right)\left(\frac{dy}{dx}\right)}$

or $\ln x^2 = \frac{y}{x}\frac{dy}{dx}$

or $\int x\ln x^2 dx = \int y\,dy$

Putting $x^2 = t$, we get $2xdx = dt$

or $\frac{1}{2}\int \ln t\,dt = \frac{y^2}{2}$

or $c + t\ln t - t = y^2$

or $y^2 = x^2\ln x^2 - x^2 + c$

21. (2) Applying componendo and divedendo, we get

$$\frac{dy}{dx} = \frac{e^{-x}}{e^x} = e^{-2x}$$

or $2y = -e^{-2x} + C$

or $2y\,e^{2x} = C\,e^{2x} - 1$

22. (2) The given equation is reduced to $x = e^{xy\,(dy/dx)}$

or $\log x = xy\,\frac{dy}{dx}$

or $\int ydy = \int\frac{1}{x}\log x dx$

or $\frac{y^2}{2} = \frac{(\log x)^2}{2} + C$

3. (4) $\int_0^x t\,y(t)\,dt = x^2 y(x)$

Differentiating w.r.t. x, we get

$x\,y(x) = x^2\,y'(x) + 2x\,y(x)$

or $x\,y(x) + x^2\,y'(x) = 0$

or $x\dfrac{dy}{dx} + y = 0$

or $\log y + \log x = \log c$

or $xy = c$

4. (1) Integrating the given differential equation, we have

$$\frac{dy}{dx} = \frac{-\cos 3x}{3} + e^x + \frac{x^3}{3} + C_1$$

But $y_1(0) = 1$

So, $1 = \left(-\dfrac{1}{3}\right) + 1 + C_1$ or $C_1 = 1/3$

Again integrating, we get

$$y = \frac{-\sin 3x}{9} + e^x + \frac{x^4}{12} + \frac{1}{3}x + C_2$$

But $y(0) = 0$. So, $0 = 0 + 1 + C_2$ or $C_2 = -1$.

Thus, $y = \dfrac{-\sin 3x}{9} + e^x + \dfrac{x^4}{12} + \dfrac{1}{3}x - 1$

5. (1) $y'\,y''' = 3(y'')^2$

or $\int \dfrac{y'''}{y''}\,dx = 3\int \dfrac{y''}{y'}\,dx$

or $\ln y'' = 3\ln y' + \ln c$

or $y'' = c(y')^3$

or $\int \dfrac{y''}{(y')^2}\,dx = \int cy'\,dx$

or $-\dfrac{1}{y'} = cy + d$

or $-dx = (cy + d)\,dy$

or $-x = \dfrac{cy^2}{2} + dy + e$

26. (3) $\dfrac{y'''}{y''} = 8$ or $\log y'' = 8x + c$

When $x = 0$, $y'' = 1$ or $\log 1 = 0$. Thus, $c = 0$.

$\therefore \quad y'' = e^{8x}$

Integrating again, we get

$$y' = \frac{e^{8x}}{8} + \lambda$$

When $x = 0$, $y'(0) = 0$

$\therefore \quad \lambda = -1/8$

$\therefore \quad y' = \dfrac{e^{8x}}{8} - \dfrac{1}{8}$.

Integrate again. Then,

$$y = \frac{e^{8x}}{64} - \frac{x}{8} + k$$

Also, when $x = 0$, $y = \dfrac{1}{8}$. Thus, $k = \dfrac{7}{64}$.

$$\therefore \quad y = \frac{1}{8}\left(\frac{e^{8x}}{8} - x + \frac{7}{8}\right)$$

27. (3) We have $\dfrac{dy}{dx} = \dfrac{y}{x} - \cos^2\left(\dfrac{y}{x}\right)$

Putting $y = vx$ so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$, we get

$$v + x\frac{dv}{dx} = v - \cos^2 v$$

or $\dfrac{dv}{\cos^2 v} = -\dfrac{dx}{x}$

or $\sec^2 u\,du = -\dfrac{1}{x}dx$

On integration, we get

$\tan u = -\log x + \log C$

or $\tan\left(\dfrac{y}{x}\right) = -\log x + \log C$

This passes through $(1, \pi/4)$. Therefore, $1 = \log C$.

So, $\tan\left(\dfrac{y}{x}\right) = -\log x + 1 = -\log x + \log e$

or $y = x\tan^{-1}\left(\log\left(\dfrac{e}{x}\right)\right)$

28. (3) $\dfrac{dy}{dx} = \dfrac{y}{x}\left[\log\dfrac{y}{x} + 1\right]$

Put $y = vx$

$$v + x\frac{dv}{dx} = v\log v + v$$

$\therefore \quad \dfrac{dv}{v\log v} = \dfrac{dx}{x}$

$\therefore \quad \log(\log v) = \log x + \log c = \log cx$

$\therefore \quad \log\dfrac{y}{x} = cx$

29. (1) The given equation can be written as

$$\frac{y}{x}\frac{dy}{dx} = \left\{\frac{y^2}{x^2} + \frac{f(y^2/x^2)}{f'(y^2/x^2)}\right\}$$

The above equation is a homogeneous equation.

Putting $y = vx$, we get

$$v\left[v + x\frac{dv}{dx}\right] = v^2 + \frac{f(v^2)}{f'(v^2)}$$

or $vx\dfrac{dv}{dx} = \dfrac{f(v^2)}{f'(v^2)}$ (variable separable)

or $\dfrac{2vf'(v^2)}{f(v^2)}\,dv = 2\dfrac{dx}{x}$

Now, integrating both sides, we get

$\log f(v^2) = \log x^2 + \log c$ [$\log c$ = constant]

or $\log f(v^2) = \log cx^2$

or $f(v^2) = cx^2$

or $f(y^2/x^2) = cx^2$

30. (2) $(x^2 + xy)\,dy = (x^2 + y^2)dx$ or $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{x^2 + xy}$

Let $\dfrac{y}{x} = v$. Then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

Thus, equation reduces to

$$x\frac{dv}{dx} = \frac{1+v^2}{1+v} - v = \frac{1+v^2-v-v^2}{1+v} = \frac{1-v}{1+v}$$

or $\int \frac{1+v}{1-v}dv = \int \frac{dx}{x}$

or $-\int \left(1-\frac{2}{1-v}\right)dv = \int \frac{dx}{x}$

or $-v - 2\log(1-v) = \log x + \log c$

or $-\frac{y}{x} - 2\log\left(\frac{x-y}{x}\right) = \log x + \log c$

or $\frac{-y}{x} - 2\log(x-y) + 2\log x = \log x + \log c$

or $\log x = 2\log(x-y) + \frac{y}{x} + k$ where $k = \log c$

31. (3) The intersection of $y - x + 1 = 0$ and $y + x + 5 = 0$ is $(-2, -3)$.
Put $x = X - 2$, $y = Y - 3$.

The given equation reduces to $\frac{dY}{dX} = \frac{Y-X}{Y+X}$.
Putting $Y = vX$, we get

$$X\frac{dv}{dX} = -\frac{v^2+1}{v+1}$$

or $\left(-\frac{v}{v^2+1} - \frac{1}{v^2+1}\right)dv = \frac{dX}{X}$

or $-\frac{1}{2}\log\left(v^2+1\right) - \tan^{-1} v = \log|X| + \text{constant}$

or $\log(Y^2+X^2) + 2\tan^{-1}\frac{Y}{X} = \text{constant}$

or $\log((y+3)^2 + (x+2)^2) + 2\tan^{-1}\frac{y+3}{x+2} = C$

32. (1) $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$ (1)

Put $y = vx$, i.e., $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Thus, equation (1) transforms to

$$v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{2xvx} = \frac{1+v^2}{2v}$$

or $x\frac{dv}{dx} = \frac{1+v^2}{2v} - v = \frac{1-v^2}{2v}$

or $\frac{2vdv}{1-v^2} = \frac{dx}{x}$

or $\log x + \log(1-v^2) = \log C$

or $x(1-v^2) = C$

or $x\left(1-\frac{y^2}{x^2}\right) = C$

or $x^2 - y^2 = Cx$

It passes through (2, 1).

$\therefore \quad 4 - 1 = 2C$ or $C = \frac{3}{2}$

$\therefore \quad x^2 - y^2 = \frac{3}{2}x$ or $2(x^2-y^2) = 3x$

33. (2) The given equation is written as

$$y\,dx - x\,dy + x\sqrt{xy}\,(x+y)dx + y\sqrt{xy}\,(x+y)\,dy = 0$$

or $ydx - xdy + (x+y)\sqrt{xy}\,(xdx + ydy) = 0$

or $\frac{ydx - xdy}{y^2} + \left(\frac{x}{y}+1\right)\sqrt{\frac{x}{y}}\left(d\left(\frac{x^2+y^2}{2}\right)\right) = 0$

or $d\left(\frac{x^2+y^2}{2}\right) + \frac{d\left(\frac{x}{y}\right)}{\left(\frac{x}{y}+1\right)\sqrt{\frac{x}{y}}} = 0$

or $\frac{x^2+y^2}{2} + 2\tan^{-1}\sqrt{\frac{x}{y}} = c$

34. (1) $x^2\frac{dy}{dx} - xy = 1 + \cos\frac{y}{x}$

or $\frac{x(xdy - ydx)}{dx} = 1 + \cos\frac{y}{x}$

or $\frac{\frac{xdy - ydx}{x^2}}{1+\cos\frac{y}{x}} = \frac{dx}{x^3}$

or $\int \frac{d\left(\frac{y}{x}\right)}{1+\cos\frac{y}{x}} = \int\frac{dx}{x^3}$

or $\frac{1}{2}\int \frac{d\left(\frac{y}{x}\right)}{\cos^2\frac{y}{2x}} = \int\frac{dx}{x^3}$

or $\frac{1}{2}\int \sec^2\frac{y}{2x}.d\left(\frac{y}{x}\right) = \int\frac{dx}{x^3}$

or $\frac{1}{2}\cdot\frac{\tan\frac{y}{2x}}{\frac{1}{2}} = \frac{x^{-2}}{-2} + c$

or $\tan\frac{y}{2x} + \frac{1}{2x^2} = c$

35. (4) $2x^2y\frac{dy}{dx} = \tan(x^2y^2) - 2xy^2$

or $x^2\,2y\frac{dy}{dx} + y^2\,2x = \tan(x^2y^2)$

or $\frac{d}{dx}(x^2y^2) = \tan(x^2y^2)$

or $\int\cot(x^2y^2)d(x^2y^2) = \int dx$

or $\log(\sin(x^2y^2)) = x + c$

When $x = 1$, $y = \sqrt{\frac{\pi}{2}}$, then $c = -1$.

Thus, equation of curve is

$\log\sin(x^2y^2) = x - 1$

or $\sin(x^2y^2) = e^{x-1}$

36. (1) $\left\{\frac{1}{x} - \frac{y^2}{(x-y)^2}\right\}dx + \left\{\frac{x^2}{(x-y)^2} - \frac{1}{y}\right\}dy = 0$

or $\left(\frac{dx}{x} - \frac{dy}{y}\right) + \left(\frac{x^2dy - y^2dx}{(x-y)^2}\right) = 0$

or $\left(\frac{dx}{x} - \frac{dy}{y}\right) + \left(\frac{dy/y^2 - dx/x^2}{(1/y - 1/x)^2}\right) = 0$

Integrating, we get $\ln|x| - \ln|y| - \dfrac{1}{(1/x - 1/y)} = c$

or $\ln\left|\dfrac{x}{y}\right| + \dfrac{xy}{x-y} = c$

37. (1) $(2y + xy^3)dx + (x + x^2y^2)dy = 0$

or $(2y\,dx + x\,dy) + (xy^3dx + x^2y^2dy) = 0$

Multiplying by x, we get

$(2xy\,dx + x^2\,dy) + (x^2y^3\,dx + x^3y^2\,dy) = 0$

or $d(x^2y) + \dfrac{1}{3}d(x^3y^3) = 0$

Integrating, we get $x^2y + \dfrac{x^3y^3}{3} = c$

38. (3) $ye^{-x/y}dx - (xe^{(-x/y)} + y^3)dy = 0$

or $(ydx - xdy)\,e^{-x/y} - y^3dy = 0$

or $\dfrac{ydx - xdy}{y^2}e^{-x/y} = ydy$

or $d(x/y)e^{-x/y} = ydy$

or $-e^{-x/y} = \dfrac{y^2}{2} + C$

or $2e^{-x/y} + y^2 = C$

39. (3) $(xy^3 - x^2)\,dy - (xy + y^4)\,dx = 0$

or $y^3\,(x\,dy - y\,dx) - x\,(x\,dy + y\,dx) = 0$

or $x^2y^3\,\dfrac{(x\,dy - y\,dx)}{x^2} - x\,(x\,dy + y\,dx) = 0$

or $x^2y^3\,d\left(\dfrac{y}{x}\right) - xd\,(xy) = 0$

Dividing by x^3y^2, we get

$\dfrac{y}{x}\,d\left(\dfrac{y}{x}\right) - \dfrac{d\,(xy)}{x^2y^2} = 0$

Now, integrating, we get $\dfrac{1}{2}\left(\dfrac{y}{x}\right)^2 + \dfrac{1}{xy} = c$

It passes through the point $(4, -2)$. Thus,

$\dfrac{1}{8} - \dfrac{1}{8} = c$ or $c = 0$

$\therefore \quad y^3 = -2x$

40. (1) The given equation can be written as

$$\frac{x\,dx + y\,dy}{(y\,dx - x\,dy)/\,y^2} = y^2\,\frac{x}{y^3}\cos^2(x^2 + y^2)$$

or $$\frac{xdx + ydy}{\cos^2(x^2 + y^2)} = \frac{x}{y}\left(\frac{ydx - xdy}{y^2}\right)$$

or $$\frac{1}{2}\sec^2(x^2 + y^2)\;d(x^2 + y^2) = \frac{x}{y}d\left(\frac{x}{y}\right)$$

On integrating, we get

$$\frac{1}{2}\tan(x^2 + y^2) = \frac{1}{2}\left(\frac{x}{y}\right)^2 + \frac{c}{2}$$

or $$\tan(x^2 + y^2) = \frac{x^2}{y^2} + c$$

41. (3) Rewrite the differential equation as

$(2xy\,dx - x^2\,dy) + y^2\,(3x^2\,y^2\,dx + 2x^3y\,dy) = 0$

Dividing by y^2, we get

$\dfrac{y\,2x\,dx - x^2\,dy}{y^2} + y^2\,3x^2\,dx + x^3\,2y\,dy = 0$

or $d\left(\dfrac{x^2}{y}\right) + d\left(x^3y^2\right) = 0$

Integrating, we get the solution

$\dfrac{x^2}{y} + x^3y^2 = c$

42. (3) $\left\{1 + x\sqrt{(x^2 + y^2)}\right\}dx + \left\{\sqrt{(x^2 + y^2)} - 1\right\}y\,dy = 0$

or $dx - y\,dy + \sqrt{(x^2 + y^2)}\,(x\,dx + y\,dy) = 0$

or $dx - y\,dy + \dfrac{1}{2}\sqrt{(x^2 + y^2)}\;d(x^2 + y^2) = 0$

Integrating, we have

$x - \dfrac{y^2}{2} + \dfrac{1}{2}\int \sqrt{t}\;dt = c, \left\{t = \sqrt{(x^2 + y^2)}\right\}$

or $x - \dfrac{y^2}{2} + \dfrac{1}{3}\left(x^2 + y^2\right)^{3/2} = c$

43. (1) $y\,(2x^4 + y)\,\dfrac{dy}{dx} = (1 - 4xy^2)x^2$

or $2x^4y\,dy + y^2\,dy + 4x^3y^2dx - x^2\,dx = 0$

or $2x^2y\,(x^2\,dy + 2xy\,dx) + y^2\,dy - x^2\,dx = 0$

or $2x^2\,y\,d(x^2y) + y^2\,dy - x^2\,dx = 0$

Integrating, we get $(x^2y)^2 + \dfrac{y^3}{3} - \dfrac{x^3}{3} = c$

or $3(x^2y)^2 + y^3 - x^3 = c$

44. (2) $(x\cot y + \log\cos x)\,dy + (\log\sin y - y\tan x)dx = 0$

or $(x\cot y\,dy + \log\sin y\,dx) + (\log\cos x\,dy - y\tan x\,dx) = 0$

or $\int d(x\log\sin y) + \int d(y\log\cos x) = 0$

or $x\log\sin y + y\log\cos x = \log c$

or $(\sin y)^x\,(\cos x)^y = c$

45. (3) We have $\dfrac{dy}{dx} = (e^y - x)^{-1} \Rightarrow \dfrac{dx}{dy} = e^y - x$

$\Rightarrow \quad \dfrac{dx}{dy} + x = e^y;$

So I.F. $= e^{\int dy} = e^y$

$\therefore$ General solution is given by

$xe^y = \dfrac{1}{2}e^{2y} + C$

$\Rightarrow \quad x = \dfrac{e^y}{2} + Ce^{-y}$

As $y(0) = 0$, so $C = \dfrac{-1}{2}$

$\therefore \quad x = \dfrac{e^y}{2} - \dfrac{1}{2}e^{-y}$

$\Rightarrow \quad e^y - e^{-y} = 2x$

$\Rightarrow \quad e^{2y} - 2xe^y - 1 = 0$

$\Rightarrow \quad 2e^y = 2x \pm \sqrt{4x^2 + 4}\,.$

But $e^y = x - \sqrt{x^2 + 1}$ (Rejected)

Hence $y = \log_e\left(x + \sqrt{x^2 + 1}\right)$

46. (1) $\frac{dy}{dx} + y\,\phi'(x) = \phi(x)\,\phi'(x)$

I.F. $= e^{\int \phi'(x)\,dx} = e^{\phi(x)}$

Hence, the solution is

$$ye^{\phi(x)} = \int e^{\phi(x)}\,\phi(x)\,\phi'(x)\,dx$$
$$= \int e^t\,t\,dt, \text{ where } \phi(x) = t$$
$$= te^t - e^t + c$$
$$= \phi(x)\,e^{\phi(x)} - e^{\phi(x)} + c$$

$\therefore \quad y = ce^{-\phi(x)} + \phi(x) - 1$

47. (3) $\because \frac{dy}{dx} + \frac{y}{x\log_e x} = \frac{2}{x}$

$\therefore$ I.F. $= e^{\int \frac{1}{x\log_e x}\,dx}$
$= e^{\log_e(\log_e x)}$
$= \log_e x$

48. (3) The given equation can be rewritten as

$$\frac{dy}{dx} + \frac{x^2-1}{x\left(x^2+1\right)}y = \frac{x^2\log x}{\left(x^2+1\right)} \qquad (1)$$

which is linear. Also,

$$P = \frac{x^2-1}{x\left(x^2+1\right)} \text{ and } Q = \frac{x^2\log x}{\left(x^2+1\right)}$$

$$\int P\,dx = \int \left[\frac{2x}{x^2+1} - \frac{1}{x}\right]dx \qquad \text{[resolving into partial fractions]}$$
$$= \log(x^2+1) - \log x$$

$\therefore$ I.F. $= e^{\log[(x^2+1)/x]} = \frac{x^2+1}{x}$

Hence, the required solution of equation (1) is

$$\frac{y(x^2+1)}{x} = \int \frac{\left(x^2+1\right)}{x}\,\frac{x^2\log x}{\left(x^2+1\right)}\,dx + c$$
$$= \int x\log x\,dx + c$$
$$= \frac{1}{2}x^2\log x - \int \frac{1}{x}\,\frac{x^2}{2}\,dx + c$$

$\therefore \quad y(x^2+1)/x = \frac{1}{2}x^2\log x - \frac{1}{4}x^2 + c$

49. (3) $\cos x\,\frac{dy}{dx} + y\sin x = 1$

or $\quad \frac{dy}{dx} + y\,\frac{\sin x}{\cos x} = \sec x$

$\therefore \quad \int P\,dx = \int \frac{\sin x}{\cos x}\,dx$
$= -\log\cos x$
$= \log\sec x$

$\therefore$ I.F. $= e^{\int P dx} = e^{\log\sec x} = \sec x$

50. (3) The given differential equation can be written as

$$\frac{dy}{dx} - \frac{\tan 2x}{\cos^2 x}y$$

which is linear differential equation of first order.

$$\int P\,dx = \int \frac{-\sin 2x}{\cos 2x\cos^2 x}\,dx$$
$$= -\int \frac{2\sin 2x\,dx}{\cos 2x(1+\cos 2x)}$$
$$= \int \frac{dt}{t(1+t)}$$
$$= \int \left(\frac{1}{t} - \frac{1}{1+t}\right)dt$$
$$= \log\frac{t}{1+t} \text{ where } t = \cos 2x$$
$$= \log\frac{\cos 2x}{1+\cos 2x} \qquad \left[\because -\frac{\pi}{2} < 2x < \frac{\pi}{2}\right]$$

$\therefore \quad e^{\int P\,dx} = e^{\log\frac{\cos 2x}{1+\cos 2x}} = \frac{\cos 2x}{1+\cos 2x} = \frac{\cos 2x}{2\cos^2 x}$

Thus, the solution is

$$y\,\frac{\cos 2x}{2\cos^2 x} = \int \frac{\cos^2 x\cos 2x}{2\cos^2 x}\,dx + C$$
$$= \frac{1}{4}\sin 2x + C$$

When $x = \frac{\pi}{6}$, $y = \frac{3\sqrt{3}}{8}$

$\therefore \quad \frac{3\sqrt{3}}{8}\,\frac{4}{2\times 2\times 3} = \frac{1}{4}\,\frac{\sqrt{3}}{2} + C \text{ or } C = 0$

$\therefore \quad y = \frac{1}{2}\tan 2x\cos^2 x$

51. (4) $x(1-x^2)dy + (2x^2y - y - ax^3)dx = 0$

or $\quad x(1-x^2)\frac{dy}{dx} + 2x^2y - y - ax^3 = 0$

or $\quad x(1-x^2)\frac{dy}{dx} + y(2x^2-1) = ax^3$

or $\quad \frac{dy}{dx} + \frac{2x^2-1}{x(1-x^2)}y = \frac{ax^3}{x(1-x^2)}$

which is of the form $\frac{dy}{dx} + Py = Q$.

Its integrating factor is $e^{\int P dx}$.

Here, $P = \frac{2x^2-1}{x(1-x^2)}$

52. (2) $f'(x) - \frac{2x(x+1)}{x+1}f(x) = \frac{e^{x^2}}{(x+1)^2}$

I.F. $= e^{\int -2x\,dx} = e^{-x^2}$

Thus, solution is $f(x)\,e^{-x^2} = \int \frac{dx}{(x+1)^2} + C$

or $\quad f(x)e^{-x^2} = -\frac{1}{x+1} + C$

Given $f(0) = 5$. Thus, $C = 6$

$\therefore \quad f(x) = \left(\frac{6x+5}{x+1}\right)e^{x^2}$

53. (4) $\frac{dy}{dx} = 1 + xy$

or $\frac{dy}{dx} - xy = 1$

I.F. $= e^{\int -x\,dx} = e^{-x^2/2}$

Hence solution is $y \cdot e^{-x^2/2} = \int e^{-x^2/2}\,dx + c$.

$\int e^{-x^2/2}\,dx$ is not further integrable.

54. (2) $\frac{dx}{dy} = \frac{x + 2y^3}{y}$

or $\frac{dx}{dy} - \frac{1}{y}x = 2y^2$ which is linear

I.F. $= e^{\int -\frac{1}{y}dy} = e^{-\log y} = \frac{1}{y}$

Thus, solution is $\frac{1}{y}x = \int \frac{1}{y} 2y^2\,dy + c = y^2 + c$

or $\frac{x}{y} = y^2 + c$

55. (1) $x^2\frac{dy}{dx}\cos\frac{1}{x} - y\sin\frac{1}{x} = -1$

or $\frac{dy}{dx} - \frac{y}{x^2}\tan\frac{1}{x} = -\sec\frac{1}{x}\frac{1}{x^2}$ (linear)

I.F. $= e^{-\int \frac{1}{x^2}\tan\frac{1}{x}\,dx} = \sec\frac{1}{x}$

Thus, solution is $y\sec\frac{1}{x} = -\int \sec^2\left(\frac{1}{x}\right)\frac{1}{x^2}\,dx = \tan\frac{1}{x} + c$

Given $y \to -1, x \to \infty$. Thus, $c = -1$.

Hence, equation of curve is $y = \sin\frac{1}{x} - \cos\frac{1}{x}$.

56. (3) Rewriting the given equation as

$2xy\frac{dy}{dx} - y^2 = 1 + x^2$

or $2y\frac{dy}{dx} - \frac{1}{x}y^2 = \frac{1}{x} + x$

Putting $y^2 = u$, we have

$\frac{du}{dx} - \frac{1}{x}u = \frac{1}{x} + x$

I.F. $= e^{-\int \frac{1}{x}dx} = \frac{1}{x}$

Thus, solution is $u\frac{1}{x} = \int\left(\frac{1}{x^2} + 1\right)dx = -\frac{1}{x} + x + C$

or $y^2 = (x^2 - 1) + Cx$

Since $y(1) = 1$, we get $C = 1$.

Hence, $y^2 = x(1 + x) - 1$ which represents a system of hyperbola.

57. (1) $\frac{dy}{dx} = \frac{1}{xy[x^2\sin y^2 + 1]}$

or $\frac{dx}{dy} = xy[x^2\sin y^2 + 1]$

or $\frac{1}{x^3}\frac{dx}{dy} - \frac{1}{x^2}y = y\sin y^2$

Putting $-1/x^2 = u$, we get

$\frac{du}{dy} + 2uy = 2y\sin y^2$.

I.F. $= e^{y^2}$

Thus, solution is $ue^{y^2} = \int 2y\sin y^2 e^{y^2}\,dy + C$

$= \int (\sin t)e^t\,dt + C$

$= \frac{1}{2}e^{y^2}(\sin y^2 - \cos y^2) + c$

or $2u = (\sin y^2 - \cos y^2) + 2Ce^{-y^2}$

or $2 = x^2[\cos y^2 - \sin y^2 - 2Ce^{-y^2}]$

58. (2) We have $\frac{dy}{dx} = 1 - \frac{1}{x^2}$ or $y = x + \frac{1}{x} + C$

This passes through (2, 7/2),

Therefore, $\frac{7}{2} = 2 + \frac{1}{2} + C$ or $C = 1$

Thus, the equation of the curve is

$y = x + \frac{1}{x} + 1$ or $xy = x^2 + x + 1$

59. (2)

$xy = C$

or $x\frac{dy}{dx} + y = 0$

or $\frac{dy}{dx} = \frac{-y}{x} = m_1$

By condition,

$\tan\frac{\pi}{4} = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

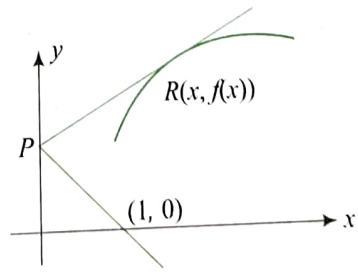
$1 = \left|\frac{-\frac{y}{x} - m_2}{1 - \frac{y}{x}m_2}\right|$

or $\frac{y}{x} + m_2 = 1 - \frac{y}{x}m_2$ or $\frac{y}{x}m_2 - 1$

or $m_2 = \frac{1 - \frac{y}{x}}{1 + \frac{y}{x}}$ or $m_2 = \frac{\frac{y}{x} + 1}{\frac{y}{x} - 1}$

or $\frac{dy}{dx} = \frac{x - y}{x + y}$ or $\frac{dy}{dx} = \frac{x + y}{y - x}$

60. (1)



The equation of the tangent at the point $R(x, f(x))$ is

$Y - f(x) = f'(x)(X - x)$

The coordinates of the point P are $(0, f(x) - xf'(x))$.

The slope of the perpendicular line through P is

$\frac{f(x) - xf'(x)}{-1} = -\frac{1}{f'(x)}$

or $f(x)f'(x) - x(f'(x))^2 = 1$

or $\frac{y\,dy}{dx} - x\left(\frac{dy}{dx}\right)^2 = 1$

which is the required differential equation to the curve at $y = f(x)$.

61. (2) $x^{2/3} + y^{2/3} = a^{2/3}$

$\Rightarrow \quad \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}}$ (1)

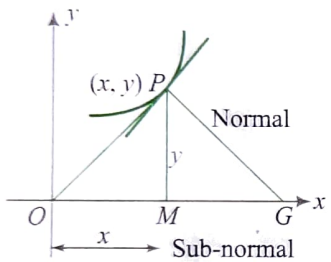
Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$\Rightarrow \quad \frac{dx}{dy} = \frac{x^{-1/3}}{y^{-1/3}}$

$\Rightarrow \quad \int x^{1/3}dx = \int y^{1/3}dy$

$\Rightarrow \quad x^{4/3} - y^{4/3} = c$

62. (2) It is given that triangle OPG is an isosceles triangle.

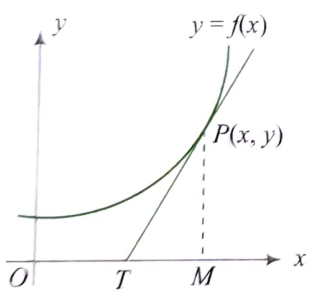


Therefore, $OM = MG$ = sub-normal

or $x = y\frac{dy}{dx}$ or $x\,dx = y\,dy$

On integration, we get $x^2 - y^2 = C$, which is a rectangular hyperbola.

63. (1) Let the equation of the curve be $y = f(x)$.



It is given that $OT \propto y$

or $OT = by$

or $OM - TM = by$

or $x - \frac{y}{dy/dx} = by$ [$\because$ TM = Length of the sub-tangent]

or $x - y\frac{dx}{dy} = by$

or $\frac{dx}{dy} - \frac{x}{y} = -b$

It is linear differential equation.

Its solution is $\frac{x}{y} = -b\log y + a$

or $x = y(a - b\log y)$

64. (2) For the family of curves represented by the first differential equation, the slope of the tangent at any point (x, y) is given by

$$\left(\frac{dy}{dx}\right)_{c_1} = \frac{x^2 + x + 1}{y^2 + y + 1}$$

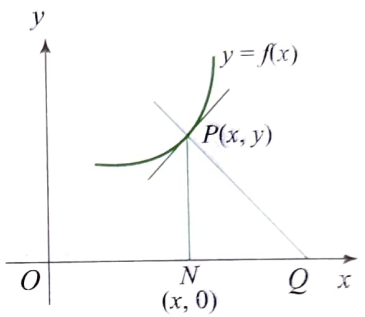
For the family of curves represented by the second differential equation, the slope of the tangent at any point is given by

$$\left(\frac{dy}{dx}\right)_{c_2} = -\frac{y^2 + y + 1}{x^2 + x + 1}$$

Clearly, $\left(\frac{dy}{dx}\right)_{c_1} \times \left(\frac{dy}{dx}\right)_{c_2} = -1$

Hence, the two curves are orthogonal.

65. (3) Equation of normal at point $P(x, y)$ is $Y - y = -\frac{dx}{dy}(X - x)$



$$NQ = y\frac{dy}{dx} = \frac{x(1 + y^2)}{1 + x^2}$$

or $\frac{x\,dx}{1 + x^2} = \frac{y\,dy}{1 + y^2}$

or $\ln(1 + x^2) = \ln(1 + y^2) + \ln c$

or $1 + y^2 = \frac{1 + x^2}{c}$

It passes through (3, 1). Thus, $1 + 1 = \frac{1 + (3)^2}{c}$ or $c = 5$

Thus, curve is $5 + 5y^2 = 1 + x^2$ or $x^2 - 5y^2 = 4$.

66. (3) The point on y-axis is $\left(0,\ y - x\frac{dy}{dx}\right)$.

According to given condition,

$$\frac{x}{2} = y - \frac{x}{2}\frac{dy}{dx}$$

or $\frac{dy}{dx} = 2\frac{y}{x} - 1$

Putting $\frac{y}{x} = v$, we get $x\frac{dv}{dx} = v - 1$

or $\ln\left|\frac{y}{x} - 1\right| = \ln|x| + c$

or $1 - \frac{y}{x} = x$ [as $y(1)$

67. (3) Slope of tangent $= \frac{dy}{dx}$

$\therefore$ Slope of normal $= -\frac{dx}{dy}$

Thus, the equation of normal is

$$Y - y = -\frac{dx}{dy}(X - x)$$

This meets x-axis, where

$$-y = -\frac{dx}{dy}(X - x) \text{ or } X = x + y\frac{dy}{dx}$$

$\therefore$ G is $\left(x + y\frac{dy}{dx}, 0\right)$

$\therefore$ $OG = 2x$

$\therefore$ $x + y\frac{dy}{dx} = 2x$

or $y\frac{dy}{dx} = x$ or $y\,dy = x\,dx$

Integrating, we get $\frac{y^2}{2} = \frac{x^2}{2} + \frac{C}{2}$

or $y^2 - x^2 = C$, which is a hyperbola.

68. (1) Equation of tangent is $Y - y = \frac{dy}{dx}(X - x)$

For x-intercept, $Y = 0$. Thus, $X = x - y\frac{dx}{dy}$.

According to question, $x - y\frac{dx}{dy} = y$

or $\frac{dy}{dx} = \frac{y}{x - y}$

Putting $y = vx$, we get

$$v + x\frac{dv}{dx} = \frac{v}{1 - v}$$

or $x\frac{dv}{dx} = \frac{v}{1 - v} - v = \frac{v - v + v^2}{1 - v}$

or $\int \frac{1 - v}{v^2}dv = \int \frac{dx}{x}$

or $-\frac{1}{v} - \log v = \log x + c$

or $-\frac{x}{y} - \log\frac{y}{x} = \log x + c$

or $-\frac{x}{y} = \log y + c$

Given when $x = 1$, $y = 1$, then $c = -1$.

Hence, equation of curve is $1 - \frac{x}{y} = \log y$

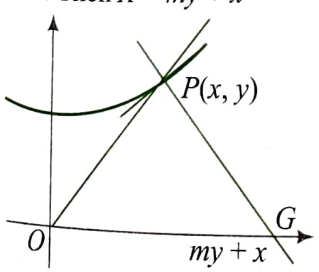
or $y = e\,e^{-x/y}$

or $e^{x/y} = \frac{e}{y}$

or $ye^{x/y} = e$

(1) Tangent at point P is $Y - y = -\frac{1}{m}(X - x)$ where $m = \frac{dy}{dx}$.

Let $Y = 0$. Then $X = my + x$



According to question, $x(my + x) = 2(x^2 + y^2)$

or $\frac{dy}{dx} = \frac{x^2 + 2y^2}{xy}$ (homogeneous)

Putting $y = vx$, we get

$$v + x\frac{dv}{dx} = \frac{1 + 2v^2}{v}$$

or $x\frac{dv}{dx} = \frac{1 + 2v^2}{v} - v = \frac{1 + v^2}{v}$

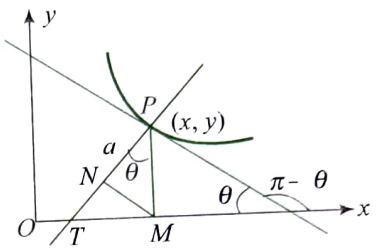
or $\int \frac{v\,dv}{1 + v^2} = \int \frac{dx}{x}$

or $\frac{1}{2}\log(1 + v^2) = \log x + \log c,\ c > 0$

or $x^2 + y^2 = cx^4$

Also, it passes through $(1, 0)$. Then $c = 1$.

70. (1)



Ordinate $= PM$. Let $P \equiv (x, y)$.

Projection of ordinate on normal $= PN$. (Given)

$\therefore$ $PN = PM\cos\theta = a$

$\therefore$ $\frac{y}{\sqrt{1 + \tan^2\theta}} = a$

or $y = a\sqrt{1 + (y_1)^2}$

or $\frac{dy}{dx} = \frac{\sqrt{y^2 - a^2}}{a}$

or $\int \frac{a\,dy}{\sqrt{y^2 - a^2}} = \int dx$

or $a\ln|y + \sqrt{y^2 - a^2}| = x + c$

71. (1) $\frac{dV}{dt} = -K4\pi r^2$

But $V = \frac{4}{3}\pi r^3$

or $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$

Hence, $\frac{dr}{dt} = -K$.

72. (3) According to the question,

$$\frac{dy}{dt} = -k\sqrt{y}$$

or $\int_4^y \frac{dy}{\sqrt{y}} = -k\int_0^t dt$

or $2\sqrt{y}\Big|_4^0 = -kt = -\frac{t}{15}$

or $0 - 4 = -\frac{t}{15}$

or $t = 60$ min

73. (3) Let population = x, at time t years. Given $\frac{dx}{dt} \propto x$

or $\frac{dx}{dt} = kx$, where k is a constant of proportionality

or $\frac{dx}{x} = kdt$.

Integrating, we get $\ln x = kt + \ln c$

or $\frac{x}{c} = e^{kt}$ or $x = ce^{kt}$

If initially, i.e., when time $t = 0$, $x = x_0$, then $x_0 = ce^0 = c$

$\therefore \quad x = x_0 e^{kt}$

Given $x = 2x_0$ when $t = 30$. Then $2x_0 = x_0 e^{30k}$ or $2 = e^{30k}$.

$\therefore \quad \ln 2 = 30\,k$ (1)

To find t, when t triples, $x = 3x_0$. Thus, $3x_0 = x_0 e^{kt}$ or $3 = e^{kt}$.

$\therefore \quad \ln 3 = kt$ (2)

Dividing equation (2) by (1), $\frac{t}{30} = \frac{\ln 3}{\ln 2}$

or $t = 30 \times \frac{\ln 3}{\ln 2} = 30 \times 1.5849 = 48$ years (approx.)

74. (1) Let $V(t)$ be the velocity of the object at time t.

Given $\frac{dV}{dt} = 9.8 - kV$ or $\frac{dV}{9.8 - kV} = dt$.

Integrating, we get $\log(9.8 - kV) = -kt + \log C$

or $9.8 - kV = C\,e^{-kt}$

But $V(0) = 0$ or $C = 9.8$

Thus, $9.8 - kV = 9.8\,e^{-kt}$

or $kV = 9.8\,(1 - e^{-kt})$

or $V(t) = \frac{9.8}{k}(1 - e^{-kt}) < \frac{9.8}{k}$ for all t

Hence, $V(t)$ cannot exceed $\frac{9.8}{k}$ m/s.

Multiple Correct Answers Type

1. (1), (2), (3)

a. $f(\lambda x, \lambda y) = \frac{\lambda(x - y)}{\lambda^2(x^2 + y^2)} = \lambda^{-1} f(x, y)$

Thus, it is homogeneous of degree –1.

b. $f(\lambda x, \lambda y) = (\lambda x)^{1/3}(\lambda y)^{-2/3} \tan^{-1}\frac{x}{y}$

$= \lambda^{-1/3} x^{1/3} y^{-2/3} \tan^{-1}\frac{x}{y}$

$= \lambda^{-\frac{1}{3}} f(x, y)$

Thus, it is homogeneous.

c. $f(\lambda x, \lambda y) = \lambda x\left(\ln\sqrt{\lambda^2(x^2 + y^2)} - \ln \lambda y\right) + \lambda y e^{x/y}$

$= \lambda x\left[\ln\left(\frac{\lambda\sqrt{(x^2 + y^2)}}{\lambda y}\right)\right] + \lambda y e^{x/y}$

$= \lambda\left[x\left(\ln\sqrt{x^2 + y^2} - \ln y\right) + y e^{x/y}\right]$

$= \lambda f\,(x, y)$

Thus, it is homogeneous.

d. $f(\lambda x, \lambda y) = \lambda x\left[\ln\frac{2\lambda^2x^2 + \lambda^2y^2}{\lambda x\,\lambda(x + y)}\right] + \lambda^2x^2 \tan\frac{x + 2y}{3x - y}$

$= \lambda x\left[\ln\frac{2x^2 + y^2}{x(x + y)}\right] + \lambda^2x^2 \tan\frac{x + 2y}{3x - y}$

Thus, it is non-homogeneous.

2. (1), (3)

We have $(x - h)^2 + (y - k)^2 = a^2$

Differentiating w.r.t. x, we get (1)

$2(x - h) + 2(y - k)\frac{dy}{dx} = 0$

or $(x - h) + (y - k)\frac{dy}{dx} = 0$ (2)

Differentiating w.r.t. x, we get

$1 + \left(\frac{dy}{dx}\right)^2 + (y - k)\frac{d^2y}{dx^2} = 0$ (3)

From equation (3),

$y - k = -\left(\frac{1 + p^2}{q}\right)$, where $p = \frac{dy}{dx}$, $q = \frac{d^2y}{dx^2}$

Putting the value of $y - k$ in equation (2), we get

$x - h = \frac{(1 + p^2)p}{q}$

Substituting the values of $x - h$ and $y - k$ in equation (1), we get

$\left(\frac{1 + p^2}{q}\right)^2(1 + p^2) = a^2$

or $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$

which is the required differential equation.

3. (1), (2)

$y\left(\frac{dy}{dx}\right)^2 + (x - y)\frac{dy}{dx} - x = 0$

or $\frac{dy}{dx} = \frac{(y - x) \pm \sqrt{(x - y)^2 + 4xy}}{2y}$

So, $\frac{dy}{dx} = 1$ which gives straight line

or $\frac{dy}{dx} = -\frac{x}{y}$ which gives circle.

4. (1), (3)

Obviously (1) is linear differential equation with $P = \frac{1}{x}$ and $Q = \log x$

$y\left(\frac{dy}{dx}\right) + 4x = 0$ or $\frac{dy}{dx} + \frac{4x}{y} = 0$.

Hence, it is not linear.

$(2x + y^3)\left(\frac{dy}{dx}\right) = 3y$

or $\frac{dx}{dy} = \frac{2x}{3y} + \frac{y^2}{3}$

or $\frac{dx}{dy} - \frac{2x}{3y} = \frac{y^2}{3}$ which is linear with $P = \frac{2}{3y}$ and $Q = \frac{y^2}{3}$

5. (1), (3)

$\frac{dy}{dx} = \frac{ax + h}{by + k}$ or $(by + k)\,dy = (ax + h)\,dx$

or $b\frac{y^2}{2} + ky = \frac{a}{2}x^2 + hx + C$

For this to represent a parabola, either $a = 0, b \neq 0$ or $a \neq 0, b = 0$.

6. (1), (4)

The given differential equation is

$$y_2(x^2 + 1) = 2xy_1 \text{ or } \frac{y_2}{y_1} = \frac{2x}{x^2+1}$$

Integrating both sides, we get

$$\log y_1 = \log (x^2 + 1) + \log C$$

or $y_1 = C(x^2 + 1)$ (1)

It is given that $y_1 = 3$ at $x = 0$.

Putting $x = 0, y_1 = 3$ in equation (1), we get $3 = C$.

Substituting the value of C in (1), we obtain

$$y_1 = 3(x^2 + 1) \quad (2)$$

Integrating both sides w.r.t. to x, we get

$$y = x^3 + 3x + C_2$$

This passes through the point (0, 1). Therefore, $1 = C_2$.

Hence, the required equation of the curve is $y = x^3 + 3x + 1$.

Obviously, it is strictly increasing from equation (2).

Also, $f(0) = 1 > 0$. Then the only root is negative.

7. (1), (2), (3)

8. (1), (2), (4)

$$\frac{dy}{dx} + y\cos x = \cos x \text{ (linear)}$$

I.F. $= e^{\int \cos x\, dx} = e^{\sin x}$

Thus, solution is $y\, e^{\sin x} = \int e^{\sin x} \cos x dx = e^{\sin x} + c$

When $x = 0, y = 1$, then $c = 0$

Thus, $y = 1$. Hence, options (1), (2), (4) are true.

9. (1), (2), (3)

We have $f''(x) = g''(x)$. On integration, we get

$$f'(x) = g'(x) + C \quad (1)$$

Putting $x = 1$, we get

$$f'(1) = g'(1) + C \text{ or } 4 = 2 + C \text{ or } C = 2$$

$\therefore \quad f'(x) = g'(x) + 2$

Integrating w.r.t. x, we get $f(x) = g(x) + 2x + c_1$ (2)

Putting $x = 2$, we get

$$f(2) = g(2) + 4 + c_1 \text{ or } 9 = 3 + 4 + c_1 \text{ or } c_1 = 2$$

$\therefore \quad f(x) = g(x) + 2x + 2.$

Putting $x = 4$, we get $f(4) - g(4) = 10$

$$|f(x) - g(x)| < 2 \text{ or } |2x + 2| < 2$$

or $|x + 1| < 1$ or $-2 < x < 0$

Also, $f(2) = g(2)$ or $x = -1$

$f(x) - g(x) = 2x$ has no solution.

10. (2) $(x^2y^2 - 1)\, dy + 2xy^3\, dx = 0$

or $x^2y^2 dy + 2xy^3\, dx = dy$

or $x^2\, dy + 2xy\, dx = \frac{dy}{y^2}$

or $\int d(x^2 y) = \int \frac{dy}{y^2} + c$

or $x^2 y = \frac{y^{-1}}{-1} + c$

or $x^2y^2 = -1 + cy$

i.e., $1 + x^2y^2 = cy$

11. (1), (2)

$$\frac{dy}{dx} = \frac{y}{x^2}$$

or $\frac{dy}{y} = \frac{dx}{x^2}$

or $\ln y = -\frac{1}{x} + \ln c$ or $\frac{y}{c} = e^{-\frac{1}{x}}$

or $y = ce^{-\frac{1}{x}}$

Comparing with $y = a\, e^{-1/x} + b, a \in R - \{0\}, b = 0$.

12. (2) We have $y\frac{dy}{dx} = k$ (constant)

or $y\, dy = k\, dx$ or $\frac{y^2}{2} = kx + C$ or $y^2 = 2kx + 2C$

13. (1), (4)

The differential equation can be rewritten as

$$\frac{x\, dx + y\, dy}{\sqrt{1-(x^2+y^2)}} = \frac{x\, dy - y\, dx}{\sqrt{x^2+y^2}}$$

Since $d \tan^{-1}(y/x) = \frac{x\, dy - y\, dx}{x^2 + y^2}$

and $d(x^2 + y^2) = 2(x\, dx + y\, dy)$,

we have $\frac{\frac{1}{2} d(x^2+y^2)}{\sqrt{x^2+y^2}\sqrt{1-(x^2+y^2)}} = \frac{x\, dy - y\, dx}{x^2+y^2}$

$$= d\{\tan^{-1}(y/x)\}$$

Put $x^2 + y^2 = t^2$ in the L.H.S. Then

$$\frac{t\, dt}{t\sqrt{1-t^2}} = d\{\tan^{-1}(y/x)\}$$

Integrating both sides, we get

$$\sin^{-1} t = \tan^{-1}(y/x) + c$$

i.e., $\sin^{-1}\sqrt{(x^2+y^2)} = \tan^{-1}(y/x) + c$

14. (1), (2)

We have length of the normal = radius vector

or $y\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{x^2+y^2}$

or $y^2\left(1+\left(\frac{dy}{dx}\right)^2\right) = x^2 + y^2$

or $y^2\left(\frac{dy}{dx}\right)^2 = x^2$

or $x = \pm y\frac{dy}{dx}$

i.e., $x = y\frac{dy}{dx}$ or $x = -y\frac{dy}{dx}$

i.e., $x\, dx - y\, dy = 0$ or $x\, dx + y\, dy = 0$

i.e., $x^2 - y^2 = c_1$ or $x^2 + y^2 = c_2$

Clearly, $x^2 - y^2 = c_1$ represents a rectangular hyperbola and $x^2 + y^2 = c_2$ represents circles.

15. (1), (2)

$$x = \sin\left(\frac{dy}{dx} - 2y\right) \Rightarrow \frac{dy}{dx} - 2y = \sin^{-1} x$$

$$x - 2y = \log\left(\frac{dy}{dx}\right) \Rightarrow \frac{dy}{dx} = e^{x-2y}$$

16. (1), (4)

$$ydx + dy = -e^x y^2 dy$$

$$\Rightarrow \quad \frac{e^{-x}ydx + e^{-x}dy}{y^2} = -dy$$

$$\Rightarrow \quad d\left(\frac{e^{-x}}{y}\right) = dy$$

$$\Rightarrow \quad e^{-x} = y^2 + cy$$

$$\because \quad f(0) = 1, \therefore c = 0$$

$$\Rightarrow \quad e^{-x} = y^2$$

$$\Rightarrow \quad y = e^{-x/2}$$

$$\Rightarrow \quad A = \int_0^1 \left(e^x - e^{-x/2}\right)dx = e + \frac{2}{\sqrt{e}} - 3$$

17. (2), (4)

$$\frac{dv}{dt} = g - kv^2$$

$$\Rightarrow \quad \frac{dv}{g - kv^2} = dt$$

$$\Rightarrow \quad \frac{1}{k}\int \frac{dv}{\frac{g}{k} - v^2} = \int dt + C$$

$$\Rightarrow \quad \frac{1}{2\sqrt{gk}} \log\left|\frac{\sqrt{g/k} + v}{\sqrt{g/k} - v}\right| = t + C$$

At $t = 0, v = 0 \Rightarrow C = 0$

$$\Rightarrow \quad \frac{1}{2\sqrt{gk}} \log\left|\frac{\sqrt{g/k} + v}{\sqrt{g/k} - v}\right| = t$$

$$\Rightarrow \quad \log\left|\frac{\sqrt{g/k} + v}{\sqrt{g/k} - v}\right| = 2\sqrt{gk}t$$

$$\Rightarrow \quad \frac{\sqrt{g/k} + v}{\sqrt{g/k} - v} = e^{2\sqrt{gk}t}$$

$$\Rightarrow \quad v = \sqrt{\frac{g}{k}} \frac{e^{2t\sqrt{gk}} - 1}{e^{2t\sqrt{gk}} + 1}$$

$$\Rightarrow \quad v = \sqrt{\frac{g}{k}} \frac{1 - \frac{1}{e^{2t\sqrt{gk}}}}{1 - \frac{1}{e^{2t\sqrt{gk}}}}$$

Clearly when $v \to \sqrt{\frac{g}{k}}$ as $t \to \infty$

Linked Comprehension Type

For Problems 1–3

1. (3), 2. (1), 3. (2)

Integrating $\frac{d^2y}{dx^2} = 6x - 4$, we get $\frac{dy}{dx} = 3x^2 - 4x + A$.

When $x = 1$, $\frac{dy}{dx} = 0$, so that $A = 1$. Hence,

$$\frac{dy}{dx} = 3x^2 - 4x + 1$$

Integrating, we get $y = x^3 - 2x^2 + x + B$.

When $x = 1, y = 5$, so that $B = 5$.

Thus, we have $y = x^3 - 2x^2 + x + 5$.

From equation (1), we get the critical points $x = 1/3, x = 1$.

At the critical point $x = \frac{1}{3}$, $\frac{d^2y}{dx^2}$ is negative.

Therefore, at $x = 1/3$, y has a local maximum.

At $x = 1$, $\frac{d^2y}{dx^2}$ is positive.

Therefore, at $x = 1$, y has a local minimum.

Also, $f(1) = 5, f\left(\frac{1}{3}\right) = \frac{139}{27}, f(0) = 5, f(2) = 7$

Hence, the global maximum value = 7
and the global minimum value = 5.

For Problems 4–6

4. (3) Given equation can be rewritten as

$$y = xp + \sqrt{\left(1 + p^2\right)}, p = \frac{dy}{dx} \qquad (1)$$

Differentiating w.r.t. x, we get

$$p = p + x\frac{dp}{dx} + \frac{1}{2\sqrt{1+p^2}} 2p \frac{dp}{dx}$$

i.e., $\frac{dp}{dx} = 0$ or $\frac{p}{\sqrt{1+p^2}} = -x$

i.e., $p = c$ or $p = \frac{x}{\sqrt{1-x^2}}$

Thus, $y = cx + \sqrt{\left(1 + c^2\right)}$ gives the general solution and $x^2 + y^2 = 1$ as singular solution.

5. (2) $y = xp + p^2 \qquad \left(p = \frac{dy}{dx}\right) \qquad (1)$

Differentiating equation (1) w.r.t. x, we get

$$p = p + x\frac{dp}{dx} + 2p\frac{dp}{dx}$$

or $\frac{dp}{dx}(x + 2p) = 0$

or $\frac{dp}{dx} = 0$ or $p = -\frac{x}{2}$

Eliminating p from equation (1), we get

$$y = -\frac{x^2}{2} + \frac{x^2}{4} = -\frac{x^2}{4} \qquad (2)$$

Clearly, $f'(x) = -\frac{x^2}{4} = -1$ has two solutions. (1)

6. (4) $y = mx + m - m^3$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = m + x\frac{dm}{dx} + \frac{dm}{dx} - 3m^2\frac{dm}{dx}$$

$$m = m + x\frac{dm}{dx} + \frac{dm}{dx} - 3m^2\frac{dm}{dx}$$

$$\frac{dm}{dx}(x+1-3m^2)=0$$

$$\frac{dm}{dx}=0 \quad \text{or } m=c \qquad (2)$$

or $\quad x+1-3m^2=0 \quad \text{or } m^2=\frac{x+1}{3} \qquad (3)$

Eliminating m between equations (1) and (3), we get

$$y=m(x+1-m^2)$$

$$=\left(\frac{x+1}{3}\right)^{1/2}\left(x+1-\frac{x+1}{3}\right)$$

$$=\left(\frac{x+1}{3}\right)^{1/2}\frac{2}{3}(x+1)=2\left(\frac{x+1}{3}\right)^{3/2}$$

or $\quad y^2=\frac{4}{27}(x+1)^3$

or $\quad 27y^2=4(x+1)^3$

For Problems 7–9

7. (2) $f(x)\le 0$ and $F'(x)=f(x)$

Now, $f(x)\ge cF(x)$

or $\quad F'(x)-cF(x)\ge 0$

or $\quad e^{-cx}F'(x)-c\,e^{-cx}F(x)\ge 0$

or $\quad \frac{d}{dx}\left(e^{-cx}F(x)\right)\ge 0$

Thus, $e^{-cx}F(x)$ is an increasing function

$\therefore \quad e^{-cx}F(x)\ge e^{-c(0)}F(0)$

or $\quad e^{-cx}F(x)\ge 0$

or $\quad F(x)\ge 0$

or $\quad f(x)\ge 0 \qquad$ [as $f(x)\ge cF(x)$ and c is positive]

$\therefore \quad f(x)=0$

Also, $\left(\frac{d\,g(x)}{dx}\right)<g(x)\ \forall\ x>0$

or $\quad e^{-x}\frac{d(g(x))}{dx}-e^{-x}g(x)<0$

or $\quad \frac{d}{dx}(e^{-x}g(x))<0$

Thus, $e^{-x}g(x)$ is a decreasing function

$\therefore \quad e^{-x}g(x)<e^{-(0)}g(0)$

or $\quad g(x)<0 \qquad$ [as $g(0)=0$]

Thus, $f(x)=g(x)$ has one solution, $x=0$.

(3) $|x^2+x-6|=f(x)+g(x)$ or $|x^2+x-6|=g(x)$

Thus, no solution exists.

(1) $g(x)(\cos^{-1}x-\sin^{-1}x)\le 0$

or $(\cos^{-1}x-\sin^{-1}x)\ge 0$ or $x\in\left[-1,\frac{1}{\sqrt{2}}\right]$

Problems 10–12

(3) Tangent at any point $P(x, y)$ on curve 'C' is

$$Y-y=\frac{dy}{dx}(X-x)$$

$$\therefore \quad Q=\left(x-\frac{y}{\frac{dy}{dx}},0\right)$$

Given $PQ^2=1$

$$\Rightarrow \quad \left(\frac{y}{\frac{dy}{dx}}\right)^2+y^2=1$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{-y}{\sqrt{1-y^2}} \qquad (1)$$

Putting $y=\sin\theta$, we get

$$\Rightarrow \quad \frac{d(\sin\theta)}{dx}=\frac{-\sin\theta}{\cos\theta}$$

$$\Rightarrow \quad \cos\theta\frac{d\theta}{dx}=\frac{-\sin\theta}{\cos\theta}$$

$$\Rightarrow \quad \frac{\cos^2\theta}{\sin\theta}d\theta=-dx$$

$$\Rightarrow \quad \frac{1-\sin^2\theta}{\sin\theta}d\theta=-dx$$

$$\Rightarrow \quad \int(\operatorname{cosec}\theta-\sin\theta)d\theta=-\int dx$$

$$\Rightarrow \quad x+c=-\cos\theta-\log\tan(\theta/2)$$

Given $y=1$ when $x=0$

$\Rightarrow \quad x=0,\ \theta=\pi/2 \quad \Rightarrow c=0$

$\therefore$ The curve in parametric form is

$$x=-\cos\theta-\log_e\tan\left(\frac{\theta}{2}\right),\quad y=\sin\theta$$

11.(3) Area, $A=\int_0^{\infty}y\,dx$

Where $y=\sin\theta$ and $dx=(\sin\theta-\operatorname{cosec}\theta)\,d\theta$

$$\therefore \quad A=-\int_0^{\pi/2}\sin\theta(\sin\theta-\operatorname{cosec}\theta)d\theta$$

$$=-\int_o^{\pi/2}(\sin^2\theta-1)\,d\theta=\frac{\pi}{4}$$

12.(1) Replacing $\frac{dy}{dx}$ by $\frac{-dx}{dy}$ in (1), we get

$$-\frac{dx}{dy}=\frac{-y}{\sqrt{1-y^2}}$$

$$\Rightarrow \quad dx=\frac{y\,dy}{\sqrt{1-y^2}}$$

$$\Rightarrow \quad -2\int dx=\int\frac{-2y\,dy}{\sqrt{1-y^2}}$$

$$\Rightarrow \quad -2x-2c=2\sqrt{1-y^2}$$

$\Rightarrow \quad (x+c)^2+y^2=1$, which are required orthogonal trajectories.

For Problems 13–15

13. (1), 14. (2), 15. (3)

We have $f(x)=(e^{-x}+e^{x})\cos x-2x-\int_0^x(x-t)\,f'(t)\,dt$

$$\Rightarrow \quad f(x)=(e^{x}+e^{-x})\cos x-2x-\left[x\int_0^x f'(t)\,dt-\int_0^x tf'(t)\,dt\right]$$

$$\Rightarrow \quad f(x)=(e^{x}+e^{-x})\cos x-2x$$

$$-\left[x(f(x)-f(0))-\left\{\left[t.f(t)\right]_0^x-\int_0^x f(t)\,dt\right\}\right]$$

$$\Rightarrow \quad f(x)=(e^x+e^{-x})\cos x-2x-xf(x)+2x+\left[xf(x)-\int_0^x f(t)\,dt\right]$$

(as $f(0)=2$)

$$\Rightarrow \quad f(x)=(e^x+e^{-x})\cos x-\int_0^x f(t)\,dt \qquad \ldots(i)$$

Differentiating w.r.t. x,

$$\Rightarrow \quad f'(x)+f(x)=\cos x(e^x-e^{-x})-(e^x+e^{-x})\sin x \qquad \ldots(ii)$$

$$\text{or} \quad \frac{dy}{dx}+y=e^x(\cos x-\sin x)-e^{-x}(\cos x+\sin x) \qquad \ldots(iii)$$

Above is linear differential equation

I.F. $=e^x$

$\therefore$ Solution is

$$y\cdot e^x=\int e^{2x}(\cos x-\sin x)\,dx-\int(\cos x+\sin x)dx$$

$$\Rightarrow \quad y\cdot e^x=\int e^{2x}(\cos x-\sin x)\,dx-(\sin x-\cos x)+C$$

$$\Rightarrow \quad y\cdot e^x=e^{2x}\left(\frac{3}{5}\cos x-\frac{1}{5}\sin x\right)-(\sin x-\cos x)+C$$

When $x=0$, $y=2$

$$\Rightarrow \quad 2=\frac{3}{5}+1+C$$

$$\Rightarrow \quad C=\frac{2}{5}$$

$$\therefore \quad y=e^x\left(\frac{3}{5}\cos x-\frac{1}{5}\sin x\right)-(\sin x-\cos x)\,e^{-x}+\frac{2}{5}\,e^{-x}$$

For Problems 16–18

16. (1) Let N denote the amount of material present at time t. Then,

$$\frac{dN}{dt}-kN=0$$

This differential equation is separable and linear. Its solution is $N=ce^{kt}$ (1)

At $t=0$, we are given that $N=50$. Therefore, from equation (1), $50=ce^{k(0)}$ or $c=50$.

Thus, $N=50\,e^{kt}$ (2)

At $t=2$, 10% of the original mass of 50 mg or 5 mg has decayed.

Hence, at $t=2$, $N=50-5=45$.

Substituting these values into equation (2) and solving for k, we have $45=50e^{2k}$ or $k=\frac{1}{2}\log\frac{45}{50}$

Substituting this value into (2), we obtain the amount of mass present at any time t as

$$N=50e^{(1/2)(\ln 0.9)\,t} \qquad (3)$$

where t is measured in hours.

17. (3) We require N at $t=4$. Substituting $t=4$ into (3) and then solving for N, we find

$$N=50e^{2\ln 0.9}$$

18. (3) We require t when $N=50/2=25$. Substituting $N=25$ into equation (3) and solving for t, we find

$$25=50\,e^{(1/2)(\ln 0.9)\,t} \text{ or } t=\frac{\ln 0.5}{\ln 0.81}\text{h}$$

For Problems 19–21

19. (1) Here, $V_0=100$, $a=20$, $b=0$, and $e=f=5$. Hence,

$$\frac{dQ}{dt}+\frac{1}{20}Q=0$$

The solution of this linear equation is $Q=ce^{-t/20}$

At $t=0$, we are given that $Q=a=20$. (1)

Substituting these values into equation (1), we find that $c=20$, so that equation (1) can be rewritten as $Q=20\,e^{-t/20}$.

For $t=20$, $Q=20/e$.

20. (2) Here, $a=0$, $b=1$, $e=4$, $f=2$, and $V_0=10$.

The volume of brine in the tank at any time t is given as

$$V_0+et-ft=10+2t.$$

We require t when $10+2t=50$. Hence, $t=20$ min.

21. (4) $\frac{dQ}{dt}+\frac{2}{10+2t}Q=4$

This is a linear equation. Its solution is $Q=\frac{40t+4t^2+c}{10+2t}$ (1)

At $t=0$, $Q=a=0$. Substituting these values into equation (1), we find that $c=0$. We require Q at the moment of overflow, which from part (a) is $t=20$. Thus,

$$Q=\frac{40(20)+4(20)^2}{10+2(20)}=40 \text{ lb}$$

Matrix Match Type

1. a → q, s; b → p; c → p; d → q, r, s

p. Equation of the required parabola is of the form $y^2=4a(x-h)$. Differentiating, we have

$$2y\frac{dy}{dx}=4a \text{ or } y\frac{dy}{dx}=2a$$

$$\text{or} \quad \left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}=0$$

The degree of this differential equation is 1 and the order is 2.

q. We have $y=a(x+a)^2$ (1)

$$\therefore \quad \frac{dy}{dx}=2a(x+a) \qquad (2)$$

Dividing equations (1) by (2), we get

$$\frac{y}{\frac{dy}{dx}}=\frac{x+a}{2}$$

$$\text{or} \quad x+a=\frac{2y}{y_1}, \text{ where } y_1=\frac{dy}{dx}$$

Substituting $a=\frac{2y}{y_1}-x$ in equation (1), we get

$$y=\left(\frac{2y}{y_1}-x\right)\left(\frac{2y}{y_1}\right)^2$$

$$\text{or} \quad y_1^3\,y=4(2y-xy_1)y^2$$

Clearly, it is a differential equation of degree 3.

r. The given equation is $\left(1+3\frac{dy}{dx}\right)^{2/3}=4\frac{d^3y}{dx^3}$

Cubing, we get $\left(1+3\frac{dy}{dx}\right)^2=64\left(\frac{d^3y}{dx^2}\right)^3$

Hence, order = degree = 3

s. We have $y^2=2c\left(x+\sqrt{c}\right)$

Differentiating w.r.t. x, we get $2y\frac{dy}{dx}=2c$

or $c = y\dfrac{dy}{dx}$

Putting in equation (1), we get

$$y^2 = 2\left(y\frac{dy}{dx}\right)x + 2\left(y\frac{dy}{dx}\right)^{3/2}$$

or $\left(y^2 - 2xy\dfrac{dy}{dx}\right)^2 = 4y^3\left(\dfrac{dy}{dx}\right)^3$

Its order is 1 and degree is 3.

2. **a → q; b → r; c → p; d → s.**

a. $y = e^{4x} + 2e^{-x}$; $y_1 = 4e^{4x} - 2e^{-x}$; $y_2 = 16e^{4x} + 2e^{-x}$;
$y_3 = 64e^{4x} - 2e^{-x}$

Now, $y_3 - 13y_1 = (64e^{4x} - 2e^{-x}) - 13(4e^{4x} - 2e^{-x})$
$= 12e^{4x} + 24e^{-x}$

$y_3 - 13y = 12(e^{4x} + 2e^{-x}) = 12y$

$\therefore$ $K = 12$ and $K/3 = 4$

b. $y\dfrac{dy}{dx} = x$

$\Rightarrow$ $ydy = xdx$

Integrating, we get

$y^2 = x^2 + c$

$\Rightarrow$ $f(1) = 3 \Rightarrow c = 8$

$\Rightarrow y^2 = x^2 + 8$

$f(x) = \sqrt{x^2 + 8}$

$\Rightarrow$ $f(4) = \sqrt{16 + 8} = 2\sqrt{6}$

c. $y = u^m$ or $\dfrac{dy}{dx} = m\,u^{m-1}\dfrac{du}{dx}$

Substituting the value of y and $\dfrac{dy}{dx}$ in $2x^4y\dfrac{dy}{dx} + y^4 = 4x^6$, we have

$$2x^4\,u^m\,m\,u^{m-1}\frac{du}{dx} + u^{4m} = 4x^6$$

or $\dfrac{du}{dx} = \dfrac{4x^6 - u^{4m}}{2m\,x^4\,u^{2m-1}}$

For homogeneous equation, $4m = 6$ or $m = \dfrac{3}{2}$

and $2m - 1 = 2$ or $m = \dfrac{3}{2}$.

d. $y = Ax^m + Bx^{-n}$

$\therefore$ $\dfrac{dy}{dx} = Amx^{m-1} - nBx^{-n-1}$

$\therefore$ $\dfrac{d^2y}{dx^2} = Am(m-1)x^{m-2} + n(n+1)Bx^{-n-2}$

Putting these values in $x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} = 12y$, we get

$m(m+1)Ax^m + n(n-1)Bx^{-n} = 12(Ax^m + Bx^{-n})$

i.e., $m(m+1) = 12$ and $n(n-1) = 12$

i.e., $m = 3, -4$ and $n = 4, -3$

(2) a. $y = kx + k^2$

$\Rightarrow$ $\dfrac{dy}{dx} = k$

Put the value of k

$\Rightarrow$ $y = x\left(\dfrac{dy}{dx}\right) + \left(\dfrac{dy}{dx}\right)^2$

b. $y = ae^{2x} + be^{3x}$

Differentiating w.r.t. x,

$$-\frac{5}{2}e^{-5x/2}y + e^{-5x/2}\frac{dy}{dx} = \frac{1}{2}\left[-ae^{-x/2} + be^{x/2}\right]$$

Again differentiating w.r.t. x,

$$\frac{25}{4}e^{\frac{-5x}{2}}y - \frac{5}{2}e^{-\frac{5x}{2}}\frac{dy}{dx}$$

$$-\frac{5}{2}e^{-\frac{5x}{2}}\frac{dy}{dx} + e^{-\frac{5x}{2}}\frac{d^2y}{dx^2} = \frac{1}{4}\left[ae^{\frac{x}{2}} + be^{\frac{x}{2}}\right]$$

$\Rightarrow$ $\dfrac{25}{4}e^{-5x/2}y - 5e^{-5x/2}\dfrac{dy}{dx} + e^{-5x/2}\dfrac{d^2y}{dx^2} = \dfrac{1}{4}e^{-5x/2}y$

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + \frac{25}{4}y - \frac{1}{4}y = 0$$

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

c. $y^2 = 4a(x + a)$

$\Rightarrow$ $yy' = 2a$

Substitute the value of a in the equation of the curve.

d. $xy = ae^x + be^{-x} + x^2$

$\Rightarrow$ $x\dfrac{dy}{dx} + y = ae^x - be^{-x} + 2x$

$\Rightarrow$ $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + \dfrac{dy}{dx} = ae^x + be^{-x} + 2$

$\Rightarrow$ $x\dfrac{d^2y}{dx^2} + \dfrac{2dy}{dx} - xy + x^2 = 2$

Numerical Value Type

1. (4) We have $4xe^{xy} = y + 5\sin^2 x$ (1)

Putting $x = 0$, in equation (1), we get $y = 0$.

Therefore, (0, 0) lies on the curve.

Now, on differentiating equation (1) w.r.t. x, we get

$$4e^{xy} + 4xe^{xy}\left(x\frac{dy}{dx} + y\right) = \frac{dy}{dx} + 10\sin x\cos x$$

or $y'(0) = 4$

2. (2) Given $\dfrac{dy}{dx} - \dfrac{1}{x}y = \left(x - \dfrac{2}{x}\right)$

$$\text{I.F.} = e^{-\int\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

Now, general solution is given by

$$\frac{y}{x} = \int\left(x - \frac{2}{x}\right)\frac{1}{x}dx \text{ or } \frac{y}{x} = x + \frac{2}{x} + C$$

As $y(1) = 1$, $C = -2$

$\therefore$ $\dfrac{y}{x} = x + \dfrac{2}{x} - 2$ or $y = x^2 - 2x + 2$

Hence, $y(2) = (2)^2 - 2(2) + 2 = 2$

3. (2) Given $y = \tan z$ (1)

$\therefore$ $\dfrac{dy}{dx} = \sec^2 z\cdot\dfrac{dz}{dx}$

Now, $\dfrac{d^2y}{dx^2} = \sec^2 z\cdot\dfrac{d^2z}{dx^2} + \dfrac{dz}{dx}\cdot\dfrac{d}{dx}(\sec^2 z)$ [using product rule]

$$= \sec^2 z\cdot\frac{d^2z}{dx^2} + \frac{dz}{dx}\cdot\frac{d}{dz}(\sec^2 z)\frac{dz}{dx}$$

$$\frac{d^2y}{dx^2} = \sec^2 z \cdot \frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2 \cdot 2\sec^2 z \cdot \tan z \qquad (2)$$

Now, $1 + \frac{2(1+y)}{1+y^2}\left(\frac{dy}{dx}\right)^2$

$$= 1 + \frac{2(1+\tan z)}{\sec^2 z} \cdot \sec^4 z \cdot \left(\frac{dz}{dx}\right)^2$$

$$= 1 + 2(1+\tan z)\cdot \sec^2 z \cdot \left(\frac{dz}{dx}\right)^2$$

$$= 1 + 2\sec^2 z\left(\frac{dz}{dx}\right)^2 + 2\tan z \cdot \sec^2 z \left(\frac{dz}{dx}\right)^2 \qquad (3)$$

From (2) and (3), we have RHS of (2) = RHS of (3)

or $\sec^2 z \cdot \frac{d^2z}{dx^2} = 1 + 2\sec^2 z \left(\frac{dz}{dx}\right)^2$

or $\frac{d^2z}{dx^2} = \cos^2 z + 2\left(\frac{dz}{dx}\right)^2$

or $k = 2$

4. (8) $\frac{dy}{dt} + 2ty = t^2$

I.F. $= e^{t^2}$

Thus, solution is $y \cdot e^{t^2} = \int t^2 e^{t^2}\, dt = \frac{1}{2}\int t \cdot (2t \cdot e^{t^2})\, dt$

$\therefore \quad y \cdot e^{t^2} = t \cdot \frac{e^{t^2}}{2} - \frac{1}{2}\int e^{t^2}\, dt + C$

$\therefore \quad y = \frac{t}{2} - e^{-t^2}\int \frac{e^{t^2}}{2}\, dt + Ce^{-t^2}$

$\therefore \quad \lim_{t\to\infty} \frac{y}{t} = \frac{1}{2} - \lim_{t\to\infty} \frac{\int \frac{e^{t^2}}{2}}{te^{t^2}} + \frac{C}{t.e^{t^2}} = \frac{1}{2}$

5. (2) $\frac{dy}{dx} = \frac{1}{x\cos y + 2\sin y \cos y}$

$\therefore \quad \frac{dx}{dy} = x\cos y + 2\sin y \cos y$

$\therefore \quad \frac{dx}{dy} + (-\cos y)\, x = 2\sin y \cos y$

$\therefore \quad$ I.F. $= e^{-\int \cos y\, dx} = e^{-\sin y}$

Thus, the solution is

$$x \cdot e^{-\sin y} = 2\int e^{-\sin y} \cdot \sin y \cos y\, dy$$

$$= -2\sin y\, e^{-\sin y} + 2\int e^{-\sin y} \cos y\, dy$$

$$= -2\sin y\, e^{-\sin y} - 2e^{-\sin y} + c$$

i.e., $x = -2\sin y - 2 + ce^{\sin y} = ce^{\sin y} - 2(1+\sin y)$

$\therefore \quad k = 2$

6. (1) $\frac{dy}{dx} = \frac{1}{dx/dy}; \frac{d^2y}{dx^2} = \frac{d}{dy}\left(\frac{1}{dx/dy}\right)\cdot\frac{dy}{dx} = -\frac{1}{(dx/dy)^3}\frac{d^2x}{dy^2}$

Hence, $x\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 - \frac{dy}{dx} = 0$

becomes $-x \cdot \frac{1}{(dx/dy)^3}\frac{d^2x}{dy^2} + \frac{1}{(dx/dy)^3} - \frac{1}{(dx/dy)} = 0$

or $x\frac{d^2x}{dy^2} - 1 + \left(\frac{dx}{dy}\right)^2 = 0$ or $x\frac{d^2x}{dy^2} + \left(\frac{dx}{dy}\right)^2 = 1$

$\therefore \quad k = 1$

7. (1) Given that y_1 and y_2 are two different solutions of the differential equation

$$\frac{dy}{dx} + P(x)\cdot y = Q(x) \qquad (1)$$

$$\Rightarrow \quad \frac{dy_1}{dx} + P(x)\cdot y_1 = Q(x) \qquad (2)$$

and $\frac{dy_2}{dx} + P(x)\cdot y_2 = Q(x)$ (3)

Now, $\alpha y_1 + \beta y_2$ will be a solution

if $\frac{d}{dx}(\alpha y_1 + \beta y_2) + P(x)\cdot(\alpha y_1 + \beta y_2) = Q(x)$

$\Rightarrow \alpha\left\{\frac{dy_1}{dx} + P(x)\cdot y_1\right\} + \beta\left\{\frac{dy_2}{dx} + P(x)\cdot y_2\right\} = Q(x)$

$\Rightarrow \alpha Q(x) + \beta Q(x) = Q(x)$

$\Rightarrow \alpha + \beta = 1$

8. (8) Equation of tangent at $P(x_1, y_1)$ of $y = f(x)$ is

$$y - y_1 = \frac{dy}{dx}(x - x_1) \qquad (1)$$

This tangent cuts the x-axis. So,

$$x_2 = x_1 - \frac{y_1}{\left(\frac{dy}{dx}\right)}$$

$x_1, x_2, x_3, \ldots, x_n$ are in A.P.

$$x_2 - x_1 = -\frac{y_1}{\frac{dy}{dx}} = \log_2 e \qquad \text{(Given)}$$

or $-y = \log_2 e \frac{dy}{dx}$

or $\frac{dy}{y}\log_2 e = -dx$

Integrating both sides, we get

$\log_e y = -x\log e^2 + c$

or $y = ke^{-x\log_e 2}$

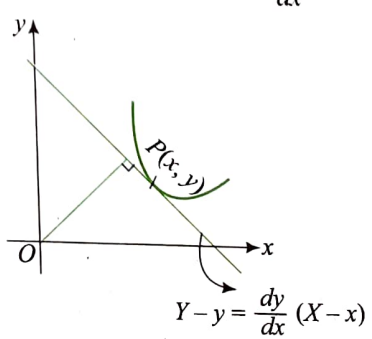
Since $y = f(x)$ passes through (0, 2),

$k = 2$

$\therefore \quad y = 2\cdot e^{-x\log_e 2}$

$\therefore \quad y = 2^{1-x}$

9. (2) Equation of tangent is $X\frac{dy}{dx} - Y - X\frac{dy}{dx} + y = 0$.



$$Y - y = \frac{dy}{dx}(X - x)$$

According to question,

$$\left|\frac{0-0-x\frac{dy}{dx}+y}{\sqrt{\left(\frac{dy}{dx}\right)^2+1}}\right| = x$$

$$\therefore \quad \left|\frac{x\frac{dy}{dx}-y}{\sqrt{\left(\frac{dy}{dx}\right)^2+1}}\right| = x$$

or $$\left(x\frac{dy}{dx}-y\right)^2 = x^2\left(1+\left(\frac{dy}{dx}\right)^2\right)$$

or $$x^2\left(\frac{dy}{dx}\right)^2 + y^2 - 2xy\frac{dy}{dx} = x^2 + x^2\left(\frac{dy}{dx}\right)^2$$

or $$\frac{y^2-x^2}{2xy} = \frac{dy}{dx}$$ (Homogeneous) (1)

Put $y = vx$ in (1). Then

$$v + x\frac{dv}{dx} = \frac{v^2-1}{2v}$$

$$\int \frac{2v}{v^2+1}dv = -\int\frac{dx}{x}$$

$$\ln(v^2+1) = -\ln x + \ln c$$

$$v^2+1 = \frac{c}{x}$$

$$\frac{y^2+x^2}{x^2} = \frac{c}{x} \quad \text{or} \quad y^2+x^2 = cx$$

It passes through (1, 1). Then $c = 2$.

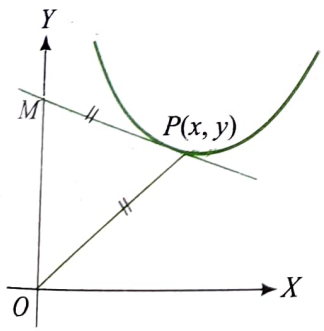
$x^2 + y^2 - 2x = 0$.

For intercept of curve on x-axis, put $y = 0$

We have $x^2 - 2x = 0$ or $x = 0, 2$.

Hence, length of intercept is 2.

10. (5)



$\because \quad OM = OP$

$$y - x\frac{dy}{dx} = \sqrt{x^2+y^2}$$

$$\frac{dy}{dx} = \frac{y-\sqrt{x^2+y^2}}{x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{y}{x} - \sqrt{1+\left(\frac{y}{x}\right)^2}$$

Put $\frac{y}{x} = v$ or $y = v + x\frac{dv}{dx}$

$$\therefore \quad v + x\frac{dv}{dx} = v - \sqrt{1+v^2}$$

$$\therefore \int \frac{dv}{\sqrt{1+v^2}} = -\int\frac{dx}{x}$$

$$\therefore \quad \log(v+\sqrt{1+v^2}) = \log\frac{c}{x}$$

$$\therefore \quad v+\sqrt{1+v^2} = \frac{c}{x}$$

$$\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}} = \frac{c}{x}$$

$$y + \sqrt{x^2+y^2} = c$$

Hence, curve is parabola, which has eccentricity 1.

11. (4) $\frac{dy}{dx} - y = 1 - e^{-x}$

$P = -1, Q = 1 - e^{-x}$

I.F. $= e^{\int P dx} = e^{\int -1 dx} = e^{-x}$

$$\therefore \quad y\cdot e^{-x} = \int e^{-x}(1-e^{-x})\,dx + C$$

$$= -e^{-x} + \frac{1}{2}e^{-2x} + C$$

$$\therefore \quad y = -1 + \frac{1}{2}e^{-x} + Ce^{x}$$

When $x = 0, y = y_0$.

So, $C = y_0 + \frac{1}{2}$

$$y = -1 + \frac{1}{2}e^{-x} + (y_0 + 1/2)e^{x}$$

when $x \to \infty$ y takes finite value.

So, $y_0 + 1/2 = 0$

or $y_0 = -1/2$

12. (1.125) We have

$$\int_0^x (f'(t) - \sin 2t)dt = \int_x^0 f(t)\tan t\,dt$$

Differentiate w.r.t. x, we get

$f'(x) - \sin 2x = 0 - f(x)\tan x$

or $y' + y\tan x = \sin 2x$, where $y = f(x)$

Above is a linear differential equation.

I.F. $= e^{\int \tan x\,dx} = \sec x$

Therefore, solution is

$$y\sec x = \int 2\sin x dx$$

or $y\sec x = -2\cos x + c$

or $y = c\cos x - 2\cos^2 x$

$f(0) = 1 \Rightarrow c = 3$

$\therefore f(x) = 3\cos x - 2\cos^2 x$

$$= -2\left[\cos x - \frac{3}{4}\right]^2 + \frac{9}{8}$$

$$\therefore f(x)\max = \frac{9}{8}$$

13. (1) We have $\frac{d^2y}{dx^2} - \frac{dy}{dx} + e^{2x} = 0$

Put $\frac{dy}{dx} = t$

$\therefore \quad \frac{dt}{dx} - t = -e^{2x}$, which is linear differential equation

$\Rightarrow$ solution is $te^{-x} = -\int e^{2x}\, e^{-x} dx + c$

$\Rightarrow \quad \frac{dy}{dx} e^{-x} = -e^{x} + c$

$y'(0) = 1 \Rightarrow c = 2$

$\therefore \quad \frac{dy}{dx} e^{-x} = 2 - e^{x}$

$\therefore \quad y = 2e^{x} - \frac{e^{2x}}{2} + c'$

$y(0) = 2 \Rightarrow c' = \frac{1}{2}$

$\therefore \quad y(x) = 2e^{x} - \frac{e^{2x}}{2} + \frac{1}{2}$

$y(x) \le \frac{5}{2}$ for $x = \log_e 2$

So, $[2\alpha] = [\log_e 4] = 1$

14. (1) We have $\cos y\left(\frac{dy}{dx} + e^{-x}\right) + \sin y\left(e^{-x} - \frac{dy}{dx}\right) = e^{e^{-x}}$

$\Rightarrow \quad (\cos y - \sin y)\frac{dy}{dx} + (\cos y + \sin y)e^{-x} = e^{e^{-x}}$ (1)

Let $\cos y + \sin y = u$

$\therefore$ equation (1) reduces to

$\Rightarrow \quad \frac{du}{dx} + e^{-x}u = e^{e^{-x}}$,

which is linear differential equation whose solution is

$ue^{-e^{-x}} = x$ or $(\cos y + \sin y)e^{-e^{-x}} = x$

Putting $(t, 0)$,

$\Rightarrow \quad e^{-e^{-t}} = t$ (putting $(t, 0)$)

$\Rightarrow \quad t \cdot e^{e^{-t}} = 1$

15. (8) $\int_0^x f(t)\, dt + \int_0^x t\, f(x-t)\, dt = e^{-x} - 1$

$\Rightarrow \quad \int_0^x f(t)\, dt + \int_0^x (x - t) f(t)\, dt = e^{-x} - 1$

$\Rightarrow \quad \int_0^x f(t)\, dt + x\int_0^x f(t)\, dt - \int_0^x tf(t)\, dt = e^{-x} - 1$

Differentiate w.r.t. x, we get

$f(x) + xf(x) + \int_0^x f(t)\, dt - xf(x) = -e^{-x}$ ...(1)

$\Rightarrow \quad f(x) + \int_0^x f(t)\, dt = -e^{-x}$

Again differentiate w.r.t. x, we get

$\Rightarrow \quad f'(x) + f(x) = e^{-x}$

$\Rightarrow \quad e^{x}f'(x) + e^{x}f(x) = 1$

$\Rightarrow \quad (e^{x}f(x))' = 1$

$\Rightarrow \quad e^{x}f(x) = x + c$

From (1), $f(0) = -1$, $\therefore c = -1$

$\therefore \quad f(x)\, e^{x} = x - 1$

$\therefore \quad e^{9}f(9) = 8$

Archives

JEE MAIN

Single Correct Answer Type

1. (4) $y = c_1 e^{c_2 x}$ (i)

$\therefore \quad y' = c_2 c_1 e^{c_2 x} = c_2 y$ (ii)

$\therefore \quad y'' = c_2 y'$

From (ii), we get

$c_2 = \frac{y'}{y}$ (iii)

From (ii) and (iii),

we get $\frac{y''}{y'} = \frac{y'}{y}$ or $yy'' = (y')^2$

2. (2) $\cos x\, dy = y(\sin x - y)dx$

$\Rightarrow \quad \frac{dy}{dx} = y \tan x - y^2 \sec x$

$\Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\tan x = -\sec x$

Let $\frac{1}{y} = t$

$\therefore \quad -\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

$\Rightarrow \quad -\frac{dt}{dx} - t\tan x = -\sec x$

$\Rightarrow \quad \frac{dt}{dx} + (\tan x)\, t = \sec x$

I.F. $= e^{\int \tan x\, dx} = \sec x$

The solution is

t (I.F.) $= \int$ (I.F.) $\sec x\, dx$

$\Rightarrow \quad \frac{1}{y}\sec x = \tan x + c$

3. (3) Since total life is T, scrap value is $V(T)$.

We have

$dV(t) = -k(T - t)dt$

$\Rightarrow \quad \int_I^{V(T)} dV(t) = \int_{t=0}^{T} -k(T - t)dt$

$\Rightarrow \quad V(T) - I = k\left[\frac{(T-t)^2}{2}\right]_0^T$

$\Rightarrow \quad V(T) - I = -k\left[\frac{T^2}{2}\right]$

$\Rightarrow \quad V(T) = I - \frac{kT^2}{2}$

4. (2) We have

$\frac{dy}{dx} = y + 3$

$\Rightarrow \quad \frac{1}{y+3}dy = dx$

$\Rightarrow \quad \ln|(y+3)| = x + \ln c$, where $\ln c$ is a constant of integra

$\Rightarrow \quad (y + 3) = c\, e^{x}$

Initially, when $x = 0$, $y = 2$

$\Rightarrow \quad c = 5$

Finally, the required solution is $y + 3 = 5e^{x}$.

$\Rightarrow \quad y(\ln 2) = 5e^{\ln 2} - 3 = 10 - 3 = 7$

5. (3) $V = \frac{4}{3}\pi r^3$

Initially volume of gas in balloon is $4500\,\pi$ cubic meters.

$\therefore \quad 4500\,\pi = \frac{4\pi r^3}{3}$

$\therefore \quad 45 \times 25 \times 3 = r^3$

$\therefore \quad r = 15$ m.

Given $\frac{dV}{dt} = 72\pi$ cubic meter per minute

$\therefore$ After 49 minutes, volume of gas leaked

$= 72\pi \times 49$

$\therefore$ Volume of gas in balloon after 49 minutes

$= (4500 - 49 \times 72)\pi = 972\,\pi\ \text{m}^3$

$\therefore \quad 972\,\pi = \frac{4\pi r^3}{3}$

$\therefore \quad r^3 = 3 \times 243 = 3 \times 3^5$

$\therefore \quad r = 9$

Now $\frac{dV}{dt} = 4\pi r^2 \left(\frac{dr}{dt}\right)$

For the rate of decrease in radius after 49 minutes,

$$72\pi = 4\pi \times 9 \times 9 \left(\frac{dr}{dt}\right)$$

$\therefore \quad \frac{dr}{dt} = \frac{2}{9}$ meter per minute

6. (1) We have, $2\frac{dp(t)}{900 - p(t)} = -dt$

Integrating, we get

$$-2\ln(900 - p(t)) = -t + c$$

when $t = 0$, $p(0) = 850$

$\therefore \quad -2\ln(50) = c$

$\therefore \quad 2\ln\left(\frac{50}{900 - p(t)}\right) = -t$

$\therefore \quad 900 - p(t) = 50\,e^{t/2}$

$\therefore \quad p(t) = 900 - 50\,e^{t/2}$

let $p(t_1) = 0$

$\therefore \quad 0 = 900 - 50\,e^{t/2}$

$\therefore \quad t_1 = 2\ln 18$

7. (3) $\frac{dP}{dx} = 100 - 12\sqrt{x}$

Integrating, we get

$$P = 100x - 12 \cdot \frac{2}{3}x^{3/2} + C$$

$\therefore \quad P = 100x - 8x^{3/2} + C$ (1)

Given if $x = 0$, $P = 2000$

$\Rightarrow \quad C = 2000$

So, (1) becomes

$$P = 100x - 8x^{3/2} + 2000$$

If $x = 25$ (25 more workers)

$$P = 100 \times 25 - 8 \times (5^2)^{3/2} + 2000$$

$$= 2500 - 1000 + 2000 = 3500$$

8. (1) $\frac{dp}{dt} = \frac{p - 400}{2}$

$\Rightarrow \quad \frac{dp}{p - 400} = \frac{1}{2}dt$

Integrating, we get

$$\ln|p - 400| = \frac{1}{2}t + c$$

When $t = 0$, $p = 100$, we have $\ln 300 = c$

$\therefore \quad \ln\left|\frac{p - 400}{300}\right| = \frac{t}{2}$

$\Rightarrow \quad |p - 400| = 300\,e^{t/2}$

$\Rightarrow \quad 400 - p = 300\,e^{t/2}$ (as $p < 400$)

$\Rightarrow \quad p = 400 - 300\,e^{t/2}$

9. (3) We have $\frac{dy}{dx} + \frac{y}{x\log x} = 2$

$$\text{I.F.} = e^{\int \frac{1}{x\log x}dx} = e^{\log(\log x)} = \log x$$

Now solution is

$$y(\log x) = \int 2(\log x)\,dx$$

$\Rightarrow \quad y(\log x) = 2[x\log x - x] + c$

when $x = 1$, $\quad c = 2$.

when $x = e$ then

$$y = 2(e - e) + 2$$

or $\quad y = 2$

10. (3) $y(1 + xy)\,dx = x\,dy$

$\Rightarrow \quad y\,dx - x\,dy + xy^2\,dx = 0$

$\Rightarrow \quad y^2 d\left(\frac{x}{y}\right) + xy^2\,dx = 0$

$\Rightarrow \quad \frac{x}{y} + \frac{x^2}{2} = C$ ...(

Curve passes through $(1, -1)$

$\Rightarrow \quad -1 + \frac{1}{2} = C$ (from (

$\Rightarrow \quad C = -\frac{1}{2}$

$\Rightarrow \quad \frac{x}{y} + \frac{x^2}{2} = -\frac{1}{2}$

Now put $x = -\frac{1}{2}$

$\Rightarrow \quad \frac{-\frac{1}{2}}{y} + \frac{\frac{1}{4}}{2} = -\frac{1}{2}$

$\Rightarrow \quad \frac{-1}{2y} = \frac{-1}{2} - \frac{1}{8}$

$\Rightarrow \quad y = \frac{4}{5}$

11. (2) We have

$$(2 + \sin x)\frac{dy}{dx} + (y + 1)\cos x = 0$$

$\Rightarrow \quad \frac{d}{dx}[(2 + \sin x)(y + 1)] = 0$

$\Rightarrow \quad (2 + \sin x)(y + 1) = c$

Since $y(0) = 1$, $c = 4$

So, $\quad y + 1 = \frac{4}{2 + \sin x}$

$\therefore \quad y\left(\frac{\pi}{2}\right) = \frac{4}{3} - 1 = \frac{1}{3}$

12. (4) $\sin x \frac{dy}{dx} + y \cos x = 4x$

$\therefore \quad \frac{dy}{dx} + y \cdot \cot x = \frac{4x}{\sin x}$

This is a linear differential equation.

I.F. $= e^{\int P dx} = e^{\int \cot x dx} = e^{\log(\sin x)} = \sin x$

Therefore, solution is

$y \sin x = \int 4x\, dx + C$

$\therefore \quad y \sin x = 2x^2 + C$

Given that $y\left(\frac{\pi}{2}\right) = 0$.

$\therefore \quad 0 \times \sin \frac{\pi}{2} = 2 \times \left(\frac{\pi}{2}\right)^2 + C$

or $\quad C = -\frac{\pi^2}{2}$

$\therefore \quad y \sin x = 2x^2 - \frac{\pi^2}{2}$

Putting $x = \frac{\pi}{6}$, we get

$\frac{y}{2} = 2\left(\frac{\pi}{6}\right)^2 - \frac{\pi^2}{2}$

$\therefore \quad y = \frac{-8\pi^2}{9}$

JEE ADVANCED

Single Correct Answer Type

1. (1) $\frac{dy}{dx} = \frac{y}{x} + \sec\frac{y}{x}$.

Let $y = vx$. Then given equation reduces to $\frac{dv}{\sec v} = \frac{dx}{x}$

or $\quad \int \cos v dv = \int \frac{dx}{x}$

or $\quad \sin v = \ln x + c$ or $\sin\left(\frac{y}{x}\right) = \log x + c$

The curve passes through $\left(1, \frac{\pi}{6}\right)$.

Thus, $\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$.

2. (2) $\frac{dy}{dx} + \frac{x}{x^2-1} y = \frac{x^4 + 2x}{\sqrt{1-x^2}}$

This is a linear differential equation.

$\text{I.F.} = e^{\int \frac{x}{x^2-1} dx}$

$= e^{\frac{1}{2}\ln|x^2-1|} = \sqrt{1-x^2} \qquad (\because x \in (-1, 1))$

Therefore, solution is:

$y\sqrt{1-x^2} = \int \frac{x(x^3+2)}{\sqrt{1-x^2}} \cdot \sqrt{1-x^2} dx$

or $\quad y\sqrt{1-x^2} = \int (x^4 + 2x) dx = \frac{x^5}{5} + x^2 + C$

Since, $f(0) = 0$, $C = 0$

$\therefore \quad f(x)\sqrt{1-x^2} = \frac{x^5}{5} + x^2$

$\Rightarrow \quad f(x) = \frac{x^5}{5\sqrt{1-x^2}} + \frac{x^2}{\sqrt{1-x^2}}$

$\therefore \quad \int_{-\sqrt{3}/2}^{\sqrt{3}/2} f(x) dx = \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \left(\frac{x^5}{5\sqrt{1-x^2}} + \frac{x^2}{\sqrt{1-x^2}}\right) dx$

$= 2\int_0^{\sqrt{3}/2} \frac{x^2}{\sqrt{1-x^2}} dx$

$= 2\int_0^{\pi/3} \frac{\sin^2\theta}{\cos\theta} \cos\theta d\theta \qquad$ (Taking $x = \sin\theta$)

$= 2\int_0^{\pi/3} \sin^2\theta d\theta$

$= 2\int_0^{\pi/3} \frac{1-\cos 2\theta}{2} d\theta$

$= 2\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/3}$

$= 2\left(\frac{\pi}{6}\right) - 2\left(\frac{\sqrt{3}}{8}\right) = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$

3. (1) We have, $dy = \frac{dx}{8\sqrt{x}\left(\sqrt{9+\sqrt{x}}\right)\left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)}$

Let $\quad 4 + \sqrt{9+\sqrt{x}} = t$

$\Rightarrow \quad \frac{1}{2\sqrt{9+\sqrt{x}}} \cdot \frac{1}{2\sqrt{x}} dx = dt$

$\Rightarrow \quad dy = \frac{dt}{2\sqrt{t}}$

$\Rightarrow \quad 2dy = \frac{1}{\sqrt{t}} dt$

$\Rightarrow \quad 2y = 2\sqrt{t} + c$

$\Rightarrow \quad 2y = 2\sqrt{4+\sqrt{9+\sqrt{x}}} + c$

Given $y(0) = \sqrt{7} \Rightarrow c = 0$

$\therefore \quad y = \sqrt{4+\sqrt{9+\sqrt{x}}}$

$\therefore \quad y(256) = 3$

Multiple Correct Answers Type

1. (1), (4)

$\frac{dy}{dx} - y \tan x = 2x \sec x$

$\therefore \quad \cos x \frac{dy}{dx} + (-\sin x) y = 2x$

$\therefore \quad \frac{d}{dx}(y \cos x) = 2x$

Integrating, we get

$y(x) \cos x = x^2 + c$, where $c = 0$ since $y(0) = 0$

When $x = \frac{\pi}{4}, y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}$.

When $x = \frac{\pi}{3}, y\left(\frac{\pi}{3}\right) = \frac{2\pi^2}{9}$

When $x = \frac{\pi}{4}, y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}} + \frac{\pi}{\sqrt{2}}$

When $x = \frac{\pi}{3}, y'\left(\frac{\pi}{3}\right) = \frac{2\pi^2}{3\sqrt{3}} + \frac{4\pi}{3}$

2. (2), (3)

Centers of circle lie on the straight line $y = x$.

$\therefore$ Equation of family of circles is

$(x-\alpha)^2 + (y-\alpha)^2 = r^2$

$\therefore \quad x^2 + y^2 - 2\alpha x - 2\alpha y + 2\alpha^2 - r^2 = 0$

Differentiating w.r.t. x, we get

$2x + 2yy' - 2\alpha - 2\alpha y' = 0$ (1)

$\Rightarrow \quad \alpha = \frac{x + yy'}{1 + y'}$ (2)

Again differentiating w.r.t. x, we get

$2 + 2(y')^2 + 2yy'' - 2\alpha y'' = 0$

$\Rightarrow \quad 1 + (y')^2 + yy'' - \left(\frac{x + yy'}{1 + y'}\right) y'' = 0$

$\Rightarrow \quad 1 + y' + (y')^2 + (y')^3 + yy'' + yy'y'' - xy'' - yy'y'' = 0$

$\Rightarrow \quad (y - x)y'' + (1 + y' + (y')^2)y' + 1 = 0$

$\Rightarrow \quad P = y - x, Q = 1 + y' + (y')^2$

3. (1), (3)

We have $(1 + e^x)\frac{dy}{dx} + ye^x = 1$

$\Rightarrow \quad \frac{dy}{dx} + \frac{e^x}{1+e^x} y = \frac{1}{1+e^x}$ (Linear differential equation)

$\text{I.F.} = e^{\int \frac{e^x}{1+e^x}} = e^{\ln(1+e^x)} = 1 + e^x$

Therefore, solution is:

$y \cdot (1 + e^x) = \int 1 \, dx$

or $\quad (1 + e^x)y = x + c$

Given $x = 0, y = 2$ so $c = 4$

$\therefore \quad (1 + e^x)y = x + 4$

$\Rightarrow \quad y = \frac{x+4}{e^x + 1}$

$y(-4) = 0$

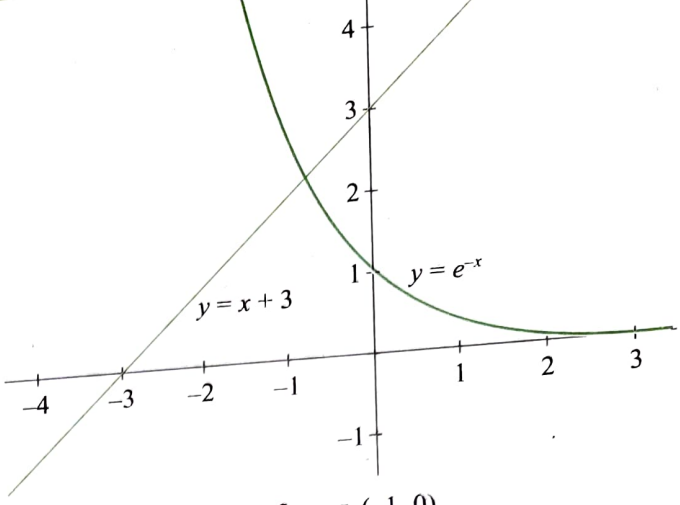
$y(-2) = \frac{2}{e^{-2} + 1}$

$\frac{dy}{dx} = \frac{(e^x + 1) \cdot 1 - (x+4)e^x}{(e^x + 1)^2}$

$= \frac{e^x(-x-3)+1}{(e^x + 1)^2}$

when $\frac{dy}{dx} = 0$ then $x + 3 = e^{-x}$

Graphs of $y = x + 3$ and $y = e^{-x}$ are as shown in the following figure.



Clearly, graphs intersect for $x \in (-1, 0)$.

Hence, $y(x)$ has a critical point in the interval $(-1, 0)$.

4. (1), (4)

$(x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 = 0, x > 0$

$\Rightarrow \quad (x + 2)^2 + y(x + 2) = y^2 \cdot \frac{dx}{dy}$

$\Rightarrow \quad \frac{dx}{dy} = \frac{(x+2)^2}{y^2} + \frac{x+2}{y}$

$\Rightarrow \quad \frac{1}{(x+2)^2}\frac{dx}{dy} = \frac{1}{y^2} + \frac{1}{y(x+2)}$

$\therefore \quad \frac{1}{(x+2)^2}\frac{dx}{dy} - \frac{1}{(x+2)y} = \frac{1}{y^2}$ ...(i)

Put $\frac{1}{x+2} = t,$

$\therefore \quad -\frac{1}{(x+2)^2}\frac{dx}{dy} = \frac{dt}{dy}$

$\therefore$ Equation (i) reduces to

$-\frac{dt}{dy} - \frac{t}{y} = \frac{1}{y^2}$

$\Rightarrow \quad \frac{dt}{dy} + \frac{t}{y} = -\frac{1}{y^2}$

Above is linear differential equation.

$\text{I.F.} = e^{\int \frac{1}{y} dy} = y$

$\therefore$ Solution is

$t.y = \int y\left(-\frac{1}{y^2}\right)dy + C$

or $\quad t.y = C - \log y$

$\therefore \quad \frac{1}{x+2} \cdot y = C - \log y$

It passes through (1, 3)

$\Rightarrow \quad 1 = C - 1\log 3$

$\Rightarrow \quad C = 1 + \log(3)$

$\therefore \quad$ Solution curve is $\frac{y}{x+2} = 1 + \log 3 - \log y$

For $y = x + 2$, we have $1 = 1 + \log 3 - \log y$

$\therefore \quad y = 3$

So, option (1) is correct.

For option (3),

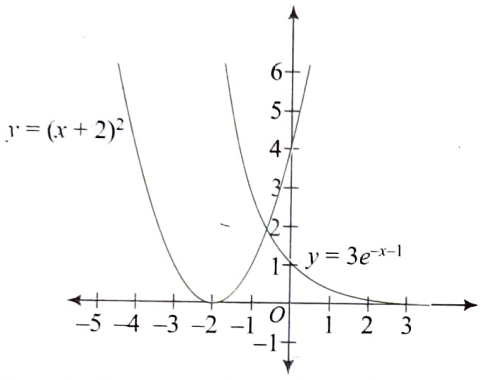
$$\frac{(x+2)^2}{x+2} = 1 - \log\left(\frac{y}{3}\right)$$

$$\therefore \quad x + 1 = \log_e\left(\frac{3}{y}\right)$$

$$\therefore \quad y = 3e^{-x-1}$$

or $\quad (x+2)^2 = 3e^{-x-1}$

Now curves $y = (x+2)^2$ and $y = 3e^{-x-1}$ intersect as shown in the figure.



From the figure no solution for $x > 0$

For option (4),

$$\frac{(x+3)^2}{x+2} - 1 = -\log\left\{\frac{(x+3)^2}{3}\right\}$$

$$\Rightarrow \quad \frac{(x+2)^2 + 1 + 2(x+2)}{x+2} - 1 + 2\log(x+3) - \log 3 = 0$$

Let $g(x) = (x+2) + \dfrac{1}{x+2} + 1 + 2\log(x+3) - \log 3$

$$\Rightarrow \quad g'(x) = 1 - \frac{1}{(x+2)^2} + \frac{2}{x+3}$$

$$= 1 + \frac{2(x+2)^2 - (x+3)}{(x+3)(x+2)^2}$$

$$= 1 + \frac{2x^2 + 7x + 5}{(x+3)(x+2)^2} > 0 \quad \forall x > 0$$

$\therefore \quad g(x)$ is an increasing function.

Also $g(0) = 2 + \dfrac{1}{2} + 1 + 2\log 3 - \log 3 > 0$

So $g(x) \neq 0$ for $x > 0$

Hence equation has no solution.

5. (2), (3)

We have $f'(x) = (e^{f(x)-g(x)})\, g'(x), f(1) = g(2) = 1$

$\therefore \quad e^{-f(x)} f'(x) = e^{-g(x)}\, g'(x)$

Integrating both sides, we get

$$e^{-f(x)} = e^{-g(x)} + c \qquad \ldots(i)$$

Putting $x = 1$, $e^{-f(1)} = e^{-g(1)} + c$

$\therefore \quad e^{-1} = e^{-g(1)} + c \qquad [\text{as } f(1) = 1] \quad \ldots(ii)$

Putting $x = 2$ in (i), we get

$$e^{-f(2)} = e^{-g(2)} + c$$

$\therefore \quad e^{-f(2)} = e^{-1} + c \qquad [\text{as } g(2) = 1] \quad \ldots(iii)$

Subtracting (iii) from (ii), we get

$$e^{-1} - e^{-f(2)} = e^{-g(1)} - e^{-1}$$

$\therefore \quad e^{-f(2)} = 2e^{-1} - e^{-g(1)} < 2e^{-1} \qquad [\text{as } e^{-g(1)} > 0]$

$\Rightarrow \quad -f(2) < \log(2e^{-1})$

$\Rightarrow \quad f(2) > 1 - \log_e 2 > 1 - \log 2$

Similarly, $g(1) > 1 - \log_e 2$

Numerical Value Type

1. (9) Equation of tangent to $y = f(x)$ at point (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$

Put $x = 0$ to get y intercept.

$$y_1 - mx_1 = x_1^3 \quad \text{(given)}$$

$$\therefore \quad y_1 - x_1 \frac{dy}{dx} = x_1^3$$

$$\therefore \quad x\frac{dy}{dx} - y = -x^3$$

$$\therefore \quad \frac{dy}{dx} - \frac{y}{x} = -x^2$$

$$\text{I.F.} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

Thus, solution is $y \times \dfrac{1}{x} = \int -x^2 \times \dfrac{1}{x}\, dx$

or $\quad \dfrac{y}{x} = -\dfrac{x^2}{2} + c$

or $\quad f(x) = -\dfrac{x^3}{2} + \dfrac{3}{2}x \qquad [\text{as } f(1) = 1]$

$\therefore \quad f(-3) = 9$

2. (0) $y'(x) + y(x)\, g'(x) = g(x)\, g'(x)$

or $\quad e^{g(x)} y'(x) + e^{g(x)} g'(x)\, y(x) = e^{g(x)}\, g(x)\, g'(x)$

$$\therefore \quad \frac{d}{dx}\left(y(x)\, e^{g(x)}\right) = e^{g(x)}\, g(x)\, g'(x)$$

$$\therefore \quad y(x)\, e^{g(x)} = \int e^{g(x)} g(x) g'(x)\, dx$$

$$= \int e^t\, t\, dt, \text{ where } g(x) = t$$

$$= (t-1)\, e^t + c$$

$\therefore \quad y(x)\, e^{g(x)} = (g(x) - 1)\, e^{g(x)} + c$

Put $x = 0$. Then $0 = (0-1) \cdot 1 + c$ or $c = 1$.

Put $x = 2$. Then $y(2) \cdot 1 = (0-1) \cdot (1) + 1$

$$y(2) = 0.$$

3. (6) $6\int_1^x f(t)\, dt = 3x f(x) - x^3$

Differentiating w.r.t. x, we get

or $\quad 6f(x) = 3f(x) + 3x f'(x) - 3x^2$

or $\quad xf'(x) - f(x) = x^2$

or $\quad x\dfrac{dy}{dx} - y = x^2$

or $\quad \dfrac{xdy - ydx}{x^2} = dx$

or $\quad \int \dfrac{xdy - ydx}{x^2} = \int dx$

or $\quad \dfrac{y}{x} = x + c$

Given $f(1) = 2$

or $c = 1$

or $y = x^2 + x$

Note If we put $x = 1$ in the given equation, we get $f(1) = 1/3$.

4. **(0.4)** $\frac{dy}{dx} = (5y + 2)(5y - 2)$

$$\Rightarrow \quad \frac{1}{25}\int \frac{dy}{y^2 - \left(\frac{2}{5}\right)^2} = \int dx$$

$$\Rightarrow \quad \frac{1}{25}\cdot\frac{5}{4}\log_e\left|\frac{y - \frac{2}{5}}{y + \frac{2}{5}}\right| = x + c$$

$$\Rightarrow \quad \frac{1}{20}\log_e\left|\frac{5y - 2}{5y + 2}\right| = x + c$$

Given that $f(0) = y(0) = 0$.

$\therefore \quad c = 0$

Hence, $\left|\frac{2 - 5y}{2 + 5y}\right| = e^{20x}$

$$\therefore \quad \lim_{x \to -\infty}\left|\frac{2 - 5y}{2 + 5y}\right| = \lim_{x \to -\infty} e^{20x}$$

$$\Rightarrow \quad \lim_{x \to -\infty}\left|\frac{2 - 5y}{2 + 5y}\right| = 0$$

$$\Rightarrow \quad 2 - 5\lim_{x \to -\infty} y = 0$$

$$\Rightarrow \quad \lim_{x \to -\infty} y = \frac{2}{5} = 0.4$$

1 APPENDIX

Chapterwise Solved January 2019 JEE Main Questions (All Sets)

Chapter 1 (Relations and Functions)

1. For $x \in R - \{0, 1\}$, let $f_1(x) = \frac{1}{x}$, $f_2(x) = 1 - x$ and $f_3(x) = \frac{1}{1-x}$ be three given functions. If a function $J(x)$ satisfies $(f_2 o J o f_1)(x) = f_3(x)$, then $J(x)$ is equal to

(1) $f_3(x)$ (2) $f_1(x)$ (3) $f_2(x)$ (4) $\frac{1}{x} f_3(x)$

2. Let $A = \{x \in R : x \text{ is not a positive integer}\}$ define a function $f: A \to R$ such that $f(x) = \frac{2x}{x-1}$. Then f is

(1) injective but not surjective
(2) not injective
(3) surjective but not injective
(4) neither injective nor surjective

3. Let N be the set of natural numbers and two functions f and g be defined as $f, g : N \to N$ such that

$$f(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases} \quad \text{and } g(n) = n - (-1)^n. \text{ Then}$$

fog is

(1) both one–one and onto
(2) one–one but not onto
(3) neither one–one nor onto
(4) onto but not one–one

4. Let $f: R \to R$ be defined by $f(x) = \frac{x}{1+x^2}, x \in R$. Then the range of f is

(1) $(-1, 1) - \{0\}$ (2) $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(3) $R - \left[-\frac{1}{2}, \frac{1}{2}\right]$ (4) $R - [-1, 1]$

5. Let a function $f: (0, \infty) \to [0, \infty)$ be defined by $f(x) = \left|1 - \frac{1}{x}\right|$. Then f is

(1) injective only
(2) not injective but it is surjective
(3) both injective and surjective
(4) neither injective nor surjective

Chapter 2 (Limits)

6. $\lim\limits_{y \to 0} \frac{\sqrt{1 + \sqrt{1 + y^4}} - \sqrt{2}}{y^4}$

(1) exists and equals $\frac{1}{4\sqrt{2}}$
(2) does not exist
(3) exists and equals $\frac{1}{2\sqrt{2}}$
(4) exists and equals $\frac{1}{2\sqrt{2}(\sqrt{2}+1)}$

7. For each $x \in R$, let $[x]$ be the greatest integer less than or equal to x. Then $\lim\limits_{x \to 0^-} \frac{x([x] + |x|)\sin[x]}{|x|}$ is equal to

(1) $-\sin 1$ (2) 0 (3) 1 (4) $\sin 1$

8. For each $t \in R$, let $[t]$ be the greatest integer less than or equal to t. Then

$$\lim_{x \to 1^+} \frac{(1 - |x| + \sin|1 - x|)\sin\left(\frac{\pi}{2}[1-x]\right)}{|1-x|[1-x]}$$

(1) equals -1 (2) equals 1
(3) does not exist (4) equals 0

9. Let $[x]$ denote the greatest integer less than or equal to x. Then $\lim\limits_{x \to 0} \frac{\tan\left(\pi \sin^2 x\right) + \left(|x| - \sin\left(x[x]\right)\right)^2}{x^2}$

(1) equals π (2) equals 0
(3) equals $\pi + 1$ (4) does not exist

10. $\lim\limits_{x \to 0} \frac{x \cot(4x)}{\sin^2 x \cot^2(2x)}$ is equal to

(1) 2 (2) 0 (3) 4 (4) 1

11. $\lim\limits_{x \to \frac{\pi}{4}} \frac{\cot^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)}$ is

(1) 4 (2) $8\sqrt{2}$ (3) 8 (4) $4\sqrt{2}$

12. $\lim\limits_{x\to 1^-} \dfrac{\sqrt{\pi}-\sqrt{2\sin^{-1}x}}{\sqrt{1-x}}$ is equal to

(1) $\dfrac{1}{\sqrt{2\pi}}$ (2) $\sqrt{\dfrac{\pi}{2}}$

(3) $\sqrt{\dfrac{2}{\pi}}$ (4) $\sqrt{\pi}$

Chapter 3 (Differentiation)

13. If $x = 3\tan t$ and $y = 3\sec t$, then the value of $\dfrac{d^2y}{dx^2}$ at $t = \dfrac{\pi}{4}$ is

(1) $\dfrac{3}{2\sqrt{2}}$ (2) $\dfrac{1}{3\sqrt{2}}$

(3) $\dfrac{1}{6}$ (4) $\dfrac{1}{6\sqrt{2}}$

14. Let $f: R \to R$ be a function such that $f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)$, $x \in R$. Then $f(2)$ equals

(1) 8 (2) −2 (3) −4 (4) 30

15. If $x\log_e(\log_e x) - x^2 + y^2 = 4 (y > 0)$, then dy/dx at $x = e$ is equal to

(1) $\dfrac{e}{4+e^2}$ (2) $\dfrac{(1+2e)}{2\sqrt{4+e^2}}$

(3) $\dfrac{(2e-1)}{2\sqrt{4+e^2}}$ (4) $\dfrac{(1+2e)}{\sqrt{4+e^2}}$

16. For $x > 1$, if $(2x)^{2y} = 4e^{2x-2y}$, then $(1+\log_e 2x)^2 \dfrac{dy}{dx}$ is equal to

(1) $\log_e 2x$ (2) $\dfrac{x\log_e 2x + \log_e 2}{x}$

(3) $x\log_e 2x$ (4) $\dfrac{x\log_e 2x - \log_e 2}{x}$

17. Let f be a differentiable function such that $f(1) = 2$ and $f'(x) = f(x)$ for all $x \in R$. If $h(x) = f(f(x))$, then $h'(1)$ is equal to

(1) $4e$ (2) $4e^2$ (3) $2e$ (4) $2e^2$

Chapter 4 (Continuity and Differentiability)

18. Let $f: R \to R$ be a function defined as

$$f(x) = \begin{cases} 5, & \text{if } x \le 1 \\ a + bx, & \text{if } 1 < x < 3 \\ b + 5x, & \text{if } 3 \le x < 5 \\ 30, & \text{if } x \ge 5 \end{cases}$$

. Then f is

(1) continuous if $a = 5$ and $b = 5$

(2) continuous if $a = -5$ and $b = 10$

(3) continuous if $a = 0$ and $b = 5$

(4) not continuous for any values of a and b

19. Let $f(x) = \begin{cases} \max.\{|x|, x^2\}, & |x| \le 2 \\ 8 - 2|x|, & 2 < |x| \le 4 \end{cases}$. Let S be the set of points in the interval $(-4, 4)$ at which f is not differentiable. Then S

(1) is an empty set

(2) equals $\{-2, -1, 1, 2\}$

(3) equals $\{-2, -1, 0, 1, 2\}$

(4) equals $\{-2, 2\}$

20. Let $f: (-1, 1) \to R$ be a function defined by $f(x) = \max.\ \{-|x|, -\sqrt{1-x^2}\}$. If K is the set of all points at which f is not differentiable, then K has exactly

(1) three elements (2) one element

(3) five elements (4) two elements

21. Let $f(x) = \begin{cases} -1, & -2 \le x < 0 \\ x^2 - 1, & 0 \le x \le 2 \end{cases}$ and $g(x) = |f(x)| + f(|x|)$. Then, in the interval $(-2, 2)$, g is

(1) differentiable at all points

(2) not differentiable at two points

(3) not continuous

(4) not differentiable at one point

22. Let K be the set of all real values of x where the function $f(x) = \sin|x| - |x| + 2(x - \pi)\cos|x|$ is not differentiable. Then the set K is equal to

(1) $\{\pi\}$ (2) $\{0\}$

(3) ϕ (an empty set) (4) $\{0, \pi\}$

23. Let S be the set of all points in $(-\pi, \pi)$ at which the function $F(x) = \min.\{\sin x, \cos x\}$ is not differentiable. Then S is a subset of which of the following?

(1) $\left\{-\dfrac{3\pi}{4}, -\dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{\pi}{4}\right\}$ (2) $\left\{-\dfrac{3\pi}{4}, -\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{4}\right\}$

(3) $\left\{-\dfrac{\pi}{2}, -\dfrac{\pi}{4}, \dfrac{\pi}{4}, \dfrac{\pi}{2}\right\}$ (4) $\left\{-\dfrac{\pi}{4}, 0, \dfrac{\pi}{4}\right\}$

Chapter 5 (Application of Derivatives)

24. If θ denotes the acute angle between the curves, $y = 10 - x^2$ and $y = 2 + x^2$ at a point of their intersection, then $|\tan\theta|$ is equal to

(1) 4/9 (2) 7/17

(3) 8/17 (4) 8/15

25. The tangent to the curve $y = xe^{x^2}$ passing through the point $(1, e)$ also passes through the point

(1) $\left(\dfrac{4}{3}, 2e\right)$ (2) $(2, 3e)$

(3) $\left(\dfrac{5}{3}, 2e\right)$ (4) $(3, 6e)$

26. A helicopter is flying along the curve given by $y - x^{3/2} = 7$, $(x \geq 0)$. A soldier positioned at the point $\left(\frac{1}{2}, 7\right)$ wants to shoot down the helicopter when it is nearest to him. Then this nearest distance is

(1) $\frac{1}{2}$ (2) $\frac{1}{3}\sqrt{\frac{7}{3}}$ (3) $\frac{1}{6}\sqrt{\frac{7}{3}}$ (4) $\frac{\sqrt{5}}{6}$

Chapter 6
(Monotonicity and Maxima-Minima of Functions)

27. The maximum volume (in cu. m) of the right circular cone having slant height 3 m is

(1) $3\sqrt{3}\pi$ (2) 6π (3) $2\sqrt{3}\pi$ (4) $\frac{4}{3}\pi$

28. The shortest distance between the point $\left(\frac{3}{2}, 0\right)$ and the curve $y = \sqrt{x}$, $(x > 0)$ is

(1) $\frac{\sqrt{5}}{2}$ (2) $\frac{5}{4}$ (3) $\frac{3}{2}$ (4) $\frac{\sqrt{3}}{2}$

29. The maximum value of the function $f(x) = 3x^3 - 18x^2 + 27x - 40$ on the set $S = \{x \in R : x^2 + 30 \leq 11x\}$ is

(1) 122 (2) –222 (3) –122 (4) 222

30. Let $f(x) = \frac{x}{\sqrt{a^2 + x^2}} - \frac{d - x}{\sqrt{b^2 + (d - x)^2}}$, $x \in R$, where a, b and d are non-zero real constants. Then

(1) f is a decreasing function of x

(2) f is neither increasing nor decreasing function of x

(3) f' is not a continuous function of x

(4) f is an increasing function of x

31. The maximum area (in sq. units) of a rectangle having its base on the x-axis and its other two vertices on the parabola $y = 12 - x^2$ such that the rectangle lies inside the parabola is

(1) $20\sqrt{2}$ (2) $18\sqrt{2}$ (3) 32 (4) 36

32. If the function f given by $f(x) = x^3 - 3(a - 2)x^2 + 3ax + 7$, for some $a \in R$ is increasing in $(0,1]$ and decreasing in $[1, 5)$, then a root of the equation $\frac{f(x) - 14}{(x-1)^2} = 0 (x \neq 1)$ is

(1) 6 (2) 5 (3) 7 (4) –7

Chapter 7
(Indefinite Integration)

33. For $x^2 \neq n\pi + 1$, $n \in N$ (the set of natural numbers), the integral $\int x \sqrt{\frac{2\sin(x^2 - 1) - \sin 2(x^2 - 1)}{2\sin(x^2 - 1) + \sin 2(x^2 - 1)}}\, dx$ is equal to (where c is a constant of integration)

(1) $\log_e \left|\sec\left(\frac{x^2 - 1}{2}\right)\right| + c$

(2) $\log_e \left|\frac{1}{2}\sec^2(x^2 - 1)\right| + c$

(3) $\frac{1}{2}\log_e \left|\sec^2\left(\frac{x^2 - 1}{2}\right)\right| + c$

(4) $\frac{1}{2}\log_e |\sec(x^2 - 1)| + c$

34. If $f(x) = \int \frac{5x^8 + 7x^6}{(x^2 + 1 + 2x^7)}\, dx$, $(x \geq 0)$ and $f(0) = 0$, then the value of $f(1)$ is

(1) $-\frac{1}{2}$ (2) $\frac{1}{2}$ (3) $-\frac{1}{4}$ (4) $\frac{1}{4}$

35. Let $n \geq 2$ be a natural number and $0 < \theta < \pi/2$. Then $\int \frac{(\sin^n \theta - \sin\theta)^{1/n} \cos\theta}{\sin^{n+1}\theta} d\theta$ is equal to (where C is a constant of integration)

(1) $\frac{n}{n^2 - 1}\left(1 - \frac{1}{\sin^{n+1}\theta}\right)^{\frac{n+1}{n}} + C$

(2) $\frac{n}{n^2 + 1}\left(1 - \frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}} + C$

(3) $\frac{n}{n^2 - 1}\left(1 - \frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}} + C$

(4) $\frac{n}{n^2 - 1}\left(1 + \frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}} + C$

36. If $\int x^5 e^{-4x^3}\, dx = \frac{1}{48} e^{-4x^3} f(x) + C$, where C is a constant of integration, then $f(x)$ is equal to

(1) $-4x^3 - 1$ (2) $4x^3 + 1$

(3) $-2x^3 - 1$ (4) $-2x^3 + 1$

37. If $\int \frac{\sqrt{1 - x^2}}{x^4} dx = A(x)\left(\sqrt{1 - x^2}\right)^m + C$, for a suitable chosen integer m and a function $A(x)$, where C is a constant of integration, then $(A(x))^m$ equals

(1) $\frac{-1}{3x^3}$ (2) $\frac{-1}{27x^9}$ (3) $\frac{1}{9x^4}$ (4) $\frac{1}{27x^6}$

38. If $\int \frac{x + 1}{\sqrt{2x - 1}} dx = f(x)\sqrt{2x - 1} + C$, where C is a constant of integration, then $f(x)$ is equal to

(1) $\frac{1}{3}(x + 4)$ (2) $\frac{1}{3}(x + 1)$

(3) $\frac{2}{3}(x + 2)$ (4) $\frac{2}{3}(x - 4)$

39. The integral $\int \cos(\log_e x)dx$ is equal to (where C is a constant of integration)

(1) $\frac{x}{2}\left[\sin(\log_e x)-\cos(\log_e x)\right]+C$

(2) $\frac{x}{2}\left[\cos(\log_e x)+\sin(\log_e x)\right]+C$

(3) $x\left[\cos(\log_e x)+\sin(\log_e x)\right]+C$

(4) $x\left[\cos(\log_e x)-\sin(\log_e x)\right]+C$

40. The integral $\int \frac{3x^{13}+2x^{11}}{\left(2x^4+3x^2+1\right)^4}dx$ is equal to (where C is a constant of integration)

(1) $\frac{x^4}{\left(2x^4+3x^2+1\right)^3}+C$ (2) $\frac{x^{12}}{6\left(2x^4+3x^2+1\right)^3}+C$

(3) $\frac{x^4}{6\left(2x^4+3x^2+1\right)^3}+C$ (4) $\frac{x^{12}}{\left(2x^4+3x^2+1\right)^3}+C$

Chapter 8 (Definite Integration)

41. The value of $\int_0^{\pi} |\cos x|^3 dx$ is

(1) 2/3 (2) 0 (3) –4/3 (4) 4/3

42. Let f be a differentiable function from R to R such that $|f(x)-f(y)| \le 2|x-y|^{3/2}$, for all $x, y \in R$. If $f(0) = 1$, then $\int_0^1 f^2(x)dx$ is equal to

(1) 0 (2) $\frac{1}{2}$ (3) 2 (4) 1

43. If $\int_0^{\frac{\pi}{3}} \frac{\tan\theta}{\sqrt{2k\sec\theta}}d\theta = 1-\frac{1}{\sqrt{2}}$, $(k>0)$, then the value of k is

(1) 2 (2) $\frac{1}{2}$ (3) 4 (4) 1

44. Let $I = \int_a^b (x^4 - 2x^2)dx$. If I is minimum, then the ordered pair (a, b) is

(1) $(-\sqrt{2}, 0)$ (2) $(-\sqrt{2}, \sqrt{2})$

(3) $(0, \sqrt{2})$ (4) $(\sqrt{2}, -\sqrt{2})$

45. The value of $\int_{-\pi/2}^{\pi/2} \frac{dx}{[x]+[\sin x]+4}$, where $[t]$ denotes the greatest integer less than or equal to t, is

(1) $\frac{1}{12}(7\pi+5)$ (2) $\frac{3}{10}(4\pi-3)$

(3) $\frac{1}{12}(7\pi-5)$ (4) $\frac{3}{20}(4\pi-3)$

46. If $\int_0^x f(t)dt = x^2 + \int_x^1 t^2 f(t)dt$, then $f'(1/2)$ is

(1) $\frac{6}{25}$ (2) $\frac{24}{25}$

(3) $\frac{18}{25}$ (4) $\frac{4}{5}$

47. The value of the integral $\int_{-2}^{2} \frac{\sin^2 x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}}dx$ (where $[x]$ denotes the greatest integer less than or equal to x) is

(1) 4 (2) 4 – sin 4

(3) sin 4 (4) 0

48. The integral $\int_{\pi/6}^{\pi/4} \frac{dx}{\sin 2x\left(\tan^5 x+\cot^5 x\right)}$ equals

(1) $\frac{1}{10}\left(\frac{\pi}{4}-\tan^{-1}\left(\frac{1}{9\sqrt{3}}\right)\right)$

(2) $\frac{1}{5}\left(\frac{\pi}{4}-\tan^{-1}\left(\frac{1}{3\sqrt{3}}\right)\right)$

(3) $\frac{\pi}{10}$

(4) $\frac{1}{20}\tan^{-1}\left(\frac{1}{9\sqrt{3}}\right)$

49. Let f and g be continuous functions on $[0, a]$ such that $f(x) = f(a-x)$ and $g(x) + g(a-x) = 4$, then $\int_0^a f(x)g(x)dx$ is equal to

(1) $4\int_0^a f(x)dx$ (2) $2\int_0^a f(x)dx$

(3) $-3\int_0^a f(x)dx$ (4) $\int_0^a f(x)dx$

50. The integral $\int_1^e \left\{\left(\frac{x}{e}\right)^{2x} - \left(\frac{e}{x}\right)^x\right\}\log_e x dx$ is equal to

(1) $\frac{1}{2}-e-\frac{1}{e^2}$ (2) $\frac{3}{2}-\frac{1}{e}-\frac{1}{2e^2}$

(3) $-\frac{1}{2}+\frac{1}{e}-\frac{1}{2e^2}$ (4) $\frac{3}{2}-e-\frac{1}{2e^2}$

51. $\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\ldots+\frac{1}{5n}\right)$ is equal to

(1) $\frac{\pi}{4}$ (2) $\tan^{-1}(2)$

(3) $\tan^{-1}(3)$ (4) $\frac{\pi}{2}$

Chapter 9 (Area)

52. The area (in sq. units) bounded by the parabola $y = x^2 - 1$, the tangent at the point (2, 3) to it and the y-axis is

(1) $\frac{14}{3}$ (2) $\frac{56}{3}$ (3) $\frac{8}{3}$ (4) $\frac{32}{3}$

53. The area (in sq. units) of the region $A = [(x, y) : 0 \le y \le x|x| + 1 \text{ and } -1 \le x \le 1]$ is

(1) $\frac{2}{3}$ (2) $\frac{1}{3}$ (3) 2 (4) $\frac{4}{3}$

54. The area enclosed between the curves $y = kx^2$ and $x = ky^2$, $(k > 0)$ is 1 sq. unit. Then k is

(1) $\frac{1}{\sqrt{3}}$ (2) $\frac{2}{\sqrt{3}}$ (3) $\frac{\sqrt{3}}{2}$ (4) $\sqrt{3}$

55. The area (in sq. units) of the region bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$ is

(1) $\frac{5}{4}$ (2) $\frac{9}{8}$ (3) $\frac{3}{4}$ (4) $\frac{7}{8}$

56. The area (in sq. units) in the first quadrant bounded by the parabola $y = x^2 + 1$, the tangent to it at the point (2, 5) and the coordinate axes is

(1) $\frac{14}{3}$ (2) $\frac{187}{24}$ (3) $\frac{37}{24}$ (4) $\frac{8}{3}$

57. The area (in sq. units) of the region bounded by the parabola $y = x^2 + 2$ and the lines $y = x + 1$, $x = 0$ and $x = 3$, is

(1) $\frac{15}{4}$ (2) $\frac{15}{2}$ (3) $\frac{21}{2}$ (4) $\frac{17}{4}$

Chapter 10 (Differential Equations)

58. If $y = y(x)$ is the solution of the differential equation, $x\frac{dy}{dx} + 2y = x^2$ satisfying $y(1) = 1$, then $y\left(\frac{1}{2}\right)$ is equal to

(1) $\frac{7}{64}$ (2) $\frac{13}{16}$ (3) $\frac{49}{16}$ (4) $\frac{1}{4}$

59. Let $f : [0, 1] \to R$ be such that $f(xy) = f(x) \times f(y)$ for all x, $y \in [0, 1]$, and $f(0) \neq 0$. If $y = y(x)$ satisfies the differential equation, $\frac{dy}{dx} = f(x)$ with $y(0) = 1$, then $y\left(\frac{1}{4}\right) + y\left(\frac{3}{4}\right)$ is equal to

(1) 4 (2) 3 (3) 5 (4) 2

60. If $\frac{dy}{dx} + \frac{3}{\cos^2 x}y = \frac{1}{\cos^2 x}$, $x \in \left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$ and $y\left(\frac{\pi}{4}\right) = \frac{4}{3}$, then $y\left(-\frac{\pi}{4}\right)$ equals

(1) $\frac{1}{3} + e^6$ (2) $\frac{1}{3}$

(3) $-\frac{4}{3}$ (4) $\frac{1}{3} + e^3$

61. Let f be a differentiable function such that $f'(x) = 7 - \frac{3}{4}\frac{f(x)}{x}$, $(x > 0)$ and $f(1) \neq 4$. Then

$\lim\limits_{x \to 0^+} xf\left(\frac{1}{x}\right)$

(1) exists and equals 4

(2) does not exist

(3) exist and equals 0

(4) exists and equals 4/7

62. The curve amongst the family of curves, represented by the differential equation $(x^2 - y^2)dx + 2xy\,dy = 0$ which passes through (1, 1) is

(1) a circle with centre on the y-axis

(2) a circle with centre on the x-axis

(3) an ellipse with major axis along the y-axis

(4) a hyperbola with transverse axis along the

63. If $y(x)$ is the solution of the differential equation $\frac{dy}{dx} + \left(\frac{2x+1}{x}\right)y = e^{-2x}, x > 0$, where $y(1) = \frac{1}{2}e^{-2}$, then

(1) $y(x)$ is decreasing in (0, 1)

(2) $y(x)$ is decreasing in $\left(\frac{1}{2}, 1\right)$

(3) $y(\log_e 2) = \frac{\log_e 2}{4}$

(4) $y(\log_e 2) = \log_2 4$

64. The solution of the differential equation $\frac{dy}{dx} = (x - y)^2$, when $y(1) = 1$, is

(1) $\log_e\left|\frac{2-y}{2-x}\right| = 2(y-1)$

(2) $\log_e\left|\frac{2-x}{2-y}\right| = x - y$

(3) $-\log_e\left|\frac{1+x-y}{1-x+y}\right| = x + y - 2$

(4) $-\log_e\left|\frac{1-x+y}{1+x-y}\right| = 2(x-1)$

65. Let $y = y(x)$ be the solution of the differential equation $x\frac{dy}{dx} + y = x\log_e x, (x > 1)$. If $2y(2) = \log_e 4 - 1$, then $y(e)$ is equal to

(1) $\frac{e^2}{4}$ (2) $\frac{e}{4}$ (3) $-\frac{e}{2}$ (4) $-\frac{e^2}{2}$

66. If a curve passes through the point (1, –2) and has slope of the tangent at any point (x, y) on it as $\frac{x^2 - 2y}{x}$, then the curve also passes through the point

(1) $(-\sqrt{2}, 1)$ (2) $(\sqrt{3}, 0)$

(3) $(-1, 2)$ (4) $(3, 0)$

Answers Key

1. (1)	**2.** (1)	**3.** (4)	**4.** (2)	**5.** (2)
6. (1)	**7.** (1)	**8.** (4)	**9.** (4)	**10.** (4)
11. (3)	**12.** (3)	**13** (4)	**14.** (2)	**15.** (3)
16. (4)	**17.** (1)	**18.** (4)	**19.** (3)	**20.** (1)
21. (4)	**22.** (3)	**23.** (1)	**24.** (4)	**25.** (1)
26. (3)	**27.** (3)	**28.** (1)	**29.** (1)	**30.** (4)
31. (3)	**32.** (3)	**33.** (1)	**34.** (4)	**35.** (3)

36. (1)	**37.** (2)	**38.** (1)	**39.** (2)	**40.** (2)
41. (4)	**42.** (4)	**43.** (1)	**44.** (2)	**45.** (4)
46. (2)	**47.** (4)	**48.** (1)	**49.** (2)	**50.** (4)
51. (2)	**52.** (3)	**53.** (3)	**54.** (1)	**55.** (2)
56. (3)	**57.** (2)	**58.** (3)	**59.** (2)	**60.** (1)
61. (1)	**62.** (2)	**63.** (2)	**64.** (4)	**65.** (2)
66. (2)				